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NONSYMMETRIC FINITE INTEGRAL TRANSFORMATIONS AND THEIR APPLICATION IN THERMOVISCOELASTICITY

V. Kovalev[1], S. Lychev[2]
[1]Moscow City Government University of Management, Russia, [2]Ishlinsky Institute for Problems in Mechanics of the Russian Academy of Sciences, Russia

Corresponding author: S. Lychev, Institute for Problems in Mechanics of the Russian Academy of Sciences, prosp. Vernadskogo 101, block 1, Moscow, Russian Federation lychev@ipmnet.ru

Abstract. A finite integral transformation method for the solution of initial-boundary value problems for hyperbolic systems of PDEs recently proposed by Y. Senitskii is generalized to nonselfadjoint case. The new method is applicable to dissipative non-symmetrical visco- and thermovisco-elastic dynamic problems for which classical approaches apparently fail. The obtained solutions are of the form of spectral expansions based on complete biorthogonal sets of eigenfunctions and associated functions, corresponding to adjoint pairs of matrix operator pencils. The co-ordinate functions of mentioned expansions, namely transforms, one can obtain by applying the special integral transformation with the matrix kernel, thus reducing the problem to the sequence of initial problems for ODEs. As illustrative example the coupled dynamic thermoviscoelastic problem for a finite cylinder is solved.

1 Nonsymmetrical finite integral transformations

A fundamental contribution to the theory of strongly damped nonselfadjoint initially-boundary value problems (IBVP) is due to Keldysh, Markus, Krein, Langer *et al.* Recent contribution may be found *e.g.* [1, 2, 3]. The minimality, completeness and basis properties of the eigenfunctions and associative functions corresponding to the wide range of IBVP are proved in the cited papers. It affords the theoretical background for the representations of solutions in the form of spectral expansions, particular, by the integral transformations.

Let $\mathbf{f}(\mathbf{x},t)$ and $\mathbf{y}_0^{(i)}(\mathbf{x})\,(i=0,\dots,m-1)$ be the square integrable vector-functions defined in $V\times[0,\infty[$ and V respectively; $V\subset\mathbb{R}^n$. Assume that V is compact. Let $\mathcal{A}_i\,(i=0,\dots,m)$ be the nonselfadjoint differential operators in the Hilbert space L^2_μ with scalar product $\langle\mathbf{v},\mathbf{w}\rangle=\int_V \mathbf{v}^{\mathrm{T}}\mu\,\overline{\mathbf{w}}\,dV(\mathbf{x})$, where μ is the metric matrix-function. Consider the following IBVP:

$$\sum_{i=0}^{m}\mathcal{A}_i\frac{\partial^i}{\partial t^i}\mathbf{y}(\mathbf{x},t)=\mathbf{f}(\mathbf{x},t),\quad \left.\frac{\partial^i}{\partial t^i}\mathbf{y}(\mathbf{x},t)\right|_{t=0}=\mathbf{y}_0^{(i)}(\mathbf{x}),\quad i=0,\dots,m-1,\quad \mathbf{y}(\mathbf{x},t)\in\mathcal{D},\tag{1}$$

where $\mathcal{D}=\{\mathbf{y}|\mathbf{y}\in L^2_\mu\cap C^{(n)}_x\wedge\mathcal{B}(\mathbf{y})=0\}$, $\mathcal{B}$ is the boundary operator, defined by prescribed boundary conditions.

The obtained solutions of IBVP (1) are of the form of spectral expansions based on complete biorthogonal sets of eigenfunctions and associated functions, corresponding to the conjugate pairs of matrix operator pencils $\mathcal{L}_\nu$, $\mathcal{L}^*_{\overline{\nu}}$:

$$\mathcal{L}_\nu=\sum_{i=0}^{m}\nu^i\mathcal{A}_i,\ \mathcal{L}^*_{\overline{\nu}}=\sum_{i=0}^{m}\overline{\nu}^i\mathcal{A}^*_i,\quad (\mathbf{u}\in\mathcal{D}\wedge\mathbf{v}\in\mathcal{D}^*)\Leftrightarrow(\langle\mathcal{L}^*_{\overline{\nu}}\mathbf{u},\mathbf{v}\rangle-\langle\mathbf{u},\mathcal{L}_\nu\mathbf{v}\rangle=0).\tag{2}$$

Here $\mathcal{A}^*_i$ are conjugate to $\mathcal{A}_i$ differential operators, defined in the domain $\mathcal{D}^*$, that defined by boundary operator $\mathcal{B}^*$ conversely. The explicit analytical formulas for $\mathcal{A}^*_i,\mathcal{B}^*$ are given in [4].

The coefficients of expansions referred to as transforms one can obtain by applying direct integral transformation $\mathcal{I}^*$ to (1), resulting in the reduction IBVP (1) to the sequence of initial problems for ODEs in image space [4]. The operators of direct and inverse transformations are:

$$\varphi=\mathcal{F}^*\mathbf{u}=-\frac{1}{\nu}\langle\mathcal{K}^*_{\overline{\nu}},\mathcal{A}_0\mathbf{u}\rangle,\qquad \mathbf{u}=\mathcal{F}\varphi=\sum_{i=1}^{\infty}\mathcal{K}_{\nu_i}\mathcal{Q}_{\nu_i}\varphi,\tag{3}$$

where $\mathcal{K}_\nu$, $\mathcal{K}^*_{\overline{\nu}}$ are matrix-function kernels, $\mathcal{Q}_\nu$ is the normalizing matrix and $\nu_i\,(i=1,\dots,\infty)$ are the elements of pencil discrete spectrum. The operator kernels (3) represent the biorthogonal basis of root subspaces, corresponding to spectral singularities of pencils (2), and they are formulated as compositions of eigenfunction and associated functions:

$$\mathcal{K}_\nu=\begin{pmatrix}\mathbf{G}_s & \mathbf{G}_{s-1} & \dots\mathbf{G}_1\end{pmatrix},\quad \mathcal{K}^*_{\overline{\nu}}=\begin{pmatrix}\mathbf{G}^*_s & \mathbf{G}^*_{s-1} & \dots\mathbf{G}^*_1\end{pmatrix}.\tag{4}$$

One can found $\mathbf{G}_s$, $\mathbf{G}^*_s$ by solving the coupled set of boundary value problems (s depends on multiplicity of spectral singularity):

$$\mathcal{L}_\nu\mathbf{G}_s=0,\quad \mathcal{B}\mathbf{G}_s=0,\quad \mathcal{L}_\nu\mathbf{G}_{s-i}=-\sum_{j=1}^{i}\frac{1}{j!}\mathcal{L}^{(j)}_\nu\mathbf{G}_{s-i+j},\quad \mathcal{B}\mathbf{G}_{s-i}=0,$$

$$\mathcal{L}_{\overline{\nu}}^{*}\mathbf{G}_{s}^{*}=0,\quad \mathcal{B}^{*}\mathbf{G}_{s}^{*}=0,\quad \mathcal{L}_{\overline{\nu}}^{*}\mathbf{G}_{s-i}^{*}=-\sum_{j=1}^{i}\frac{1}{j!}\mathcal{L}_{\overline{\nu}}^{*(j)}\mathbf{G}_{s-i+j}^{*},\quad \sum_{j=0}^{i}\frac{1}{j!}\left(\frac{\partial^{j}}{\partial\overline{\nu}^{j}}\mathcal{B}_{\overline{\nu}}^{*}\right)\mathbf{G}_{s-i+j}^{*}=0\quad (s-i>0).$$

The constructible representation of normalizing matrix $\mathcal{Q}_\nu$ and the exact method for evaluation of corresponding quadratures are described in [5]. Note, that operation property [4] here is in the form $\sum_{i=0}^{m}\left[\overline{\Lambda}^{\mathrm{T}}\right]^{i}\mathcal{F}_{\overline{\nu}}^{*}\mathcal{A}_i=0$, where Λ is block-diagonal (Jordan) matrix. For this reason the result of suggested integral transformation, unlike known analogues, is similar to Jordan matrix decomposition.

The corresponding to introduced transformations generalized algorithmic procedure resolving the IBVP is described in [4]. It enable us to represent the solution of (1) as follows:

$$\mathbf{y}=\mathcal{F}\left[\exp(\overline{\Lambda}^{\mathrm{T}}t)\sum_{i=1}^{m}\sum_{j=i}^{m}\overline{\Lambda}^{\mathrm{T}(j-i)}\mathcal{F}^{*}\mathcal{A}_{j}\mathbf{y}_{0}^{(i-1)}+\int_{0}^{t}\exp\left[\overline{\Lambda}^{\mathrm{T}}(t-\tau)\right]\mathcal{F}^{*}\mathbf{f}(\tau)\,d\tau\right]. \tag{5}$$

The proposed generalization of biorthogonal transformation is applicable to analysis of dissipative dynamic systems, particular for dynamic viscoelastic and coupled dynamic thermoelastic nonselfadjoint IBVP.

2 Closed solutions of thermoviscoelastic problems

Consider the coupled equations of viscoelastic motion and heat conduction in cylindrical co-ordinate system

$$\begin{pmatrix}\mathcal{L}_1 & -\gamma\mathcal{L}_2\\ 0 & \nabla^2\end{pmatrix}\mathbf{y}+\begin{pmatrix}\mathcal{L}_1' & 0\\ -\eta\mathcal{L}_3 & -1/\kappa\end{pmatrix}\frac{\partial}{\partial t}\mathbf{y}+\begin{pmatrix}-\rho\mathcal{E} & 0\\ 0 & 0\end{pmatrix}\frac{\partial^2}{\partial t}\mathbf{y}=\mathbf{f}, \tag{6}$$

wherein $\mathbf{f}=(-X_r,-X_\varphi,-X_z,-\omega)$ is prescribed vector-function, defined by volumetric force and heat sources intensity, $\mathcal{E}$ is identity operator, $\mathcal{L}_1,\dots\mathcal{L}_3$ are the following differential operators:

$$\mathcal{L}_1=\begin{pmatrix}\mu(\nabla^2-\frac{1}{r^2})+(K+\frac{\mu}{3})\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right) & \frac{K+\mu/3}{r}\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)-\frac{2\mu}{r^2}\frac{\partial}{\partial\varphi} & (K+\frac{\mu}{3})\frac{\partial^2}{\partial r\partial z}\\ \frac{2\mu}{r^2}\frac{\partial}{\partial\varphi}+\frac{K+\mu/3}{r}\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right) & \mu(\nabla^2-\frac{1}{r^2})+\frac{K+\mu/3}{r^2}\frac{\partial^2}{\partial\varphi^2} & \frac{K+\mu/3}{r}\frac{\partial^2}{\partial\varphi\partial z}\\ (K+\frac{\mu}{3})\frac{\partial}{\partial z}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right) & \frac{K+\mu/3}{r}\frac{\partial^2}{\partial z\partial\varphi} & \mu\nabla^2+(K+\frac{\mu}{3})\frac{\partial^2}{\partial z^2}\end{pmatrix},$$

$$\mathcal{L}_1'=\mu'\begin{pmatrix}\nabla^2-\frac{1}{r^2}+\frac{1}{3}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right) & \frac{1}{3r}\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)-\frac{2}{r^2}\frac{\partial}{\partial\varphi} & \frac{1}{3}\frac{\partial^2}{\partial r\partial z}\\ \frac{2}{r^2}\frac{\partial}{\partial\varphi}+\frac{1}{3r}\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right) & \nabla^2-\frac{1}{r^2}+\frac{1}{3r^2}\frac{\partial^2}{\partial\varphi^2} & \frac{1}{3r}\frac{\partial^2}{\partial\varphi\partial z}\\ \frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right) & \frac{1}{3r}\frac{\partial^2}{\partial z\partial\varphi} & \nabla^2+\frac{1}{3}\frac{\partial^2}{\partial z^2}\end{pmatrix},$$

$$\mathcal{L}_2=\begin{pmatrix}\frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z}\end{pmatrix}^{\mathrm{T}},\ \mathcal{L}_3=\begin{pmatrix}\frac{\partial}{\partial r}+\frac{1}{r} & \frac{1}{r}\frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z}\end{pmatrix},\ \nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2},$$

K,μ are the elastic modulus; γ,η are the thermomechanical constants; κ is the thermal conductivity coefficient, ρ is the density, μ' is the viscosity modulus.

The boundary conditions $\mathcal{D}$ are arbitrary on lateral area and have some restrictions on end faces (to admit the separation of variables, see [6]). In particular the boundary condition may be formulated as

$$\mathcal{D}=\{\mathbf{y}|\mathbf{y}\in L_2^4,\mathcal{B}\mathbf{y}=0,\mathbf{y}=O(1)\}, \tag{7}$$

$$\mathcal{B}\mathbf{y}=\begin{pmatrix}\mathcal{B}_1\mathbf{y}\big|_{r=R}\\ \mathcal{B}_2\mathbf{y}\big|_{z=0}\\ \mathcal{B}_2\mathbf{y}\big|_{z=H}\\ [\mathbf{y}]_0^{2\pi}\end{pmatrix},\quad \mathcal{B}_1=\begin{pmatrix}\vartheta\frac{\partial}{\partial r}+\frac{\lambda}{r} & \frac{\lambda}{r}\frac{\partial}{\partial\varphi} & \lambda\frac{\partial}{\partial z} & -\gamma\\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial r} & 0\\ \frac{1}{r}\frac{\partial}{\partial\varphi} & \frac{\partial}{\partial r}-\frac{1}{r} & 0 & 0\\ 0 & 0 & 0 & \frac{\partial}{\partial r}\end{pmatrix},\quad \mathcal{B}_2=\begin{pmatrix}0 & 0 & 1 & 0\\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial r} & 0\\ 0 & \frac{\partial}{\partial z} & \frac{1}{r}\frac{\partial}{\partial\varphi} & 0\\ 0 & 0 & 0 & \frac{\partial}{\partial r}\end{pmatrix},$$

wherein $[\mathbf{y}]_0^{2\pi}=\mathbf{y}\big|_{\varphi=0}-\mathbf{y}\big|_{\varphi=2\pi}$, $\vartheta=4\left(\mu+\mu'\partial/\partial t\right)/3+K$, $\lambda=K-2\left(\mu+\mu'\partial/\partial t\right)/3)$. Initial values are defined by initial distributions of temperature, displacements and velocities, i.e.

$$\theta\big|_{t=0}=\theta_0;\quad u\big|_{t=0}=u_0,\ v\big|_{t=0}=v_0,\ w\big|_{t=0}=w_0;\quad \dot{u}\big|_{t=0}=\dot{u}_0,\ \dot{v}\big|_{t=0}=\dot{v}_0,\ \dot{w}\big|_{t=0}=\dot{w}_0.$$

The obtained solutions of problem (6), (7) are of the form of spectral expansions based on complete biorthogonal sets of eigenfunctions (and perforce associated functions), corresponding to the mutual conjugate pairs of matrix operator pencils $\mathcal{L}_\nu$, $\mathcal{L}_{\overline{\nu}}^{*}$:

$$\mathcal{L}_\nu=\mathcal{A}_0+\mathcal{A}_1\nu+\mathcal{A}_2\nu^2,\quad \mathcal{L}_{\overline{\nu}}^{*}=\mathcal{A}_0^{*}+\mathcal{A}_1^{*}\overline{\nu}+\mathcal{A}_2\overline{\nu}^2,$$

$$\mathcal{A}_0^{*}=\begin{pmatrix}\mathcal{L}_1 & 0\\ \gamma\mathcal{L}_3 & \nabla^2\end{pmatrix},\quad \mathcal{A}_1^{*}=\begin{pmatrix}\mathcal{L}_1' & \eta\mathcal{L}_2\\ 0 & -1/\kappa\end{pmatrix}.$$

Here $\mathcal{A}_i^*$ are conjugate to $\mathcal{A}_i$ differential operators, defined in the domain $\mathcal{D}^*$, defined by boundary operator $\mathcal{B}^*$.

$$\mathcal{B}^*\mathbf{y} = \begin{pmatrix} \mathcal{B}_1^*\mathbf{y}\big|_{r=R} \\ \mathcal{B}_2\mathbf{y}\big|_{z=0} \\ \mathcal{B}_2\mathbf{y}\big|_{z=H} \\ [\mathbf{y}]_0^{2\pi} \end{pmatrix}, \quad \mathcal{B}_1^* = \begin{pmatrix} \vartheta\frac{\partial}{\partial r}+\frac{\lambda}{r} & \frac{\lambda}{r}\frac{\partial}{\partial\varphi} & \lambda\frac{\partial}{\partial z} & \nu\eta \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial r} & 0 \\ \frac{1}{r}\frac{\partial}{\partial\varphi} & \frac{\partial}{\partial r}-\frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial r} \end{pmatrix}.$$

The coefficients of expansions referred to as transforms one can obtain by applying direct integral transformation to (6), resulting in the reduction initial boundary value problem to the sequence of initial problems for ODEs in image space [4]. It enable us to represent the solution of (6), (7) in the case of simple spectrum as follows:

$$\mathbf{y} = \sum_{i=1}^{\infty}\left[\left(\langle\mathcal{A}_1^*\mathbf{k}_i^* + \overline{\nu}_i\mathcal{A}_2^*\mathbf{k}_i^*, \mathbf{y}_0\rangle + \langle\mathcal{A}_2^*\mathbf{k}_i^*, \dot{\mathbf{y}}_0\rangle\right)\exp(\overline{\nu}_i t) + \int_0^t \langle\mathbf{f}(\tau), \mathbf{k}_i^*\rangle\exp\big(\overline{\nu}_i(t-\tau)\big)\,d\tau\right]\mathbf{k}_i\mathcal{Q}_i^{-1}. \tag{8}$$

In equation (8) $\mathcal{Q}_\nu$ is the normalizing matrix

$$Q_i = \langle\mathcal{A}_1\mathbf{k}_i, \mathbf{k}_i^*\rangle + 2\nu_i\langle\mathcal{A}_2\mathbf{k}_i, \mathbf{k}_i^*\rangle$$

and $\nu_i\,(i = 1,\ldots,\infty)$ are the elements of pencil discrete spectrum. Note, that biorthogonal relations [4] here are in the form

$$\left\langle\mathcal{A}_1\mathbf{k}_i, \mathbf{k}_j^*\right\rangle + (\nu_i+\nu_j)\left\langle\mathcal{A}_2\mathbf{k}_i, \mathbf{k}_j^*\right\rangle = 0, \quad \left\langle\mathcal{A}_0\mathbf{k}_i, \mathbf{k}_j^*\right\rangle - \nu_i\nu_j\left\langle\mathcal{A}_2\mathbf{k}_i, \mathbf{k}_j^*\right\rangle = 0.$$

One can found $\mathbf{k}_i$, $\mathbf{k}_i^*$ by solving the coupled set of boundary eigenvalue problems:

$$\mathcal{L}_\nu\mathbf{k} = 0 \quad (\mathbf{k}\in\mathcal{D}), \quad \mathcal{L}_{\overline{\nu}}^*\mathbf{k}^* = 0 \quad (\mathbf{k}^*\in\mathcal{D}^*). \tag{9}$$

The solution of differential equation (9_1) may be obtained as

$$\mathbf{k} = \mathcal{Y}\mathbf{C}, \quad \mathbf{C} = (c_1, c_2, c_3, c_4), \quad \mathcal{Y} = \mathcal{Y}_{\varphi z}\mathcal{Y}_r, \tag{10}$$

$$\mathcal{Y}_{\varphi z} = \mathrm{diag}\left[\begin{Bmatrix}\sin n\varphi \\ \cos n\varphi\end{Bmatrix}\cos m'z, \begin{Bmatrix}\cos n\varphi \\ \sin n\varphi\end{Bmatrix}\cos m'z, \begin{Bmatrix}\sin n\varphi \\ \cos n\varphi\end{Bmatrix}\sin m'z, \begin{Bmatrix}\sin n\varphi \\ \cos n\varphi\end{Bmatrix}\cos m'z\right],$$

$$\mathcal{Y}_r = \begin{pmatrix} Y_r^{11} & Y_r^{12} & -\frac{n}{r}J_n(s_3r) & -\frac{m\pi}{H}J_{n+1}(s_3r) \\ \frac{n\zeta_1}{r}J_n(s_1r) & \frac{n}{r}J_n(s_2r) & s_3J_{n+1}(s_3r)-\frac{n}{r}J_n(s_3r) & \frac{m\pi}{H}J_{n+1}(s_3r) \\ -\frac{m\pi\zeta_1}{H}J_n(s_1r) & -\frac{m\pi}{H}J_n(s_2r) & 0 & s_3J_n(s_3r) \\ J_n(s_1r) & \zeta_2J_n(s_2r) & 0 & 0 \end{pmatrix},$$

where J_n are Bessel functions of the first kind, n, m are natural numbers,

$$m' = \frac{m\pi}{H}, \quad Y_r^{11} = \frac{n\zeta_1}{r}J_n(s_1r) - s_1\zeta_1J_{n+1}(s_1r),\; Y_r^{12} = \frac{n}{r}J_n(s_2r) - s_2J_{n+1}(s_2r)$$

and $\zeta_1,\ldots,\zeta_5$, $s_1,\ldots,s_3$ are the following parameters

$$\zeta_1 = \frac{\zeta_3-\zeta_4+\zeta_5}{2\zeta_3\nu^2},\; \zeta_2 = \frac{2\zeta_3\nu^2}{\zeta_3-\zeta_4-\zeta_5}, \quad s_{1,2} = \sqrt{m'^2 + \nu\frac{\zeta_3+\zeta_4\pm\zeta_5}{2\kappa\left(4(\mu+\nu\mu')/3+K\right)}}, \quad s_3 = \sqrt{m'^2 + \frac{\nu^2\rho}{\mu}},$$

$$\zeta_3 = \nu\rho\kappa, \quad \zeta_4 = \gamma\eta\kappa + 4\left(\mu+\nu\mu'\right)/3+K, \quad \zeta_5 = \sqrt{(\zeta_3+\zeta_4)^2 - 4\zeta_3\left(4\left(\mu+\nu\mu'\right)/3+K\right)}.$$

The solutions (10) of system (9_1) satisfy the boundary conditions (7) at the ends, the periodicity conditions, and the boundedness conditions; but for arbitrary values of the constants $c_1,\ldots,c_4$ and the parameter ν, conditions on the lateral surface of the cylinder are not satisfied in general. By substituting the expressions (10) into the boundary conditions (7), we obtain a homogeneous system of algebraic equations for the constants $c_1,\ldots,c_4$:

$$\mathbf{D}\mathbf{C} = 0, \quad \mathbf{C} = (c_1, c_2, c_3, c_4), \quad \mathbf{D} = \mathcal{B}_1\mathcal{Y} \tag{11}$$

After the corresponding operations of differentiation and reduction of the Bessel functions to the orders n and $n+1$, we obtain the following expression for the matrix $\mathbf{D}$:

$$\mathbf{D} = \mathbf{A}\,\mathrm{diag}\left[J_n(s_1R), J_n(s_2R), J_n(s_3R), J_n(s_3R)\right] + \mathbf{B}\,\mathrm{diag}\left[J_{n+1}(s_1R), J_{n+1}(s_2R), J_{n+1}(s_3R), J_{n+1}(s_3R)\right],$$

$$\mathbf{A}=\begin{pmatrix} \zeta_1\left(\frac{2\zeta_6 l}{R^2}-\zeta_7 s_1^2-\frac{\zeta_8 m^2\pi^2}{H^2}\right)-\gamma & \frac{2\zeta_6 l}{R^2}-\zeta_7 s_2^2-\frac{\zeta_8 m^2\pi^2}{H^2}-\gamma\zeta_2 & -\frac{2\zeta_6 l}{R^2} & -\frac{2\zeta_6 m\pi s_3}{H} \\ -2mn\frac{\pi\zeta_1}{HR} & -2mn\frac{\pi}{HR} & mn\frac{\pi}{HR} & n\frac{s_3}{R} \\ 2l\frac{\zeta_1}{R^2} & 2\frac{l}{R^2} & s_3^2-2\frac{l}{R^2} & m\frac{\pi s_3}{H} \\ \frac{n}{R} & n\frac{\zeta_2}{R} & 0 & 0 \end{pmatrix},$$

$$\mathbf{B}=\begin{pmatrix} 2\zeta_6\frac{s_1\zeta_1}{R} & 2\zeta_6\frac{s_2}{R} & 2\zeta_6\frac{ns_3}{R} & 2\zeta_6\frac{m(n+1)\pi}{HR} \\ 2\frac{m\pi s_1\zeta_1}{H} & 2\frac{m\pi s_2}{H} & 0 & \frac{m^2\pi^2}{H^2}-s_3^2 \\ -2\frac{ns_1\zeta_1}{R} & -2\frac{ns_2}{R} & -2\frac{s_3}{R} & -2\frac{m(n+1)\pi}{HR} \\ -s_1 & -s_2\zeta_2 & 0 & 0 \end{pmatrix},$$

$$\zeta_6=\mu+\nu\mu',\ \zeta_7=\frac{4}{3}\left(\mu+\nu\mu'\right)+K,\ \zeta_8=K-\frac{2}{3}\left(\mu+\nu\mu'\right),\ l=n^2-n.$$

System (11) of homogeneous equations has a nontrivial solution under the condition that the determinant:

$$|\mathbf{D}|=|\mathbf{D}(n,m,\nu)|=0 \tag{12}$$

is zero. For fixed integers $n=N$ and $m=M$, the roots of Eq. (12) form a sequence of eigenvalues

$$\left\{\nu_i^{NM}\right\}_{i=1}^{\infty}=\{\nu|\nu\in\mathbb{C},\mathbf{D}(M,N,\nu)=0\},$$

which are associated with nontrivial solutions of the system of algebraic equations (11), $\mathbf{C}_i^{NM}=(c_{1i}^{NM},\ldots,c_{4i}^{NM})$, and with nontrivial solutions of the Sturm–Liouville boundary value problem, i.e, with the eigenfunctions $\mathbf{k}_i^{NM}=\mathfrak{Y}|_{n=N,m=M,\nu=\nu_i^{NM}}\mathbf{C}_i^{NM}$ (Fig.1). Combining the sequences $\{\mathbf{k}_i^{NM}\}$ constructed for all integer N and M, we obtain a complete system of eigenfunctions, which can be linearly arranged in ascending order of absolute values of the corresponding eigenvalues ν_i^{NM}. Since the adjoint Sturm–Liouville problem (9_2) can formally be obtained from the direct problem by transposing the opposite values of the coefficients $\nu\eta$ and γ, we see that the adjoint eigenfunctions $\mathbf{k}_i^*$ are calculated by the same dependencies as $\mathbf{k}_i$ with the above rearrangement and the replacement of ν by $\overline{\nu}$. We note that when calculating the partial sums of the obtained spectral expansions, it is important to preserve the order of summation according to the position of the limit points on the complex plane. The completeness of the biorthogonal system $\mathbf{k}_i$, $\mathbf{k}_i^*$ ensures the mean square convergence of the representations for displacements.

It is important to note, that, unlike well-known transformation technique (Laplace transform, etc.), that uses numerical approach for inversion, proposed method admit to obtain solution in closed analytical form and to develop effective algorithmic realization of computer simulation.

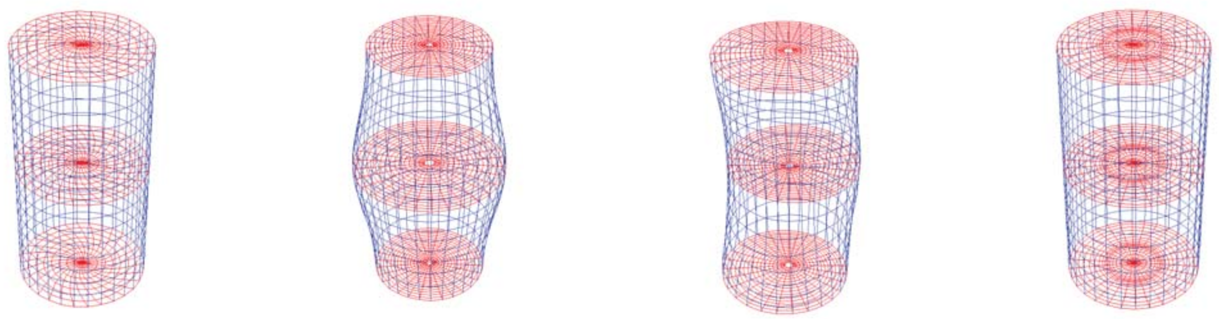

Fig. 1. 3D representation of the real parts of eigenfunctions (torsional, dilative, bending and prportional types).

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