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NUMERICAL EXPLORATION OF KALDORIAN INTERREGIONAL MACRODYNAMICS: ENHANCED STABILITY AND PREDOMINANCE OF PERIOD DOUBLING UNDER FLEXIBLE EXCHANGE RATES

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Abstract. We present a discrete two-regional Kaldorian macrodynamic model with flexible exchange rates, and explore the stability of equilibrium and the possibility of generation of business cycles. The structures of the two regional economies are assumed similar. The model is five-dimensional with three basic parameters, the common speed of adjustment of the goods markets and the degrees of economic interaction between the regions through trade and capital movement. We use a grid search method in two-dimensional parameter subspaces, and coefficient criteria for the flip and Hopf bifurcation curves, to determine the stability of equilibrium region and its boundary curves in several parameter ranges. We find that the model is characterized by enhanced stability of equilibrium, while its predominant asymptotic dynamical behavior when equilibrium is unstable is period doubling. This evolves to chaotic behavior by going through an intermediate phase of period–2 cycles. Business cycles are scarce and short-lived in parameter space, occurring at large values of the degree of capital movement. A characteristic difference from the fixed exchange rates system considered previously is that in our present system for cycles to occur sufficient amount of trade is required *together* with high levels of capital movement. Examples of bifurcation and Lyapunov exponent diagrams are given.

1 Introduction

Aspects of international macroeconomics and regional economics are studied recently by methods of nonlinear economic dynamics (see e.g. [10], [9], and [1]). In particular, the Kaldorian business cycle theory originated by Kaldor [6] has been developed by Lorenz [8], Gandolfo [5], and others. Interregional Kaldorian macrodynamic models of business cycles, based on trade interaction between the regions, have been studied in [7] and [9].

In this paper we study the economic interdependency between two regions. We present a five-dimensional nonlinear discrete time model of the economic transactions between the regions with flexible exchange rates. The present work is a sequel to our previous study of the corresponding model of two-regional Macrodynamics with fixed exchange rates [3]. The economic structures of both regions, assumed similar, are characterized by the Kaldorian business cycle model, and the two regions interact economically through trade and capital movement. These two factors of economic interaction are expressed by separate terms of the model equations and quantified by means of the basic interaction parameters, δ for trade and β for capital movement. The present model is an extended two-regional version of the Kaldorian small open economy model with flexible exchange rates, which was expressed as a three-dimensional system of nonlinear difference equations and considered in [2].

We explore our two-regional macrodynamic model numerically focusing on the stability of equilibrium under variations of the model parameters and on the asymptotic behavior of the system outside the stability region, and consider in particular the possibility of occurrence of business cycles. For a five-dimensional model with several parameters this is a formidable task. However, by means of a numerical grid search method and analytical coefficient criteria, we determine the stability region in several two-dimensional sections of the parameter space and identify the flip bifurcation curve and the Hopf-Neimark bifurcation curve as parts of the boundary of this region.

The model exhibits complex dynamics. Certain conclusions are drawn on the effects of the model parameters. These regard mainly the size of the region of stability of equilibrium in parameter space, the possibility of occurrence of business cycles at reasonably small values of the interaction parameters, and the type of predominant asymptotic dynamical behavior of the system outside the stability region.

2 Structure of the model

The following set of equations (1) – (9) is common to the two-regional Kaldorian macrodynamic model with flexible exchange rates in this paper and that with fixed exchange rates in [3].

Kaldorian quantity adjustment process in the goods market:

$$Y_i(t+1)-Y_i(t)=\alpha_i\left[C_i(t)+I_i(t)+G_i+J_i(t)-Y_i(t)\right];\quad \alpha_i>0. \tag{1}$$

Capital accumulation equation:

$$K_i(t+1)-K_i(t)=I_i(t). \tag{2}$$

Consumption function:

$$C_i(t)=c_i\left[Y_i(t)-T_i(t)\right]+C_{0i}\,;\qquad 0<c_i<1,\quad C_{0i}>0. \tag{3}$$

Investment function:

$$I_i(t)=I_i\left(Y_i(t),K_i(t),r_i(t)\right);\qquad \frac{\partial I_i}{\partial Y_i}>0,\quad \frac{\partial I_i}{\partial K_i}<0,\quad \frac{\partial I_i}{\partial r_i}<0. \tag{4}$$

Tax function:

$$T_i(t)=\tau_i Y_i(t)-T_{0i}\,;\quad 0<\tau_i<1,\quad T_{0i}>0. \tag{5}$$

Equilibrium condition for money market:

$$\frac{M_i(t)}{p_i}=L_i\left(Y_i(t),r_i(t)\right);\quad \frac{\partial L_i}{\partial Y_i}>0,\quad \frac{\partial L_i}{\partial r_i}<0. \tag{6}$$

Current account function of region 1:

$$J_1(t)=\delta H_1\left(Y_1(t),Y_2(t),E(t)\right);\quad \frac{\partial H_1}{\partial Y_1}<0,\quad \frac{\partial H_1}{\partial Y_2}>0,\quad \frac{\partial H_1}{\partial E}>0,\quad 0\le\delta\le 1. \tag{7}$$

Capital account function of region 1:

$$Q_1(t)=\beta\left[r_1(t)-r_2(t)-\frac{E^e(t)-E(t)}{E(t)}\right];\quad \beta>0. \tag{8}$$

Definition of total balance of payments of region 1:

$$A_1(t)=J_1(t)+Q_1(t). \tag{9}$$

Here t denotes the time period and $i(=1,2)$ is the index number of a region. The meanings of the symbols are as follows: $Y_i =$ real regional income, $C_i =$ real private consumption expenditure, $I_i =$ real private investment expenditure on physical capital, $G_i =$ real government expenditure (fixed), $K_i =$ real physical capital stock, $T_i =$ real income tax, $r_i =$ nominal rate of interest, $M_i =$ nominal money supply, $p_i =$ price level (fixed), $E =$ exchange rate (1 unit of currency of region 2 $= E$ units of currency of region 1), $E^e =$ expected exchange rate of near future, $J_i =$ balance of current account (net export) in real terms $(Ep_2J_2=-p_1J_1)$, $Q_i =$ balance of capital account in real terms $(Ep_2Q_2=-p_1Q_1)$, $A_i=J_i+Q_i=$ total balance of payments in real terms $(Ep_2A_2=-p_1A_1)$, $\alpha_i =$ adjustment speed in the goods market, $\beta =$ degree of capital mobility, $\delta =$ degree of interregional trade.

The following set of equations (10) – (12) is peculiar to the model with flexible exchange rates in this paper:

$$A_1(t)=0, \tag{10}$$

$$E^e(t+1)-E^e(t)=\gamma[E(t)-E^e(t)];\quad \gamma>0, \tag{11}$$

$$M_i(t)=\bar{M}_i=const. \tag{12}$$

Furthermore, we fix price levels as follows without loss of generality:

$$p_1=p_2=1. \tag{13}$$

Equation (10) means that the exchange rate $E(t)$ is determined endogenously to keep the equilibrium of the total balance of payments instantaneously. Equation (11) formalizes the adaptive expectation hypothesis of the changes of the expected exchange rate $E^e(t)$. It is worth noting that the nominal money supply of each region can be controlled by the monetary authority of each region independent of interregional trade and interregional capital movement in our model with flexible exchange rates. Equation (12) means that nominal money supply of each region is fixed by the regional monetary authority. In our model price levels, except the exchange rate, are supposed to be fixed for simplicity. Equation (13) is the normalization procedure to simplify the notation.

3 Derivation of the fundamental dynamical system of equations

Next, we derive the fundamental dynamical system of equations in this paper, which is a five-dimensional system of nonlinear difference equations. Substituting Equations (12) and (13) into Equation (6) and solving with respect to $r_i(t)$, we have the following "LM equation" (dependence of the rate of interest on income) in each region:

$$r_i(t) = r_i\left(Y_i(t)\right); \qquad \frac{\partial r_i}{\partial Y_i} = -\frac{\partial L_i / \partial Y_i}{\partial L_i / \partial r_i} > 0. \tag{14}$$

Substituting now Equations (7), (8), and (9) into Equation (10), we have:

$$\delta H_1\left(Y_1(t), Y_2(t), E(t)\right) + \beta\left[r_1\left(Y_1(t)\right) - r_2\left(Y_2(t)\right) - \frac{E^e(t)}{E(t)} + 1\right] = 0. \tag{15}$$

Solving this equation with respect to $E(t)$ we obtain an expression of the exchange rate $E(t)$ as an endogenous variable:

$$E(t) = E\left(Y_1(t), Y_2(t), E^e(t); \beta, \delta\right), \tag{16}$$

and differentiating (15) with respect to Y_1, Y_2 and E^e we obtain:

$$\frac{\partial E}{\partial Y_1} = \frac{-(\partial H_1 / \partial Y_1) - (\partial r_1 / \partial Y_1)\left(\beta / \delta\right)}{(\partial H_1 / \partial E) + (E^e / E^2)\left(\beta / \delta\right)}, \qquad \frac{\partial E}{\partial Y_2} = \frac{-(\partial H_1 / \partial Y_2) + (\partial r_2 / \partial Y_2)\left(\beta / \delta\right)}{(\partial H_1 / \partial E) + (E^e / E^2)\left(\beta / \delta\right)}, \tag{17}$$

$$\frac{\partial E}{\partial E^e} = \frac{\left(\beta / \delta\right)}{(\partial H_1 / \partial E)E + (E^e / E)\left(\beta / \delta\right)} > 0. \tag{18}$$

We note that due to (7) we have: $\partial E / \partial Y_1 > 0$, $\partial E / \partial Y_2 < 0$, for sufficiently small values of β / δ, and $\partial E / \partial Y_1 < 0$, $\partial E / \partial Y_2 > 0$, for sufficiently large values of β / δ. Substituting equations (2) – (5), (7), (14) and (16) into equations (1), (2), and (11), we obtain the following nonlinear five dimensional system of difference equations, which is our fundamental system of dynamical equations:

$$\begin{aligned} Y_1(t+1) = Y_1(t) + \alpha_1 \Big[& c_1(1-\tau_1)Y_1(t) + c_1 T_{01} + C_{01} + G_1 + I_1\left(Y_1(t), K_1(t), r_1\left(Y_1(t)\right)\right) \\ & + \delta H_1\left(Y_1(t), Y_2(t), E\left(Y_1(t), Y_2(t), E^e(t); \beta, \delta\right)\right) - Y_1(t)\Big] = F_1\left(Y_1(t), K_1(t), Y_2(t), E^e(t); \alpha_1, \beta, \delta\right), \end{aligned} \tag{19}$$

$$K_1(t+1) = K_1(t) + I_1\left(Y_1(t), K_1(t), r_1\left(Y_1(t)\right)\right) = F_2\left(Y_1(t), K_1(t)\right), \tag{20}$$

$$\begin{aligned} Y_2(t+1) = Y_2(t) + \alpha_2 \Big[& c_2(1-\tau_2)Y_2(t) + c_2 T_{02} + C_{02} + G_2 + I_2\left(Y_2(t), K_2(t), r_2\left(Y_2(t)\right)\right) \\ & - \delta H_1\left(Y_1(t), Y_2(t), E\left(Y_1(t), Y_2(t), E^e(t); \beta, \delta\right)\right) / E\left(Y_1(t), Y_2(t), E^e(t); \beta, \delta\right) - Y_2(t)\Big] \\ & = F_3\left(Y_1(t), Y_2(t), K_2(t), E^e(t); \alpha_2, \beta, \delta\right), \end{aligned} \tag{21}$$

$$K_2(t+1) = K_2(t) + I_2\left(Y_2(t), K_2(t), r_2\left(Y_2(t)\right)\right) = F_4\left(Y_2(t), K_2(t)\right), \tag{22}$$

$$E^e(t+1) = E^e(t) + \gamma\left[E\left(Y_1(t), Y_2(t), E^e(t); \beta, \delta\right) - E^e\right] = F_5\left(Y_1(t), Y_2(t), E^e(t); \beta, \delta\right). \tag{23}$$

4 Functional forms and specifications

For our numerical exploration the fundamental system of dynamical equations is employed in the form:

$$Y_1(t+1) - Y_1(t) = \alpha_1\left\{\phi_1(t) + \delta H_1(t) + \left[c_1(1-\tau_1) - 1\right]Y_1(t) + Z_1\right\}, \tag{24}$$

$$K_1(t+1) - K_1(t) = \phi_1(t), \tag{25}$$

$$Y_2(t+1) - Y_2(t) = \alpha_2\left\{\phi_2(t) - \delta H_1(t) / E(t) + \left[c_2(1-\tau_2) - 1\right]Y_2(t) + Z_2\right\}, \tag{26}$$

$$K_2(t+1) - K_2(t) = \phi_2(t), \tag{27}$$

$$E^e(t+1) - E^e(t) = \gamma\left[E(t) - E^e(t)\right], \tag{28}$$

where we have adopted the following functional forms:

$$\phi_i(t) = f\left(Y_i(t)\right) - \frac{3K_i(t)}{10} - r_i(t), \qquad i = 1, 2, \tag{29}$$

$$r_i(t) = -M_i + 10\, Y_i(t)^{1/4}, \qquad i = 1, 2, \tag{30}$$

$$H_1(t) = 100 - \frac{100}{E(t)} - \frac{3Y_1(t)}{10} + \frac{3Y_2(t)}{10}, \tag{31}$$

and the numerical specifications:

$$M_1 = M_2 = 300, \quad c_1 = c_2 = 0.8, \quad \tau_1 = \tau_2 = 0.2, \tag{32}$$

$$Z_1 = c_1 T_{01} + C_{01} + G_1 = Z_2 = c_2 T_{02} + C_{02} + G_2 = 75, \tag{33}$$

corresponding to similarly structured regional economies. The function f is a particular case of the Kaldorian sigmoid direct dependence of the investment function on income (see e.g. [4]), given by:

$$f(x) = \frac{80}{\pi} arc\tan\left[\frac{9\pi}{80}(x-250)\right] + 35. \tag{34}$$

These functional forms and specifications are taken in the present case to be essentially the same as in the case of fixed exchange rates [3], so as to facilitate comparison of results. The right-hand sides of the system now become:

$$f_1 = \alpha_1 \left\{300 + Z + f\left(Y_1(t)\right) - \frac{3K_1(t)}{10} - 10Y_1(t)^{1/4} - \frac{36}{100}Y_1(t) + \delta\left[100 - \frac{100}{E(t)} - \frac{3Y_1(t)}{10} + \frac{3Y_2(t)}{10}\right]\right\}, \tag{35}$$

$$f_2 = 300 + f\left(Y_1(t)\right) - \frac{3K_1}{10} - 10Y_1(t)^{1/4}, \tag{36}$$

$$f_3 = \alpha_2 \left\{300 + Z + f\left(Y_2(t)\right) - \frac{3K_2(t)}{10} - 10Y_2(t)^{1/4} - \frac{36}{100}Y_2(t) - \frac{\delta}{E(t)}\left[100 - \frac{100}{E(t)} - \frac{3Y_1(t)}{10} + \frac{3Y_2(t)}{10}\right]\right\}, \tag{37}$$

$$f_4 = 300 + f\left(Y_2(t)\right) - \frac{3K_2(t)}{10} - 10Y_2(t)^{1/4}, \tag{38}$$

$$f_5 = \gamma\left[E(t) - E^e(t)\right], \tag{39}$$

and the system is completed by Equation (15) which takes the form:

$$\delta\left[100 - \frac{100}{E(t)} - \frac{3Y_1(t)}{10} + \frac{3Y_2(t)}{10}\right] + \beta\left[1 - \frac{E^e(t)}{E(t)} + 10Y_1(t)^{1/4} - 10Y(t)_2^{1/4}\right] = 0. \tag{40}$$

The expression (16) of the exchange rate is now:

$$E(t) = \frac{10\left[100\,\delta + \beta\, E^e(t)\right]}{10\,\beta\left[1 + 10Y_1(t)^{1/4} - 10Y_2(t)^{1/4}\right] + \delta\left[1000 - 3Y_1(t) + 3Y_2(t)\right]}, \tag{41}$$

and the final recurrence system is obtained by substituting this into Equations (35) and (37). For simplicity, in our numerical exploration we shall further assume equal speeds of adjustment of the goods markets in the two regions $(\alpha_1 = \alpha_2 = \alpha)$, thus reducing the space of essential parameters of the model, from four-dimensional $(\alpha_1, \alpha_2, \beta, \delta)$ to three-dimensional (α, β, δ). However, variations of the quantities γ and Z, as secondary parameters, can also be discussed.

5 Position and stability of equilibrium

To find the equilibrium values of the system, denoted below by asterisks, we first observe that Equation (39) implies $E^{e*} = E^*$, and substituting this into Equation (41) we obtain:

$$E^{e*} = E^* = -\frac{1000\delta}{100\beta\left[-(Y_1^*)^{1/4} + (Y_2^*)^{1/4}\right] + \delta(-1000 + 3Y_1^* - 3Y_2^*)}. \tag{42}$$

It is then found from Equations (35) – (38) and (42) that the equilibrium values of our flexible exchange rates model under the above specifications are:

$$Y_1^* = Y_2^* = \frac{25Z}{9}, \quad K_1^* = K_2^* = \frac{10}{3}\left[300 + f(\frac{25Z}{9}) - 10(\frac{25Z}{9})^{1/4}\right], \quad E^{e*} = E^* = 1. \tag{43}$$

In particular, for $Z = 75$ we obtain the equilibrium values: $Y_1^* = Y_2^* = 625/3 \cong 208.333$, $K_1^* = K_2^* \cong 862.449$.

Stability of the equilibrium is determined by the roots of the characteristic polynomial of the Jacobian of the mapping, i.e. the matrix:

$$J^* = I + (\frac{\partial f_i}{\partial x_j}), \qquad i, j = 1, \ldots, 5, \qquad (x_1, x_2, x_3, x_4, x_5) = (Y_1, K_1, Y_2, K_2, E^e), \tag{44}$$

where I is the 5×5 unit matrix, and the superscript (*) denotes evaluation at the equilibrium. The characteristic equation is a quintic:

$$P_5(\lambda) = \lambda^5 + a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0, \tag{45}$$

and for stability all its roots, real or complex, must be inside the unit circle in the complex plane.

Our basic tool for the numerical determination of the region of stability is a two-dimensional grid-search technique. We compute the characteristic polynomial (45) and its roots at the node points of a dense grid covering a region of interest in a two-dimensional section of the space of parameters, and store for graphical representation the points at which the equilibrium is stable.

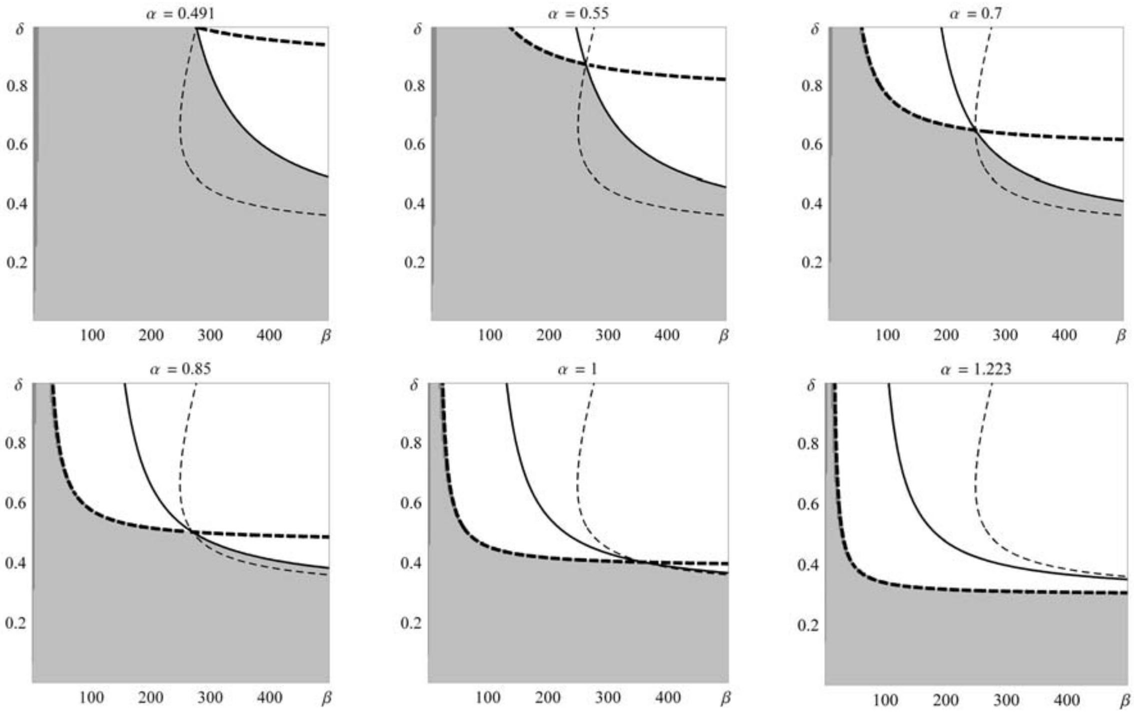

Figure 1: Region of stability of equilibrium in the (β, δ) plane for $\gamma = 1.2$ and sample values of α.

The technique is first employed to determine the stability diagrams of Figure 1 showing the stability region in the (β, δ) parameter plane for fixed $\gamma = 1.2$ and for different values of the common speed of adjustment of the goods markets α. The part of the stability region in which the roots of the characteristic equation are all real is shown dark-shaded, while the part in which some of the roots are complex conjugate is shown light-shaded. In these diagrams the flip bifurcation condition:

$$g_1(\alpha, \beta, \gamma, \delta) = P_5(-1) = a_0 + a_2 + a_4 - (1 + a_1 + a_3) = 0, \tag{46}$$

is drawn in as a bold dashed curve. The Hopf bifurcation curve is also drawn in, as a continuous curve. We have found that in all stability diagrams of this paper the Hopf bifurcation curve can be determined by requiring that two of the roots of the characteristic polynomial have unit product [3], and is given by the following relation of the coefficients of the characteristic polynomial:

$$\begin{aligned} g_2(\alpha, \beta, \gamma, \delta) &= a_0^4 + a_0^2(a_1 - 2) - a_2^2 - a_0 a_2(a_0^2 + 3a_1 - 2a_3 - 1) + (1 + a_1 - a_3)\left[(a_1 - 1)^2 + a_0^2 a_3\right] \\ &+ a_4\left\{(1 + 2a_0^2 + a_1)a_2 - a_0\left[a_0^2 + a_1^2 + 3a_3 - a_1(2 + a_3) - 1\right]\right\} - a_4^2(a_0^2 + a_1 + a_0 a_2) + a_0 a_4^3 = 0. \end{aligned} \tag{47}$$

Each one of relations (46) and (47) is the implicit equation of a surface in the four-dimensional space of the parameters $(\alpha, \beta, \gamma, \delta)$, the contours of which for fixed γ and for various levels of α provide in the (β, δ) plane one or more curves. Segments of the curves arising from (46) form the part of the boundary of the stability region that is a flip bifurcation curve, and similarly segments of the curves arising from (47) form the part of the boundary of the stability region that is a Hopf bifurcation curve. Segments of the above curves that are not parts of the boundary of the stability region do not correspond to loss of stability and can be ignored. These two relations can therefore be employed, in combination with the grid search technique, as coefficient criteria for flip bifurcations and Hopf bifurcations in the present case of our five-dimensional discrete system.

The above tools are similarly employed to determine the stability region in the (β,α) plane for $\gamma = 1.2$ and different values of the level of trade transactions δ. Some of the stability regions found are shown in Figure 2.

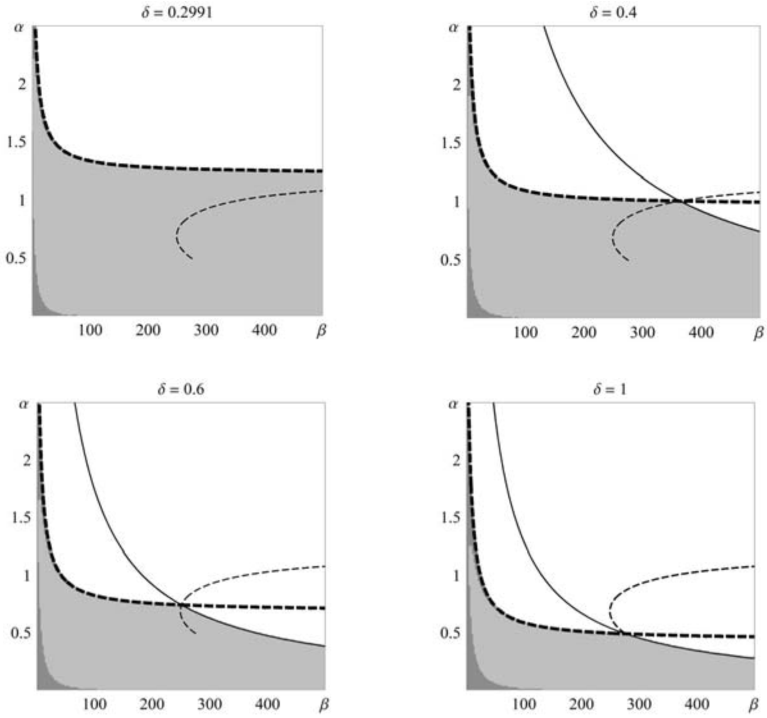

Figure 2: Region of stability of equilibrium in the (β,α) plane for $\gamma = 1.2$ and sample values of δ.

6 Geometrical aspects and implications

Let us now consider the geometrical aspects of the boundary curves of the stability region. We begin by noting that in our model Equation (46) is equivalent to a quadratic with respect to α :

$$g_1(\alpha,\beta,\gamma,\delta) = (1-\alpha_{00}\alpha)\left\{10(1-\alpha_{00}\alpha)\left[\beta-50(\gamma-2)\delta\right] - \alpha\beta\delta\left[3-10\times 3^{3/4}(\gamma-2)\right]\right\} = 0, \tag{48}$$

with roots $\alpha = \alpha_{00}{}^{-1}$, and

$$\alpha = \left\{\alpha_{00} + \beta\delta\,\frac{3-10\times 3^{3/4}(\gamma-2)}{10\left[\beta-50\,\delta\,(\gamma-2)\right]}\right\}^{-1}, \tag{49}$$

where we have abbreviated:

$$\alpha_{00} = \frac{9}{50} + \frac{3^{3/4}}{85} - \frac{23040}{4352+95625\pi^2} \cong 0.1825. \tag{50}$$

The constant root $\alpha_{00}{}^{-1}$ corresponds to the leftmost point of the flip bifurcation curve in the (β,α) plane, while (49) represents the flip bifurcation condition (46) in the form $\alpha = \alpha(\beta,\gamma,\delta)$. Substituting this expression of α into (47), we obtain an equation:

$$g_2\left(\alpha(\beta,\gamma,\delta),\beta,\gamma,\delta\right) = 0, \tag{51}$$

which, for a fixed value of γ, is satisfied by the values of β and δ representing the locus of the points of intersection of the flip bifurcation curve with the Hopf bifurcation curve in the (β,δ) plane. In Figure 1 this locus has been drawn in as a thin dashed curve.

Considering the existence of the locus for large values of β, we find that for $\beta\to\infty$ the expression in the left-hand side of (51) tends to a limit function $P_{\gamma,\delta}$, the root of which for $\gamma = 1.2$ (denoted by δ_∞) is given below together with the corresponding value of α (denoted by α_∞ and obtained from (49) for $\delta = \delta_\infty$ and $\beta\to\infty$):

$$\gamma = 1.2: \qquad \delta_\infty \cong 0.2991, \qquad \alpha_\infty \cong 1.2231. \tag{52}$$

The flip bifurcation curve and the Hopf bifurcation curve approach each other asymptotically in the (β, δ) plane for $\beta \to \infty$ when $\delta = \delta_\infty$. It follows from Figure 1 that they do not intersect when $\delta < \delta_\infty$, but they do so at a finite value of β when $\delta > \delta_\infty$. Therefore, the locus of their points of intersection does not exist when $\delta < \delta_\infty$.

Similarly, in the (β, α) plane the flip bifurcation curve and the Hopf bifurcation curve approach each other asymptotically for $\beta \to \infty$ when $\alpha = \alpha_\infty$. It follows from Figure 2 that they do not intersect when $\alpha > \alpha_\infty$, but they do so at a finite value of β when $\alpha < \alpha_\infty$. Therefore, the locus of their points of intersection does not exist when $\alpha > \alpha_\infty$.

Of importance here is also the fact that for $\gamma = 1.2$ the locus of intersections of the flip bifurcation curve and the Hopf bifurcation curve attains its minimum with respect to the parameter β of capital movement at a considerably large value of β. Specifically, this minimum occurs at:

$$\beta = \beta_{\min} \cong 249.13 \qquad (\delta \cong 0.662,\ \alpha \cong 0.688). \tag{53}$$

To find the locus of intersections in the (β,α) plane, instead of in the (β,δ) plane, we can solve (49) for δ and substitute the resulting expression:

$$\delta_1(\alpha,\beta,\gamma) = \frac{10\,(1-\alpha_{00}\,\alpha)\beta}{500\,(1-\alpha_{00}\,\alpha)(\gamma-2) - \alpha\beta\left[10\times 3^{3/4}(\gamma-2)-3\right]}, \tag{54}$$

into (47). We then obtain the following equation which, for a fixed value of γ, is satisfied by the values of β and α representing the locus curve in the (β,α) plane:

$$g_2\left(\alpha,\beta,\gamma,\delta_1(\alpha,\beta,\gamma)\right) = 0. \tag{55}$$

In Figure 2 the locus has been drawn in as in Figure 1 (thin dashed curve). However, since the maximum value of δ allowed in the model is 1, we can substitute $\delta = 1$ into (49) to find the following relation representing, for a fixed value of γ, the model restriction $\delta \le 1$ on the flip bifurcation curve in the (β,α) plane:

$$\alpha \ge \left\{\alpha_{00} + \beta\,\frac{3-10\times 3^{3/4}(\gamma-2)}{10\left[\beta-50\,(\gamma-2)\right]}\right\}^{-1}. \tag{56}$$

For the value $\gamma = 1.2$ we thus obtain:

$$\alpha \ge \left\{\alpha_{00} + \beta\,\frac{3+8\times 3^{3/4}}{10\,(\beta+40)}\right\}^{-1} \cong 0.433628 + \frac{15.9724}{\beta+3.16579}. \tag{57}$$

This expression of α, with the equality sign, represents the final location of the flip bifurcation curve (corresponding to $\delta = 1$). It follows that the actual locus of intersections of the flip bifurcation curve with the Hopf bifurcation curve is only the part of the curve (55) in which (57) is satisfied. In the diagrams of Figure 2 only that part of the curve is shown.

We can now substitute α by its lowest possible value, as given by the right-hand side of (56), into (55) to obtain an equation:

$$g_2^*(\beta,\gamma) = 0, \tag{58}$$

giving (for a fixed value of γ) the value of β at the final point of the actual locus. For $\gamma = 1.2$ we obtain $\beta \cong 276.65$, and the value of α at this point is then found from (57) to be $\alpha \cong 0.491$. Note in Figure 1 that this is the value of α for the intersection of the flip bifurcation curve with the Hopf bifurcation curve to be at the top end point of the locus in the (β,δ) plane.

Let us now discuss the implications of the geometrical aspects of the boundary curves of the stability region, including consequences on the possibility of occurrence of interregional business cycles. The minimum with respect to the parameter β of the locus of intersections of the flip bifurcation curve with the Hopf bifurcation curve in Figures 1 and 2 means that no segment of the curve (47) can be part of the boundary of the stability region for $\beta < 249.13$, equivalently that no Hopf bifurcation curve exists, and no business cycles can occur, for lower values of β. For lower values of β the flip bifurcation curve forms the boundary of the stability region, thus period doubling can be expected to be the only mode of asymptotic dynamical behavior of the system when exiting the stability region in parameter space.

The inequalities:

$$\delta > \delta_\infty,\ \ \beta > \beta_{\min},\ \ \alpha < \alpha_\infty, \tag{59}$$

represent the threshold for the occurrence of cycles in our model. For $\delta > \delta_\infty$, $\alpha < \alpha_\infty$, and large but finite values of capital movement β, the locus of intersections of the flip bifurcation curve with the Hopf bifurcation curve exists, and cycles occur when exiting the stability region in parameter space through the segment of the curve (47) which forms part of the boundary of the stability region for such values of β. The important difference between our present flexible exchange rates system and the corresponding fixed exchange rates system studied in [3], is that in our present system for cycles to occur sufficient amount of trade is required *together* with high levels of capital movement.

In Figure 3 we show the occurring cycles for $\delta = 0.6$, $\beta = 300$, and α as the bifurcation parameter varying between 0.627 and 0.629. The cycles are shown in their (Y_1, Y_2) and exchange rate (ER) versus Y_1 projections. The (Y_1, Y_2) projection in particular shows counter synchronization of regional incomes when interregional cycles appear. Note, also, that when the income of region 1 is sufficiently higher than the income of region 2, then the exchange rate is less than 1, that is the currency of region 1 is "stronger" (1 unit of currency of region 2 is < 1 unit of currency of region 1). However, the occurring cycles are small in amplitude and short-lived as α varies.

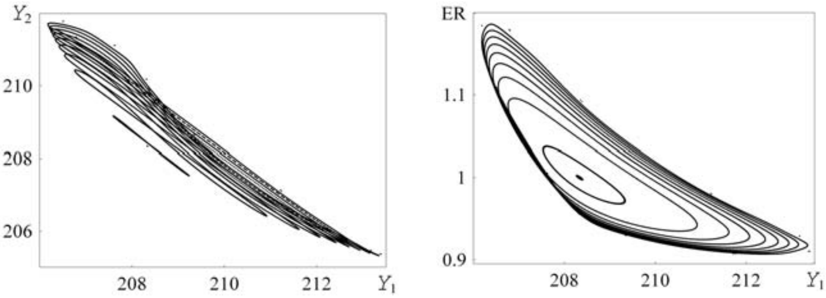

Figure 3: Interregional cycles for $\delta = 0.6$, $\beta = 300$, $0.627 \le \alpha \le 0.629$, in (Y_1, Y_2) and (Y_1, ER) projections.

We close this section with the following basic conclusions concerning the enhanced stability of equilibrium characterizing our model of flexible exchange rates. From the stability region diagrams it can be seen that high levels of capital movement β do not induce instability of the system for low levels of the speed of adjustment of the goods markets α (i.e. for prudent reactions by firms); but high levels of α induce instability even at relatively low levels of β (Figure 2). For prudent reactions by firms $(\alpha < 1)$ equilibrium remains stable at high levels of capital movement even for high levels of trade transactions δ (Figure 1).

7 Period doubling and second generation cycles

We now consider the asymptotic dynamical behavior of the system outside the stability region. For this we employ numerical simulations of the model mapping (35) – (39) to compute bifurcation and Lyapunov exponent diagrams with β as the bifurcation parameter. We choose parameter cases in which as β increases stability is lost by going through a flip bifurcation, such being the characteristic cases in our model (except for very large values of β).

Our results for $\delta = 0.6$ (see the bottom left-hand diagram of Figure 2 for the relevant stability region), and sample values of α, are shown in Figure 4. As expected period doubling occurs when stability is lost (at: $\beta \cong 7.96$ for $\alpha = 2$, and $\beta \cong 14.71$ for $\alpha = 1.5$).

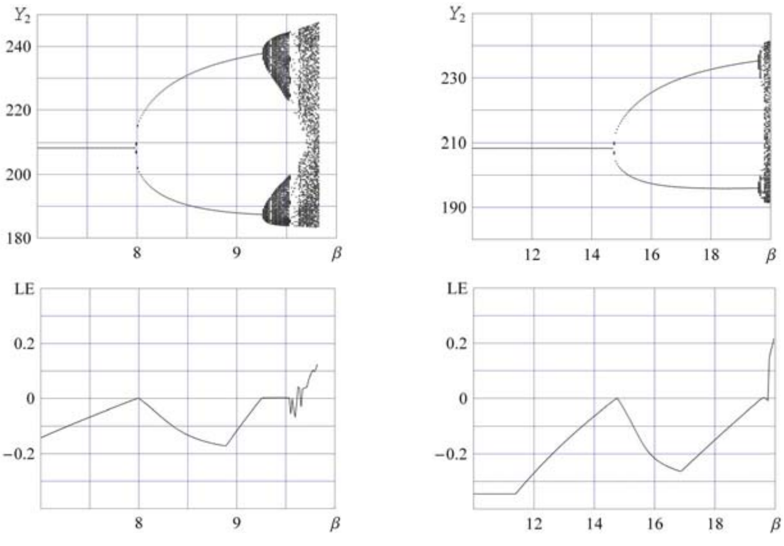

Figure 4: Bifurcation and Lyapunov exponent diagrams for $\delta = 0.6$ at: $\alpha = 2$ (left), $\alpha = 1.5$ (right).

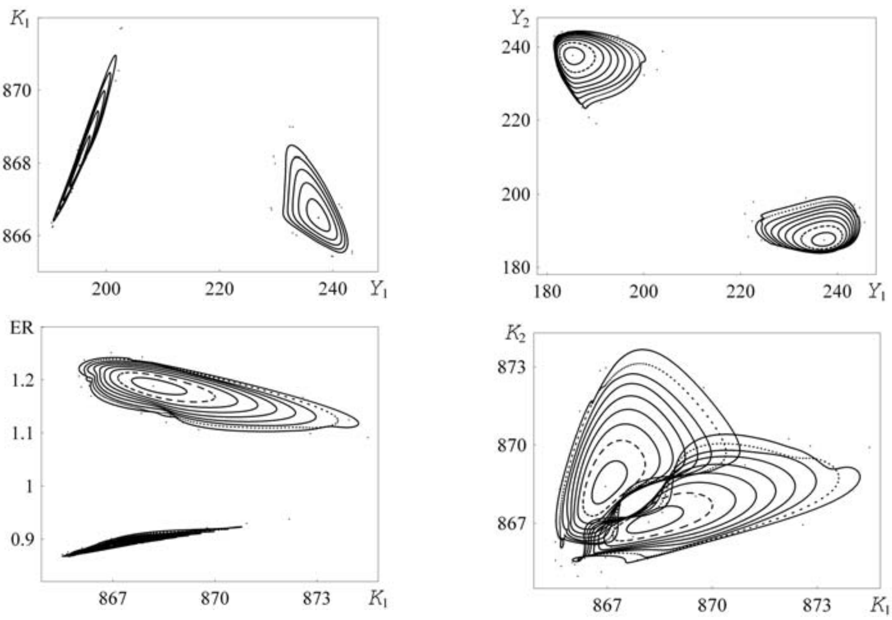

Figure 5: Period-2 cycles for $\delta = 0.6,\ \ \alpha = 2.$

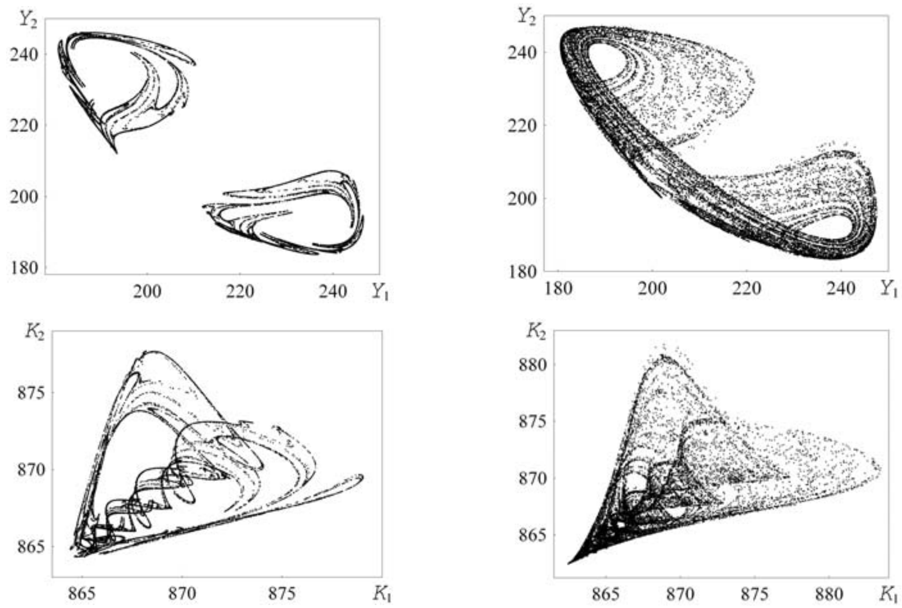

Figure 6: Development of the cycles of Figure 5, (Y_1, Y_2) and (K_1, K_2) projections, into chaotic attractors at: $\beta = 9.68$ (left) and $\beta = 9.76$ (right / mask-like and heart-like attractors).

However, in the present case the period doubling process does not develop directly into chaotic behavior. Instead, it first develops into an intermediate phase of "second generation", period-2, cycles as indicated by the characteristic flatness of the Lyapunov exponent diagram for $\alpha = 2$ and β approximately between 9.25 and 9.55. A similar situation, for a narrower interval of β, is seen to occur for $\alpha = 1.5$. The actual second generation cycles occurring for $\delta = 0.6$, $\alpha = 2$, and their development into chaotic attractors are shown in Figures 5 and 6.

8 Concluding remarks

We presented a five-dimensional discrete two-regional Kaldorian macrodynamic model with flexible exchange rates, assuming similar economies of the two regions, and carried out a numerical exploration of its dynamical behavior, considering the effects of variation of the three basic parameters, namely the common speed of adjustment of the goods markets α, the degree of capital mobility β, and the level of trade transactions between the regions δ. We employed as our basic tools a grid search method and analytical coefficient criteria for the determination of the stability of equilibrium region and its boundary curves, the flip bifurcation curve and the Hopf bifurcation curve, in two-dimensional sub-sections of the parameter space. We considered the geometrical aspects of the boundary curves and of the curve representing the locus of their intersections, and the implications of these aspects on the stability region and the occurrence of period doubling or cycles in the parameter space. Our main findings are as follows.

Compared to the corresponding model of fixed exchange rates considered in a previous paper, our present model is characterized by enhanced stability of equilibrium. High levels of capital movement β do not induce instability of the system for low levels of the speed of adjustment of the goods markets α (i.e. for prudent reactions by firms), although high levels of α induce instability even at relatively low levels of β. For prudent reactions by firms $(\alpha < 1)$ equilibrium remains stable at high levels of capital movement even for high levels of trade transactions δ.

Business cycles are generally scarce and short-lived in parameter space, occurring at large values of the degree of capital movement β. We determined the threshold for the occurrence of cycles in the form of restrictions described by inequalities that must be satisfied by the parameters δ, α and β. A characteristic difference between our present flexible exchange rates system and the corresponding fixed exchange rates system studied in [3], is that in our present system for cycles to occur sufficient trade is required *together* with high levels of capital movement. The importance of trade as a generating factor for business cycles is significantly reduced by the flexibility of exchange rates.

Further, we have considered the effects of variation of two additional parameters, the speed of adaptation γ of the expected exchange rate and the parameter Z involving government expenditure. We found that the threshold for the occurrence of cycles is relaxed, in the sense that cycles occur for larger regions in the basic parameters space (α, β, δ), when γ is increased, and that it is similarly relaxed in the space of the economic interaction parameters (β, δ) when Z is decreased (details are included in the full version of the paper to be published elsewhere). A plausible interpretation of these results is that rapid changes in exchange rates expectations and decreased government expenditure are factors contributing to the creation of Hopf bifurcations and interregional business cycles.

It may be commented here concerning the plausibility of the above results and their economic interpretations, that these do not contradict intuition and experience and are therefore suggestive of some degree of real world relevance of our formulation of the present model of Kaldorian two-regional Macrodynamics under flexible exchange rates.

We also explored the asymptotic dynamical behavior of our system outside the stability of equilibrium region by means of numerical simulations resulting in bifurcation and Lyapunov exponent diagrams, and found that in several cases of the period doubling process occurring when exiting the stability region in parameter space through flip bifurcation, the process first develops into second generation (period 2) cycles. It was noted that when exiting the stability region through Hopf bifurcation the occurring cycles exhibit counter synchronization of regional incomes. Examples of the occurring first and second generation cycles were given in terms of two-dimensional projection diagrams.

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