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APPROXIMATING DISTRIBUTIONAL BEHAVIOUR, SYSTEMS THEORY AND CONTROL

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Abstract. In many engineering and generally speaking more physical dynamic differential systems, the problem of transferring the initial state of the system to a desired in (almost) zero-time time is very significant. For instance, in biology, it is well known the famous Leslie population growth model. Thus, in this particular application the kicking of the initial state in (almost) zero time is being appeared whenever an environmental conservational organization (for instance WWF) reveals new populations in their habitats. In practice, this can be succeeded by using a linear combination of Dirac δ-function and its derivatives. Additionally, some good reasons for this choice are discussed in this paper. Obviously, such an input is very hard to imagine physically. However, we can think of it approximately as a combination of small pulses of very high magnitude and infinitely small duration. In this brief paper, an overview of the approximating distributional behaviour of descriptor differential systems is presented. Based on several new research results, a concrete mathematical framework is provided for our approximation procedure. Some important elements of system and control theory are also available. Finally, a numerical example is appeared.

1 Introduction

In the vast literature of engineering applications, the Dirac δ-function has been extensively used. This "*function*" has led to the use of the concept of impulse behaviour and it has formulated the basis of time domain as well as the transformation analysis of several kinds of systems. The Dirac δ-function is denoted as having undefined magnitude at the time of occurrence and zero elsewhere with a further property that area under this is unity. Obviously, such "*function*" is very hard to imagine physically. However, it can be considered approximately as a small pulse of very high magnitude and infinitely small duration such as the domain under the pulse is unity.

Before we go further, we can consider a *RC circuit network* (see example 2.5, [8]), see **figure 1**. A voltage δ-function is applied to this circuit.

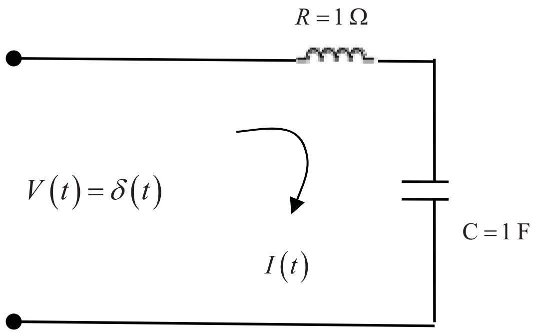

Figure 1: A simple circuit network

We have

$$\delta(t) = I(t) + \int I(t)\,dt \ \text{ or } \ \delta^{(1)}(t) = I(t) + \frac{d}{dt} I(t). \tag{1}$$

Assume that we have zero initial conditions, and the main question is to calculate the current $I(t)$. Since an impulse attitude is applied at time $t = 0$, it is wiser to assume that the current $I(t)$ has also an impulse response in its expression. However, for $t > 0$, the capacitor will discharge as e^{-t}. Hence, we can assume

$$I(t) = a1(t)e^{-t} + b\delta(t) \tag{2}$$

Substituting (2) into (1), we obtain

$$a\delta(t)e^{-t} + b\delta^{(1)}(t) + b\delta(t) = \delta^{(1)}(t),$$

and

$$(a+b)\delta(t) + (b-1)\delta^{(1)}(t) = 0, \text{ because } \delta(t)e^{-t} = \delta(t),$$

which implies that $a = -b$ and $b = 1$. Therefore,

$$I(t) = -1(t)e^{-t} + \delta(t). \tag{3}$$

The example above implies that we are able to change the voltage across the capacitor in zero time applying the Dirac δ -function. Moreover, if our circuit network has a greater number of inductors and capacitors, and we are interested in changing in zero time (under some conditions) the current in the inductors and the voltage across the capacitors, we have to apply higher order Dirac δ -functions.

In this paper, our main objective is to review, to introduce the basic elements of distributional functions, and to provide a deep concrete mathematical framework for our approximation procedure. Analytically, in Section 2, the basic problem is introduced. The problem of transferring the initial state to a desired released condition in minimum time is a valid problem in linear control theory. This question can be answered theoretically by using Dirac delta functions and its derivatives. However, only smooth approximations of this input have practical importance. Thus, in Section 3, we review the basic approximation functions o Dirac delta function. Why Dirac function is the right theoretical approach is being answered in Section 4. Moreover, in Section 5, an interesting numerical example based on [10] is discussed. Thus, the unknown coefficients are analytically determined. Section 6 concludes the whole paper. Some further topics of research are also available.

2 The Basic Problem: Transferring of the State of a System (Almost) Instantly

The last decades, the development of reliable mathematical, numerical and computational techniques for the analysis and the synthesis of *linear autonomous (descriptor-regular or singular) control (matrix) differential systems* has been a very active area of productive research. For instance, let suppose that we obtain the following matrix equation (note that our intension is to present as wide-ranging results as we can)

$$FX'(t) = AX(t) + BU(t), \tag{4}$$

where $X(t) \in \mathcal{C}^{\infty}(\mathbb{F}, \mathcal{M}(m\times k;\mathbb{F}))$ (is a smooth function over $\mathbb{F}$, which elements belong to the algebra $\mathcal{M}(m\times k;\mathbb{F})$), and $U(t) \in \mathbb{F}^{l\times k}$ state, and input (controller) matrix, respectively; the matrices $F, A \in \mathcal{M}(n\times m;\mathbb{F})$ and the input matrix $B \in \mathcal{M}(n\times l;\mathbb{F})$ are constants.

The need of such techniques arises primarily from the increasing, practical interest for a more general system description, which considers the inherent physical system model structure, see for instance [15-17], [21] et al..

Besides that, many standard state-space (matrix) systems problems might lead naturally to descriptor (matrix) systems formulations and therefore can be solved reliably only by using descriptor (matrix) systems computational techniques. Obviously, this approach is more general and includes standard state-space systems.

Nowadays, it is well known that distributional solutions and behaviour enter to the study of many areas in systems and control science such as:

1. *Controllability, Observability.*
2. *Infinite zero characteristic behaviour.*
3. *Almost invariant subspaces, almost controllability spaces.*
4. *Dynamics of singular systems, etc.*

In this part of the paper, it should be stressed out that since the four (1-4) topics above are significant in many areas of control and system theory, there are really numerous papers that we should be referred. However, due to the restriction in pages, we prefer to skip out the relevant references. Perhaps, another review paper should be written discussing each topic separately.

Moreover, the distributional characterization is also linked to the solution of a number of control problems. Although such solutions have theoretical significance, their value is limited from the practical (implementation of solutions) viewpoint, since impulses represent distributions and cannot be constructed. Only functions can.

Thus, the following very important question should be properly answered.

Question:

Can we develop an approximation to impulsive behaviour with a respective approximation of the related system and control properties?

If there is an answer to this, then we can develop a smooth version approximation of impulsive behaviour and related system and control properties. Furthermore, a number of results obtained in this area highlight the importance of initial release conditions of the state vector. Last decades, through the point of several important applications, in several fields of product research, changing the given state of a linear system to a much-desired state in minimum time is a very interesting problem in general, for control and system theory, see **figure 2**.

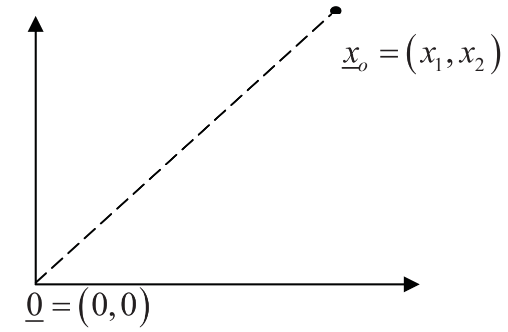

Figure 2: The instant changing of the initial condition (0,0) to the desired (x_1 ,x_2) for a two-dimensional system

Significant attention has been given to this problem in the case of linear systems; see [7-8] and [14]. [10] has further enriched these first approaches, where some previous assumptions have been completely moved out. Furthermore, the methodology has been applied to the more general class of linear descriptor (regular) systems; see for more details [11-13] and [19].

In all the above approaches, it has been assumed that for a linear differential system, the input is a linear combination of Dirac δ -function and its first δ -function $n-1$ derivatives, as follows, see Eq. (5)

$$U(t) = \sum_{i=0}^{n-1} F_i \delta^{(i)}(t) \tag{5}$$

where $\delta^{(k)}$ is the k^{th}-derivative of the Dirac δ -function, and the sequence of $F_i \in \mathcal{M}(l \times k; \mathbb{F})$ matrices for $i = 0,1,\ldots,n-1$ are the magnitudes of the delta function and its derivatives.

Furthermore, we assume that our system is *controllable*, i.e. we can transfer the state in a desired place. Let the state of the system at time $t = 0^-$ is $X(0^-) = \left[x_{ij}^o\right]_{\substack{i=1,2,\ldots,m \\ j=1,2,\ldots,k}}$ and at time $t = 0^+$ achieves $X(0^+) = \left[x_{ij}\right]_{\substack{i=1,2,\ldots,m \\ j=1,2,\ldots,k}}$.

The existence of an input that transfer the state of the system (4) from $X(0^-)$ to $X(0^+)$ requires that the matrix $\left[x_{ij}\right]_{\substack{i=1,2,\ldots,m \\ j=1,2,\ldots,k}}$ belongs to the controllable subspace of the pair (A, B), where matrices $A \in \mathcal{M}(n \times m; \mathbb{F})$ and $B \in \mathcal{M}(n \times l; \mathbb{F})$.

The necessary and sufficient condition for the state of a system (4) to be transferred from $\left[x_{ij}^o\right]_{\substack{i=1,2,\ldots,m \\ j=1,2,\ldots,k}}$ at time $t = 0^-$ to some $\left[x_{ij}\right]_{\substack{i=1,2,\ldots,m \\ j=1,2,\ldots,k}} \{A \mid \mathcal{B}\}$, where $\mathcal{B}$ is the range of matrix B, at $t = 0^+$ by the action of control input of type (5) is that the direction F_i, $i = 0,1,\ldots,n-1$ are chosen to be the components of $\left[x_{ij}\right]_{\substack{i=1,2,\ldots,m \\ j=1,2,\ldots,k}}$ along the subspace $\mathcal{B}, A\mathcal{B}, A^2\mathcal{B},\ldots, A^{n-1}\mathcal{B}$, respectively according to some projections law.

In the next section some fundamental approximations of Dirac delta function are presented.

3 Approximations of Dirac delta function

For many practical applications, see also section 1, since the $\delta(t)$ function is not an "ordinary" function, we suggest a systematic and very rigorous procedure for generating sequences of its derivatives. However, if one of the dozens approximations of Dirac δ-function is being followed, see [4-5], [7-8], [18], and [20] the change of the state in some minimum practical time depends mainly upon how well the approximations is being generated. Let the Dirac δ-function can be viewed as the limit of sequence function

$$\delta(t) = \lim_{a\to 0}\delta_a(t) \tag{6}$$

where $\delta_a(t)$ is called a *nascent* delta function. This limits is in the sense that

$$\lim_{a\to 0}\int_{-\infty}^{+\infty}\delta_a(t)f(t)\,dt = f(0). \tag{7}$$

These problems can often be circumvented by using a smooth, finite approximation of the Dirac distribution. Such approximations have an additional advantage.

Approximating the Dirac distribution by a smooth function may actually be a better representation of the system under consideration, especially if the width of the approximation function can be coupled to the physics of the problem. Following, as close as it is possible, the ideas of [4], a suitable approximating function, which is convenient for computation, should satisfy the following important properties everywhere on the domain under consideration:

1. Its limit with some defining parameter is the Dirac distribution, see Eq. (6).
2. It is positive, decreases monotonically from a finite maximum at the source point, and trends to zero at the domain extremes.
3. Its derivative exists and is continuous function.
4. It is symmetric about the source point, for instance 0 (see eqs. (6) and (7)).
5. It is representable by a reasonably simple Fourier integral (for infinite domains) or Fourier series (for finite domains).

In the next lines, the appropriate approximation of Dirac function is discussed based on the finiteness or infiniteness of the time domain.

Infinite time domain

Before, we go further we want to point out that the best nascent delta function depends mainly on the particular application. Some well known and very useful in applications nascent delta functions are the Gaussian and Cauchy distributions, the rectangular function, the derivative of the sigmoid (or Fermi-Dirac) function, the Airy function etc, see for instance [8], [17], [20] et al. and quite recently by using a finite difference formula which is converted into an appropriate sequence, see [2].

Also, some well known and very useful in applications nascent delta functions are the Normal and Cauchy distributions,

$$\delta_a(t) = \frac{1}{\pi}\frac{a}{a^2+x^2} = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{ikx-|ak|}dk\,,$$

the rectangular function,

$$\delta_a(t) = \frac{rect(x/a)}{a} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sin c\left(\frac{ak}{2\pi}\right)e^{ikx}dk\,,$$

the derivative of the sigmoid (or Fermi-Dirac) function,

$$\delta_a(t) = \partial_x\frac{1}{1+e^{-x/a}} = -\partial_x\frac{1}{1+e^{x/a}}\,,$$

the Airy function

$$\delta_a(t) = \frac{1}{a}A_i\left(\frac{x}{a}\right).$$

Following, the finite difference formula may be easily converted into a sequence that approaches a derivative of the Dirac delta function in one dimension. Thus,

$$\delta_a(t)=\begin{cases}\frac{1}{a}, -\frac{a}{2}<t<\frac{a}{2}\\ 0, \quad |t|>\frac{a}{2}\end{cases} \tag{8}$$

which approaches $\delta(t)$ as $a\to 0$. Moreover, an expression for the derivatives of the Dirac delta can be given by the following equation.

$$\frac{d^n}{dx^n}\delta(x)=\lim_{\substack{a\to 0\\ h\to 0}}\left[\left(\frac{1}{h}\right)^n\sum_{j=0}^{n}a_j\delta_a\left(x+b_jh\right)\right] \tag{9}$$

where $x=t_o-t$ and we use $\frac{d^m}{du^m}\delta(u)|_{t-t_o}=(-1)^m\frac{d^m}{du^m}\delta(u)|_x$. The eq. (9) is exactly what we might obtain by simply making the substitution $f(t)\to\delta_a(t)$ in the following finite difference approximation for the n-th derivative of a test function f evaluated at t_o which can be represented as

$$\frac{d^n}{dt^n}f(t)|_{t=t_o}\approx\left(\frac{1}{h}\right)^n\sum_{j=0}^{n}a_jf\left(t_o+b_jh\right). \tag{10}$$

Note that a_j and b_j are suitable chosen constants and eq. (10) becomes exactly in the limit $h\to 0$. Furthermore, due to the fact that f is sampled at discrete points, we can write

$$\frac{d^n}{dt^n}f(t)|_{t=t_o}=\lim_{h\to 0}\left\{\left(\frac{1}{h}\right)^n\sum_{j=0}^{n}a_j\int_{-\infty}^{+\infty}\delta\left(t-\left(t_o+b_jh\right)\right)f(t)\,dt\right\}. \tag{11}$$

Finite time domain

Unfortunately, the Gaussian distribution is not a good approximation of the Dirac distribution on a finite domain, namely that the first derivative (which is important in this paper) can be discontinuous at a special point. Thus, recently, a different approximation has been proposed by [4], which satisfies all the properties 1 through 5. This is the β-distribution of the classical probability theory. This distribution has the expression

$$\beta_\pi(\theta)=\begin{cases}\dfrac{(\pi+\theta)^{a-1}(\pi-\theta)^{b-1}}{(2\pi)^{2a-1}B(a,b)}, \forall\theta\in\mathcal{J}\\ 0 \qquad\qquad\qquad ,otherwise\end{cases} \tag{12}$$

where $\mathcal{J}$ is a finite interval, and

$$B(a,b)\triangleq\int_{\mathcal{J}}(\pi+\theta)^{a-1}(\pi-\theta)^{b-1}\,d\theta=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

where also $\Gamma(x)$ is the well-known Gamma distribution. Since, in the next few lines of the present paper, the infinite time domain is used, the interesting reader may consult [4] for further details.

The results presented in [7-8], [10-13] and [19] based on the classical normal function, i.e.

$$\delta(t)=\lim_{\sigma\to 0}\frac{1}{\sigma\sqrt{2\pi}}e^{-t^2/2\sigma^2}=\lim_{\sigma\to 0}\frac{1}{\sigma}\Phi\left(\frac{t}{\sigma}\right), \tag{13}$$

where $\Phi(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.

Consequently, the approximate expression for the controller (4) is given by

$$U_\sigma(t)=\sum_{i=0}^{n-1}F_i\frac{1}{\sigma^{i+1}}\Phi^{(i)}\left(\frac{t}{\sigma}\right) \tag{14}$$

where $\Phi^{(i)}\left(\frac{t}{\sigma}\right)=\left(\frac{d^i}{dt^i}\left(\frac{t}{\sigma}\right)^i\right)\Phi\left(\frac{t}{\sigma}\right)$.

Then, we take the limit

$$U(t)=\lim_{\sigma\to 0}U_\sigma(t). \tag{15}$$

4 Why Dirac delta functions?

Moreover, considering the results of section 1 and the whole discussion till that section, we can assume that the input for the linear descriptor differential system (4) can be given by a *single-layer distribution*. This kind of distributions has a huge importance in applications.

Lemma 1 If $\mathcal{U}$ is a bounded closed set in $\mathbb{F}$ and $\mathcal{Y}$ is a neighbourhood of $\mathcal{U}$, then there exists a function such that $n=1$ on $\mathcal{U}$, $n=0$ outside $\mathcal{Y}$, and $0\le n\le 1$ over $\mathbb{F}$.

Definition 1 Let $\mathcal{S}$ be a piecewise regular curve in $\mathbb{F}$ and σ is a locally integrable function defined on $\mathcal{S}$. The linear continuous functional $\sigma\delta_S$ on the space $\mathcal{D}$ of infinitely differentiable complex-valued functions on $\mathbb{F}$ with compact support, see [5], [18] and [20] is defined as

$$\langle\sigma\delta_S,\varphi\rangle=\int_S\varphi(\xi)\sigma(\xi)\delta S \tag{16}$$

$\forall\ \varphi\in\mathcal{D}$ and is called *single (or simple) layer on* $\mathcal{S}$ *with density* σ.

Note that $\sigma\delta_S(x)=\int_S\delta(x-\xi)\sigma(\xi)\delta S_\xi$.

Definition 2 Let $\mathcal{S}$ be a piecewise regular curve in $\mathbb{F}$ and $\mu\delta_S$. The linear continuous functional $-d/dt(\mu\delta_S)$ on the space $\mathcal{D}$ of infinitely differentiable complex-valued functions on $\mathbb{F}$ with bounded support is defined as

$$\langle -d/dt(\sigma\delta_S),\varphi\rangle=\int_S\sigma(\xi)\frac{d\varphi(x-\xi)}{dt}\delta S \tag{17}$$

$\forall\ \varphi\in\mathcal{D}$.

Consequently, *it can be easily shown that every distribution* $\sigma\delta_S(x)$ *that has compact support is of finite order, see [5] and [18]. Then, it is deduced that every distribution* $\sigma\delta_S(x)$ *whose support is the point* $x=\tau$ *has the form*

$$\sum_{i=0}^{n-1}c_i\delta^{(i)}(t-\tau),$$

i.e. a linear independent combination of Dirac δ-function and its first $n-1$ derivatives. Thus, since we are interesting to transfer the state of system (4) at time $t=0^-$ from the initial point $\left[x_{ij}^o\right]_{\substack{i=1,2,\ldots,m\\ j=1,2,\ldots,k}}$ and at time $t=0^+$ to achieve $\left[x_{ij}\right]_{\substack{i=1,2,\ldots,m\\ j=1,2,\ldots,k}}$, eq. (5) is very properly.

5 Numerical Application

In this section, a numerical example based on [10] is presented. We consider the descriptor system of the form

$$F\underline{x}'(t)=G\underline{x}(t)+\underline{b}u_o(t) \tag{18}$$

where $\underline{x}(t)\in\mathbb{R}^n$, and $u_o(t)\in\mathbb{R}$ state vector, and input respectively. We assume that $E,\ A\in\mathbb{R}^{n\times n}$ no diagonal matrices, where $\det E=0$ and input vector $\underline{b}\in\mathbb{R}^n$ to be constants. This kind of systems can be appeared in engineering, in biology, in finance, in actuarial science etc. Now, we can have the following matrices

$$F = \begin{bmatrix} 0 & 1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -2 & 1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 1 & -1 & 0 & -1 & -1 \\ 0 & -1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; G = \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & -1 & -1 \\ -3 & -4 & -3 & 1 & 2 & 4 & 5 \\ -1 & 2 & 0 & 1 & -2 & -1 & 1 \\ -1 & -1 & -1 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ -1 & 1 & 1 & 1 & 0 & 1 & 1 \\ -3 & -2 & -3 & 1 & 2 & 3 & 4 \end{bmatrix}$$

and

$$\underline{b} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}^T .$$

Let the state of the system at time

0^- is $\underline{x}(0^-) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ and at time 0^+ achieves $\underline{x}(0^+) = \begin{bmatrix} -3 & -2 & 2 & 3 & 1 & 2 & 1 \end{bmatrix}^T$.

From the regularity of matrix pencil $sF - G$, there exist non-singular $\mathbb{R}^{n\times n}$ matrices

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

such that, see for further details [10], [3] and [6]

$$PFG = F_w = \begin{bmatrix} I_2 & \mathbb{O}_{5,2} \\ \mathbb{O}_{5,2} & H_5 \end{bmatrix} \text{ and } PGQ = G_w = \begin{bmatrix} J_2(2) & \mathbb{O}_{5,2} \\ \mathbb{O}_{5,2} & I_5 \end{bmatrix}$$

where

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 2}, \quad J_2(2) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \in \mathbb{R}^{2\times 2} \text{ and } H_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{5\times 5} .$$

The system (18) can be divided into the following subsystems

$$\underline{y}_2'(t) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \underline{y}_2(t) + \underline{\beta}_2 u_o(t), \tag{19}$$

$$\begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \\ & & & 0 & 1 \\ & & & 0 & 0 \end{bmatrix} \underline{y}_5'(t) = \underline{y}_5(t) + \underline{\beta}_5 u_o(t). \tag{20}$$

Now, we assume that

$$u_o(t) = a_o\delta(t) + a_1\delta^{(1)}(t) + \ldots + a_5\delta^{(5)}(t) + a_6\delta^{(6)}(t).$$

Our objective is to determine the unknown coefficients a_i, for $i = 0,1,\ldots,6$.

Before we go further with the analytic determination of the unknown coefficients above, we should firstly verify that the important condition (2.39), see for further details [10], is always satisfied. Thus, taking into consideration the inverse matrix of Q,

$$Q^{-1} = \left(\tilde{q}_{ij}\right)_{i,j=1,2,\ldots,n} = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \end{bmatrix}$$

and

$$\underline{y}\left(0^+\right) = Q^{-1}\underline{x}\left(0^+\right) = \begin{bmatrix} \underline{y}_2\left(0^+\right) \\ \underline{y}_5\left(0^+\right) \end{bmatrix} = \begin{bmatrix} -5 & -2 & -3 & 1 & 0 & 5 & -1 \end{bmatrix}^T.$$

Finally, it is not difficult to verify that $H_5^2 \underline{y}_5\left(0^+\right) = \underline{0}_5$ (i.e. condition (2.39)).

Furthermore, the expression $\underline{\beta} \triangleq P\underline{b} = \begin{bmatrix} 0 & 1 & 2 & 2 & 1 & 0 & 1 \end{bmatrix}^T$ are easily calculated. Thus

$$\underline{\beta}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \underline{\beta}_5 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Using the approximate expression (14) for the controller u_o,

$$u(t) = \frac{a_o}{\sigma}\Phi\left(\frac{t}{\sigma}\right) + \frac{a_1}{\sigma^2}\Phi^{(1)}\left(\frac{t}{\sigma}\right) + \ldots + \frac{a_5}{\sigma^5}\Phi^{(5)}\left(\frac{t}{\sigma}\right) + \frac{a_6}{\sigma^6}\Phi^{(6)}\left(\frac{t}{\sigma}\right), \tag{21}$$

and obtaining (15), the solution (19) is transposed into

$$\underline{y}_2(t) = \lim_{\sigma\to 0}\left\{ e^{J_2(2)t} \int_{-\infty}^{t} e^{-J_2(2)\tau} \underline{\beta}_2 u(\tau)\, d\tau \right\}$$

Analytically, we have

$$\int_{-\infty}^{t} e^{-J_2(2)\tau} \underline{\beta}_2 u(\tau)\, d\tau = \begin{bmatrix} -\int_{-\infty}^{t} \tau e^{-2\tau} u(\tau)\, d\tau \\ \int_{-\infty}^{t} e^{-2\tau} u(\tau)\, d\tau \end{bmatrix}.$$

Inserting the controller u into the above expression and making all the necessary assumptions (see. Remark 2.1, [10] and what it follows). Thus, in order to make our calculations affordable due to the long number of terms involved, we assume that $\Phi(t/\sigma)$ and its derivatives tend to zero very strongly when $t \to 0$ (note also that $\sigma \to 0$). Thus, letting $t = K\cdot\sigma$, where K is chosen large enough (i.e. $K \to \infty$) that the assumption as stated above is valid, i.e.

$$\Phi(t/\sigma) \overset{t=K\sigma}{=} \Phi(K) \to 0,$$

and its derivatives $\Phi^{(m)}(t/\sigma) \overset{t=K\sigma}{=} \Phi^{(m)}(K) \overset{K\to\infty}{\to} 0$, for $m = 0,1,2,\ldots,6$.

Moreover, it is known that

$$\int_{-\infty}^{t} e^{-\lambda\tau} u(\tau)\, d\tau \approx \left\{ \sum_{k=0}^{6} \lambda^k a_k \right\} e^{\frac{\lambda^2\sigma^2}{2}} \Phi^{-1}\left(\frac{t}{\sigma} + \lambda\sigma\right)$$

and

$$\int_{-\infty}^{t}\tau e^{-\lambda\tau}u(\tau)d\tau \approx \left\{\sum_{k=1}^{6}k\lambda^{k-1}a_k\right\}e^{\frac{\lambda^2\sigma^2}{2}}\Phi^{-1}\left(\frac{t}{\sigma}+\lambda\sigma\right).$$

And also

$$\left.\begin{aligned}\frac{y_1(0^+)}{\Phi^{-1}(K)} &= -(\alpha_1+4\alpha_2+12\alpha_3+32\alpha_4+80\alpha_5+150\alpha_6)\\ \frac{y_2(0^+)}{\Phi^{-1}(K)} &= \alpha_0+2\alpha_1+4\alpha_2+8\alpha_3+16\alpha_4+32\alpha_5+64\alpha_6\end{aligned}\right\}.$$

So, we conclude to the following system

$$\begin{bmatrix}0 & -1 & -4 & -12 & -32 & -80 & -150\\ 1 & 2 & 4 & 8 & 16 & 32 & 64\end{bmatrix}\underline{a}=\begin{bmatrix}-y_1^*\\ y_2^*\end{bmatrix} \tag{22}$$

where $\underline{y}^*(K\sigma)=\left(\Phi^{-1}(K)\right)^{-1}\underline{y}(K\sigma)$; $\underline{y}=\begin{bmatrix}y_1 & y_2\end{bmatrix}^T$ and

$$u=\begin{bmatrix}0 & -1 & -4 & -12 & -32 & -80 & -150\\ 1 & 2 & 4 & 8 & 16 & 32 & 64\end{bmatrix}.$$

Note that $\underline{x}_2(0^+)=\begin{bmatrix}-3 & -2\end{bmatrix}^T$ and consequently $\underline{y}_2(k\sigma)=\left[Q^{-1}\underline{x}(0^+)\right]_2=\begin{bmatrix}-5 & -2\end{bmatrix}^T$.

As we can easily see, according to corollary 2.2 [10](see also about the generalized algebra, [1]), the system (22) has solution which is given by expression.

$$\underline{a}=u^{(1)}\begin{bmatrix}-5\\ -2\end{bmatrix}\frac{1}{\Phi^{-1}(K)}+\left(I_7-u^{(1)}u\right)\underline{Y}$$

where, the $\{1\}$ -inverse of matrices u_j is $u^{(1)}=P_2\begin{bmatrix}I_2\\ O_{5,2}\end{bmatrix}Q_2$

and
$$P_2=\begin{bmatrix}0 & 0 & 4 & 16 & 48 & 128 & 256\\ 0 & 0 & -4 & -12 & -32 & -80 & -160\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}, \qquad Q_2=\begin{bmatrix}1 & -2\\ 0 & 1\end{bmatrix}$$

Finally, the general solution is given by the expression (23)

$$\underline{a}=\begin{bmatrix}-1\\ -2\\ 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix}\frac{1}{\Phi^{-1}(K)}+\begin{bmatrix}0 & 0 & 4 & 16 & 48 & 128 & 256\\ 0 & 0 & -4 & -12 & -32 & -80 & -160\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}\underline{Y} \tag{23}$$

for arbitrary $\underline{Y}\in\mathbb{R}^7$.

Thus, we conclude with this numerical application with the determination of the unknown coefficients a_i, for $i=0,1,\ldots,6$.

$$\underline{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} \dfrac{-1}{\Phi^{-1}(K)} + 4y_3 + 16y_4 + 48y_5 + 128y_6 + 256y_7 \\ \dfrac{-2}{\Phi^{-1}(K)} - 4y_3 - 12y_4 - 32y_5 - 80y_6 - 160y_7 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix}$$

for arbitrary $y_i \in \mathbb{R}$, for $i = 3,4,5,6,7$ and $\Phi^{-1}(\mathrm{K}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{K} e^{-\frac{x^2}{2}} dx$; K is large enough (i.e. $K \to \infty$) that the assumption $\lim_{x \to K} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = 0$ is valid.

6 Conclusion – Further Research

In this paper, we have investigated how important is the approximation of distributional behaviour for the transferring of the initial state of the system (4) in (almost) zero time. For this realistic physical problems (famous in many scientific fields, such as in engineering, in biology etc), an input containing Dirac delta function and its derivatives has been applied theoretically. In all the known approaches, see [7-8], [10-13] and [19], the normal probability distribution has been used.

Until now, no other distribution has been applied. It will be very interesting if some other approximations can be also considered and some comparing results can be derived and discussed. Moreover, only in [6-7] and [13], there are some hints about the minimum time. However, there is not any known formula which can calculate the minimum time considering some other significant to the problem parameters, such as the volatility σ, the K etc.

Moreover, since we have applied an approximation for the impulse input, it is wise to believe that we actually do not transfer the initial state to a desired, but ε-close to it, see figure 3.

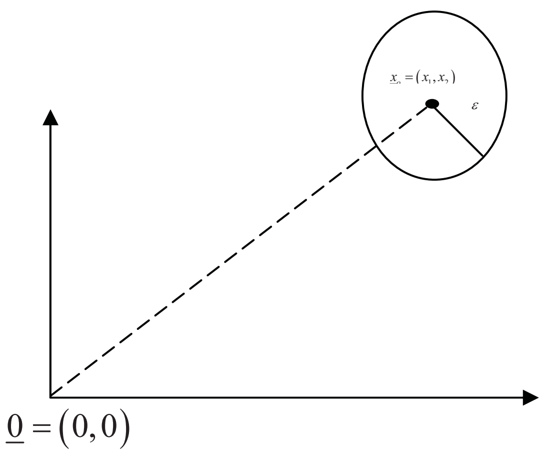

Figure 3: The desired ε-region of (x_1, x_2) for a two-dimensional system

Consequently, it is naturally derived a question about how close we can be, how we can calculate that distance, and if it is possible to minimize it.

Future plans has to do with the determination of the unknown coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ for higher order differential systems, see for instance

$$FX^{(r)}(t) = GX(t) + BU(t) \quad where \ \ r \in \mathbb{N}.$$

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