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SOME PARAXIAL APPROXIMATIONS OF VLASOV-MAXWELL EQUATIONS

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Abstract. In recent years, modelling and solving numerically problems which couple charged particle to electromagnetic fields has given rise to challenging mathematical and scientific computing developments. A variety of examples can be thought of, such as the ion or electron injectors for particle accelerators, the free electron lasers, the hyperfrequency devices, the vulnerability of spatial devices to particle flows, etc. This paper will treat about the analysis and the development of some approximate models for the Vlasov-Maxwell system of equations. These approximate models are derived by exploiting physical properties, are simpler to implement, and generally avoid expensive computations. We present some situations in which this strategy can be applied. Numerical results illustrate the possibilites of the approach.

1 Introduction

Charged particles appear essentially in two kinds of physics problems: charged particle beams, like in hyperfrequency devices or vacuum diode technology, and plasma physics, a plasma being roughly speaking a gas of quasi neutral charged particles. Over the past few years, plasmas are involved in a lot of real-life applications. They are commonly used in Science and Technology and play an important role in the energy production (for instance in the magnetic confinement fusion). Plasmas are also ingredients of instruments and others devices (see the Introduction of [8] for a survey of the applications). Moreover, all fusion applications involve non linear interaction of charge particle beams. As a consequence, there is a need in finding mathematical models which can be used for numerical simulations.

Quite complete mathematical models to solve these problems are based on the time-dependent Vlasov-Maxwell system of equations, sometimes under the relativistic assumption. Indeed, there exists a strong correlation between the Maxwell equations and models that describe the motion of particles. This correlation is at the origin of most of the coupled models, where the Maxwell equations (or any kind of equations approximating them) appear in parallel with (and depending on) other models of equations.

Hence, the Maxwell equations are related to electric charged particles, the motion of which being a source for generating an electromagnetic field. Conversely, for a population of charged particles with a mass m and a charge q, the main force field is the electromagnetic Lorentz force

$$\mathbf{F} = q\left(\mathscr{E}(\mathbf{x},t) + \mathbf{v}(t) \times \mathscr{B}(\mathbf{x},t)\right), \tag{1}$$

that describes how the electromagnetic field $\mathscr{E}(\mathbf{x},t)$ and $\mathscr{B}(\mathbf{x},t)$ acts on a particle with a velocity $\mathbf{v}(t)$.

Describe the motion of a set of N particles by a microscopic model consists in looking simultaneously at the positions $(\mathbf{x}_i)_{1\leq i\leq N}$ and the velocities $(\mathbf{v}_i)_{1\leq i\leq N}$ of these particles. An exact knowledge of this $6N$ dimensional system is in general impossible, because N is very large. So approximate model are needed. In the first level of approximation - the kinetic model - a statistical approach is used, basically by considering the probability of a particle to have a given position and velocity. This also yields a very large system, and simplifications are required. The Vlasov equation is obtained by using the mean-field interaction assumption. Other assumptions are possible. For instance, assuming that binary collisions are predominant gives the Boltzmann equation. In the second level of approximation - the fluid approach - models are obtained by taking a few moments (generally 2 or 3) of the kinetic equation, then adding a closure relation.

This paper is essentially concerned with the kinetic model, which can be viewed as an intermediate between the molecular level and the fluid one (see [4], [6] and references therein). Indeed, the kinetic approach retains information on the distribution of particle in velocity, which is lost in a fluid model.

To avoid very expensive computations when it is possible, simpler models have been developed in many situations, and will require mathematical analysis. For instance, the case of stationary beams was analyzed in [7], using a technique of asymptotic expansions. Our goal is to exploit specific properties of the beams (paraxial property, relativistic case) to derive by means of asymptotic expansions, simpler models. Mathematical analysis will be also necessary to investigate the accuracy of these proposed models. Moreover, laser-plasma interaction models (cf. [10]) could be also developed using the same tools, and would be important applications (energy production). Hence, in some cases, assuming that the problem is static allows to replaced Maxwell's equations by a *reduced model* like Poisson's equation. Following this idea, one can obtain a hierarchy of reduced models, like Vlasov-Poisson, Vlasov-Darwin, paraxial models, gyrokinetic models, laser-plasma interaction models, etc...generally obtained by exploiting specific geometries/properties of the problem. Sometimes, these models have been derived in a formal way, and there is a need first to justify them, then to precise how much accurate they are, finally to improve their accuracy. Moreover, one can hope to also obtain by these techniques new approximate models, and to develop new algorithms. This will be a significant progress for improving the comprehension of complex problems. In this paper, we will treat about the analysis and the development of some approximate models for the Vlasov-Maxwell system of equations. We present some situations in which such approximate models can be derived.

This paper is organized as follows. In the next Section, we recall the Vlasov model and the Maxwell equations, together with their coupling. In Section 3, we introduce the approximate models. First the relativistic paraxial one, then a Poisson-like model, both derived in axisymmetric geometry. Examples of well-adapted numerical schemes that lead to particularly simple algorithms are also given. Then, in Section 4, we illustrate by numerical examples the possibilities of these approximate models. Concluding remarks follow.

2 The model

2.1 The Vlasov model

Let us consider a population of charged particles, submitted to a given force field $\mathbf{F}(\mathbf{x},\mathbf{v},t)$ given by (1). Each particle is characterized by its position $\mathbf{x}$ and its velocity $\mathbf{v}$ in the so-called phase space $(\mathbf{x},\mathbf{v})$. We introduce the distribution function $f(\mathbf{x},\mathbf{v},t)$, which can be defined as the average number of particles in a volume $d\mathbf{x}d\mathbf{v}$ of the phase space. Assuming that collisions between particles can be neglected, so that all the particles which are initially "close" stay close as they move along their trajectory. Then the distribution function $f(\mathbf{x},\mathbf{v},t)$ is solution to the following transport equation, named the Vlasov equation

$$\frac{d}{dt}f(\mathbf{x}(t),\mathbf{v}(t),t) = \frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \frac{q}{m}(\mathscr{E}(\mathbf{x},t)+\mathbf{v}\times\mathscr{B}(\mathbf{x},t))\cdot\nabla_{\mathbf{v}} f = 0. \tag{2}$$

Remark 2.1 *If collisions are not neglected, they induce changes in the particle velocity. To model these collisions, one usually introduces the collision operator $Q(f)$, that can be linear, quadratic, etc., depending on the physics involved. The mathematical tools as well as the numerical methods involved in that case are fairly different from these we intend to use, and then, handle the collisions is excluded from this paper.*

Note that for the relativistic case, if we denote by $\mathbf{p}$ the momentum and introduce the distribution function $f(\mathbf{x},\mathbf{p},t)$ such that

$$\mathbf{p} = \gamma m\mathbf{v}, \quad \text{with} \quad \gamma m = \frac{\sqrt{|\mathbf{p}|^2+m^2c^2}}{c},$$

then the relativistic Vlasov equation is obtained by substituting the term $\frac{1}{m}\nabla_{\mathbf{v}} f$ in Equation (2) by the term $\nabla_{\mathbf{p}} f$.

2.2 The Maxwell equations

The expressions of the charge and the current density induced by the motion of these particles are given by

$$\rho(\mathbf{x},t) = q\int_{\mathbb{R}^3_v} f(\mathbf{x},\mathbf{v},t)\,d\mathbf{v}, \tag{3}$$

$$\mathscr{J}(\mathbf{x},t) = q\int_{\mathbb{R}^3_v} f(\mathbf{x},\mathbf{v},t)\,\mathbf{v}\,d\mathbf{v}, \tag{4}$$

that express the coupling of the Maxwell and Vlasov equations. Indeed $\rho(\mathbf{x},t)$ and $\mathscr{J}(\mathbf{x},t)$ appear as (part of[1]) the right-hand sides of the Maxwell equations (in the vacuum)

$$\frac{1}{c^2}\frac{\partial\mathscr{E}}{\partial t} - \nabla\times\mathscr{B} = -\mu_0\mathscr{J}, \tag{5}$$

$$\frac{\partial\mathscr{B}}{\partial t} + \nabla\times\mathscr{E} = 0, \tag{6}$$

$$\nabla\cdot\mathscr{E} = \frac{\rho}{\varepsilon_0}, \tag{7}$$

$$\nabla\cdot\mathscr{B} = 0, \tag{8}$$

[1] it can happen that parts of ρ and $\mathscr{J}$ are due to external charge and current sources.

where $\mathscr{E}$ denotes the electric field and $\mathscr{B}$ the magnetic induction. The constants ε_0, μ_0 are respectively the dielectric permittivity and the magnetic permeability in the vacuum. c is the speed of the light and satisfies $\varepsilon_0\mu_0 c^2 = 1$. Remark that handle inhomogeneous media is not necessary here, as we deal with Maxwell's equations coupled with a model of motion of particles.

As it is well known, the charge conservation equation, which is a consequence of the Maxwell equations[2], expresses a compatibility relation between ρ and $\mathscr{J}$ and reads

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathscr{J} = 0. \tag{9}$$

Note also that the divergence conditions (7) and (8) are satisfied for all time t if they are satisified at the initial time $t = 0$. This is essentially due to the property $\nabla \cdot (\nabla \times \cdot) = 0$ together with the charge conservation equation (9). Nevertheless, both properties are not generally valid at the discretization level, which is always a source of problems.

Remark 2.2 *One can distinguish between exterior vector fields* $(\mathscr{E}_{ext}, \mathscr{B}_{ext})$ *which are given, and self-consistent fields which are induced by the particles.*

The Vlasov and Maxwell equations *separately* are linear hyperbolic systems, but the expression of the Lorentz force $\mathbf{F}$ in a way and those of the charge and current density ρ and $\mathscr{J}$ in another way leads to a strong coupling, that makes the whole problem quadratic. Indeed, the term

$$\mathbf{F} \cdot \nabla_{\mathbf{p}} f = q(\mathscr{E}(\mathbf{x},t) + \mathbf{v} \times \mathscr{B}(\mathbf{x},t)) \cdot \nabla_{\mathbf{p}} f$$

is a quadratic term since $\mathscr{E}$ and $\mathscr{B}$ depend on the distribution function f in an affine way, through ρ and $\mathscr{J}$. In this paper, our aim is to take into account the particularities of the physical problems to derive some approximate models leading to faster and cheaper computations.

3 Approximate Models

From a numerical point of view, the Vlasov-Maxwell system is very complete but also not easy to solve numerically, in particular in a three-dimensional domain. Evenhougth this is necessary in several cases (see [1], [2]), one easy understands the need of deriving simpler (but accurate) models, by exploiting given physical assumptions. Let us give a first example. Assume that we can neglect the time derivative $\partial_t \mathscr{B}$ in Eq. (6), then $\nabla \times \mathscr{E} = 0$ yields $\mathscr{E} = -\nabla\phi$, where ϕ denotes the electrostatic potential. From the Coulomb's law $\nabla \cdot \mathscr{E} = \rho/\varepsilon_0$, we get the classical quasi-static Vlasov-Poisson model

$$\begin{cases} -\Delta\phi = \dfrac{\rho(t)}{\varepsilon_0}, \\ \mathscr{E} = -\nabla\phi, \end{cases}$$

coupled with the Vlasov equation (2), in which the Lorentz force is given by

$$\mathbf{F} = q\mathscr{E} = -q\nabla\phi.$$

The main advantage of this model is that is not *explicitly* time-dependent, the charge density $\rho(t)$ being given at each time step of the Vlasov equation solution. This model can also be derived in the context of approximate models. In [9], Sonnendrücker et. al. assume that the velocity of the particles $\mathbf{v}_p$ is very small compared to the velocity of the electromagnetic waves c, and introduce a small parameter $\eta = \dfrac{\mathbf{v}_p}{c}$. Then after a scaling and an asymptotic expansion of the solution in power of η, they proved that the quasistatic model is a first order approximation of the Maxwell equations. The main objective of this paper is to derive, to discretize and to compare approximate models in the numerical simulation of charged particle beams and plasma physics phenomena.

3.1 Relativistic Paraxial Model

Very often, simpler mathematical models can be derived by exploiting physical properties. Hence, when dealing with charged particle beams, it is fruitful to exploit the paraxial property, i.e the property that the particles of the beam remain close to an optical axis.

Hence, consider a particle beam that is highly relativistic i.e., satisfies $\gamma = (1 - \frac{\mathbf{v}^2}{c^2})^{-1/2} >> 1$. Since the velocity is thus close to light velocity c, that is $\mathbf{v} \simeq c$ for any particle in the beam, it is convenient to rewrite the Vlasov-Maxwell equations in the beam frame, i.e. in a frame which moves along the z-axis with the velocity c. According to [7], it is worthwhile to distinguish the transverse quantities from the longitudinal ones. See [3] for the resulting expression of the equations. Now, to derive a paraxial model, the first step is to introduce a scaling of the equations. Let us assume that

[2] obtained by taking the divergence of Ampere's law (5) combined with the time derivative of the divergence condition (7)

1. the dimensions of the beam are small compared to the longitudinal length of the device,
2. the longitudinal particle velocities v_z are close to the light velocity c,
3. the transverse particle velocities $\mathbf{v}_\perp$ are small compared to c,

we can introduce a small parameter $\eta = \frac{\bar{\mathbf{v}}_\perp}{c}$, where $\bar{\mathbf{v}}_\perp$ is the transverse characteristic velocity. Then, we rewrite the Vlasov-Maxwell system of equations in the beam frame, which moves along the z-axis with the light velocity c. Hence we set

$$\zeta = ct - z, \quad v_\zeta = c - v_z\,.$$

and use dimensionless variables (see again [3] for a complete expression). It remains now to develop asymptotic expansions of all the quantities $(f, \rho, \mathscr{J}, \mathscr{E}, \mathscr{B}, ...)$ in powers of the small parameter η, and to retain the first four terms (in that case) of this asymptotic expansion to obtain a new mathematical model. An important theoretical result ([7]) was to prove that such a model is accurate up to fourth order. Preliminary numerical results can be found in [3].

Let us consider now the axisymmetric counterpart. Assume that the beam of charged particles moves inside a perfectly conducting cylindrical tube, the z-axis being the axis of the tube. Using the coordinates (r, θ, ζ) (with obvious notations), we denote by R the radius of the tube, so that the boundary of the tube is $\{(r, \theta, z);\, r = R\}$. The electric field is now denoted (E_r, E_θ, E_z) and the magnetic one (B_r, B_θ, B_z). One so obtains that the electromagnetic force $\mathbf{F}$ is entirely determined by the transverse fields $E_r, E_\theta, B_r, B_\theta$, which are zero order fields, the longitudinal ones E_z, B_z, that are first order fields, and the so-called pseudo-fields $\mathscr{K}_r = E_r - cB_\theta$ and $\mathscr{K}_\theta = E_\theta + cB_r$, which are second order corrections. Hence, the relativistic paraxial model of Maxwell equations is written:
For the zero order fields:

$$\left\{ \begin{array}{l} E_r = cB_\theta = \dfrac{1}{\varepsilon_0\, r} \displaystyle\int_0^r \rho s\, ds \\[2ex] \qquad E_\theta = B_r = 0 \end{array} \right. \tag{10}$$

For the first order fields:

$$\left\{ \begin{array}{l} \dfrac{\partial E_z}{\partial r} = \dfrac{\partial B_\theta}{\partial t} \\[2ex] E_z(r = R) = 0 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \dfrac{\partial B_z}{\partial r} = \mu_0 J_\theta \\[2ex] \int_0^R B_z r dr = 0 \end{array} \right. \tag{11}$$

For the second order pseudo-fields $\mathscr{K}_r$ and $\mathscr{K}_\theta$:

$$\left\{ \begin{array}{l} \mathscr{K}_r = \dfrac{1}{r} \displaystyle\int_0^r (\mu_0 c J_\zeta - \dfrac{1}{c}\dfrac{\partial E_z}{\partial t}) s\, ds \\[2ex] \qquad \mathscr{K}_\theta = -\dfrac{1}{r} \displaystyle\int_0^r \dfrac{\partial B_z}{\partial t} s\, ds, \end{array} \right. \tag{12}$$

where J_ζ is defined by

$$J_\zeta = \rho c - J_z = q \int v_\zeta f\, d\mathbf{v}$$

As expected, this leads to far simpler model than the full Vlasov-Maxwell model.

We approximate these equations with specific numerical schemes based on a finite-difference approach. The order of the computations is induced by the asymptotic expansion. Hence, the zero order fields E_r, B_θ have to be first computed, and are necessary to obtain the first order quantities E_z. Similarly, the computation of the second order pseudo-fields $\mathscr{K}_r$ and $\mathscr{K}_\theta$ requires the first order approximate fields E_z and B_z. Note that the longitudinal magnetic component B_z only depends on the azimuthal current density J_θ. In particular, B_z is identically zero as soon as J_θ vanishes.

As we are working in the beam frame, the particles drift slowly in the direction $\zeta > 0$. As a consequence, the computational domain is defined as a simple rectangular domain in variables (r, ζ), $0 \le r \le R$, $0 \le \zeta \le Z$. The value of R is given by the radius of the cylindrical tube, and Z is chosen in such a way that the particles remain in a fixed geometrical domain (in the beam frame), during the time interval $[0, T]$ of the simulation.

As an example, we give here the numerical scheme for B_θ (or equivalently E_r). For a given or computed charge density, equation (10) can be solved by simple numerical integration methods. For instance, consider a classical

2-point Newton-Cotes formula, which is exact for the first-order degree polynoms. We thus obtain for B_θ^{n+1} (the same for E_r^{n+1})

$$B_{\theta,i,j}^{n+1} = \frac{1}{c\, r_i \varepsilon_0}\frac{\Delta r}{2}[\rho_{1,j}^{n+1} r_1 + 2\rho_{2,j}^{n+1} r_2 + ... + \rho_{i,j}^{n+1} r_i] \tag{13}$$

Similar numerical schemes can be derived for the other components.

3.2 Poisson-like model

Another approximate model can similarly be derived. Indeed, at first glance, there are interesting similarities between the zero-order terms of the above paraxial model and the Vlasov-Poisson equation, both written in the beam frame. This remark motivates the following computations.

Writing the axisymmetric Poisson equation in the beam frame (r,ζ), we obtain

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial \zeta^2} = \frac{1}{\varepsilon_0}\rho, \tag{14}$$

and the electric field verifies

$$\mathbf{E} = (\frac{\partial \phi}{\partial r}, \frac{\partial \phi}{\partial \zeta}). \tag{15}$$

Then, deriving numerical schemes for Vlasov-Poisson equation by straightforward finite differences approximation leads to

$$\frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{\Delta r^2} + \frac{1}{r_i}\frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta r} + \frac{\phi_{i,j+1} - 2\phi_{i,j} - \phi_{i,j-1}}{\Delta \zeta^2} = \frac{1}{\varepsilon_0}\rho_{i,j} \tag{16}$$

with the boundary condition

$$\rho|_{I,j} = \rho|_{0,j} = 0, \qquad \text{and } \phi|_{r=0} = 0 \tag{17}$$

The electric field is here approximated with

$$(E_r, E_\theta, E_z)_{i,j} = (\frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta r}, 0, -\frac{\phi_{i,j+1} - \phi_{i,j}}{\Delta \zeta}). \tag{18}$$

From a physical point of view, this Poisson-like model can be partially compared with the previous paraxial one. Indeed, in the paraxial model, the particle motion is first governed by the zero-order radial electric field E_r, whereas the longitudinal field E_z appears as a first-order correction. Now, to obtain a similar expression for the radial electric field E_r solution to the Vlasov-Poisson model, we have to artificially impose $E_z = 0$. In that case, E_r can be easily reduced to the zero-order paraxial model

$$E_r = \frac{1}{\varepsilon_0 r}\int_0^r \rho s\, ds. \tag{19}$$

Otherwise, E_r and E_z solution to Poisson-like model will be approximately at the same scale. Indeed, Poisson equation describes electric field that depends only on the distance between particles, without taking into account their velocity. Hence, the cornerstone of the paraxial model, the assumption that particle velocity is close to the speed of light, can not be handled properly in Poisson equation that describes in essence static electric field.

4 Numerical Results

In order to illustrate these two approximate models, we propose here results of numerical experiments and comparisons. As we are working in the beam frame, the computational domain is the rectangle $]0,R[\times]0,Z[$ in variables (r,ζ). The mesh sizes $\Delta r, \Delta\zeta$ are chosen such that $R/\Delta r = Z/\Delta\zeta = 0.01$. The time step Δt is taken in order to comply with the CFL stability condition. The Vlasov equation is discretized with a particle approach like in [2].

As a numerical example, consider a bunch of particles emitted with velocities such that the paraxial assumptions are verified. According to stability condition [4], more than 10 particles are placed in each cell, with the same weight w and a charge following

$$w = \frac{\mathbf{J}\Delta t}{Ne},$$

where $\mathbf{J}$ is the total current to be emitted, N the particle number and e the electric charge of the particles, here electrons. Figure 1 shows the electric radial field E_r obtained after 50 time steps of simulation with the relativistic

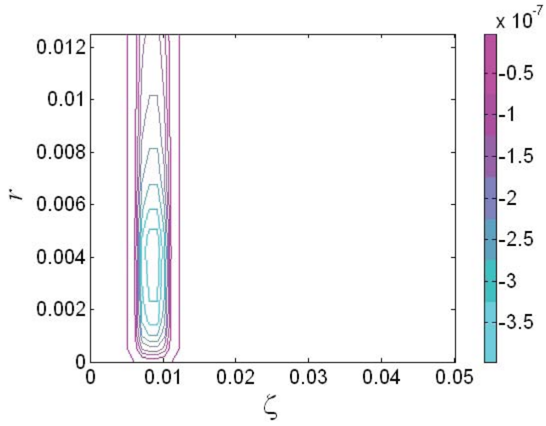

Figure 1: E_r with relativistic paraxial model

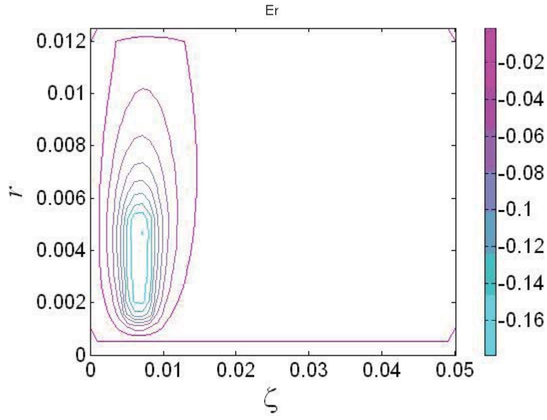

Figure 2: E_r with Poisson-like model

paraxial model. The result of a simulation obtained with the Poisson-like model (in the same conditions) is depicted on Figure 2.

As we have already mentioned, these models (and so these results) can be partially compared. Indeed, in the Poisson-like problem which is only static, E_r and E_z are approximately at the same scale. In the relativistic paraxial model, the radial electric field E_r is a zero-order field, whereas the longitudinal field E_z appears as a first-order correction. In other terms, Poisson-like equation, that describes static solution, describes electric field that depends only on the distance between particles, whereas the relativistic paraxial takes into account their velocity.

Due to its sensitivity to particle velocity, the paraxial model, which appears numerically as simple as a static or quasi-static one, is much more powerful for ultrarelativistic process simulation than a Vlasov-Poisson model.

5 Conclusion

In this paper, we are concerned with the development of paraxial approximations for Vlasov-Maxwell equations. This aim was to build a useful tool for solving particle beams and plasma physics problems. Exploiting the particularities of the physical problem, we proposed to develop reduced models. We hope this approach to be powerfull in its ability to get accurate, but fast and easy to implement algorithms. Numerical results were presented to illustrate the feasibility and the accuracy of this approach. In particular, numerical comparisons between a relativistic paraxial model and a Poisson-like one, written in the beam frame were performed. They help us to illustrate the importance of handling such models in different physical configurations.

6 References

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