EXP-FUNCTION METHOD FOR KHOKHLOV-ZABOLOTSKAYA-KUZNETSOV (KZK) EQUATION

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Abstract. The Exp-function method is used to obtain exact solution for some nonlinear evolution equations arising in mathematical physics using symbolic computation. The KZK (Khokhlov-Zaholotskaya-Kuznetsov) equation is chosen to illustrate the effectiveness and convenience of the method.
Keywords: Exp-function; nonlinear acoustic; KZK equation

1 Introduction

Many new approaches have been proposed to introduce analytical solutions to nonlinear evolution equations. Many new approaches with advantages on the one hand and disadvantages on the other hand have been suggested to solve various nonlinear equations, such as the variational iteration method [1], the homotopy perturbation method [2], the tanh-method [3], the sinh-method [4], the homogeneous balance method [5], the F-expansion method [6], the extended Fan's sub-equation method [7]. A complete review is available in Ref. [8].

The Exp-function method introduced by [9] has been developed to find solitary solutions, compact-like solutions and periodic solutions for nonlinear evolution equations in mathematical physics. In this study, we focus on KZK equation. There are some numerical methods for solving KZK equation that nearly all of them are based on finite difference method. To date, no explicit analytical solutions exist for the KZK equation.

When an acoustic source of finite size radiates into free space, the effects of diffraction must be considered. The KZK nonlinear parabolic wave equation [10] is known to describe accurately the propagation of a finite amplitude sound beam by including, to the lowest order, the combined effects of diffraction, absorption, and nonlinearity; in addition, it models the propagation of plane wave finite amplitude sound beam [11].

In the derivation of the KZK equation, the sound waves are assumed to form a directive beam that permits a parabolic approximation to be made in the terms which account for diffraction. The parabolic approximation introduces errors at field points that are far away from the acoustical axis (e.g., more than 20O off axis), and at near field (e.g. locations close to the source). However, these restrictions are relatively weak in practice, and the main regions of interest in a directive beam are accurately modelled by the KZK equation [11].

The behaviour of Burgers equation solution represents shock waves in propagation of mechanical waves; the KZK equation is somehow more general than Burgers equation and in fact it is a perturbation of Burgers equation and hence the diffraction effect is modelled in it. In previous studies some solutions are prepared for Burgers equation, but no analytical solution is proposed for KZK. Therefore, we discuss solutions of KZK equation with def nite boundary conditions by Exp-function in this letter.

2 Summary of Exp-function Method

We consider a general nonlinear partial differential equation (PDE) in the form

$$F(u,u_x,u_y,u_z,u_t,u_{xx},u_{yy},u_{zz},u_{tt},u_{xy},u_{xt},u_{yt},...)=0, \tag{1}$$

we introduce complex variation ξ defined as

$$\xi = ax+by+cz+dt+\varphi, \tag{2}$$

where $a,b,$ and c are constants, using a transformation

$$u(x,y,z,t)=u(\xi),$$

We can rewrite Eq. (1) in the following nonlinear ordinary differential equation (ODE):

$$G(u,u',u'',u''',...)=0, \tag{3}$$

where the prime denotes the derivation with respect to ξ.

The basic idea of the exp-function method is very simple; it is based on the assumption that traveling wave solutions can be expressed in the following form [12]:

$$u(\xi)=\frac{\sum_{n=-f}^{g} an\ \exp(n\xi)}{\sum_{m=-p}^{q} bm\ \exp(m\xi)} \quad , \tag{4}$$

where f, g and q, p are positive integers which are unknown and will be further determined in our presention, and a_n and b_m are unknown constants to be determined later.

Eq.(4) can be re-written in an alternative form as follows:

$$u(\xi)=\frac{a_g\ \exp(g\xi)+...+a_{-f}\ \exp(-f\xi)}{b_q\ \exp(q\xi)+...+b_{-p}\ \exp(-p\xi)} \quad , \tag{5}$$

We present Exp-function method application in the nonlinear KZK equation.

3 KZK equation

Anna Rozanova-Pierrat considers the KZK equation $(u_t - uu_x - \beta u_{xx})_x - \gamma\Delta_y u=0,$ which describes for instance the propagation of sound beam in a nonlinear media. It is used in acoustical problems as a mathematical model which describes the pulse finite amplitude sound beam nonlinear propagation in the thermo-viscous medium [13].

The KZK equation for a non-axisymmetric sound beam that propagates in the positive z direction (Fig. 1) can be written in terms of the acoustic pressure p as follows [11]:

$$p_{zt}=\frac{c_0}{2}(p_{xx}+p_{yy})+\frac{D}{2c_0^3}p_{ttt}+\frac{\beta}{\rho_0 c_0^3}(p_{tt}+p_t^2)$$

With boundary condition:

$$\begin{aligned} &p(x,y,0,t)=p_0.g(x,y).f(t) \\ &f(t)=e^{-\left(\frac{t}{T}\right)^2}.\sin(t) \\ &g(x,y)=\begin{cases} 1 & -1\le x\le 1\ ,\ -1\le y\le 1 \\ 0 & \text{Otherwise.} \end{cases} \end{aligned} \tag{6}$$

We consider the KZK equation

$$p_{zt}-\gamma(p_{xx}+p_{yy})-\beta p_{ttt}-p_t^2-pp_{tt}=0. \tag{7}$$

Introducing a complex variation defined as

$$\xi=ax+by+cz+\omega t+\varphi, \tag{8}$$

making the transformation (8), Eq.(7) becomes

$$\omega cp''-\gamma(a^2+b^2)\ p''-\beta\omega^3\ p'''-\omega^2\ p-\omega^2\ pp''=0. \tag{9}$$

For Eq.(5) to determine the values g and q, following the balancing procedure [12],we balance the linear term of highest order in Eq.(9) with the highest order nonlinear term. Similarly, to determine the values of f and p, we balance the linear term of lowest order in Eq.(9) with the lowest order nonlinear term.

The highest linear term p''' is now given by

$$p'''=\frac{C_1\exp[(7q+g)\xi]+...}{C_2\exp[8q\xi]+...} \tag{10}$$

and

$$pp'' = \frac{c_3 \exp[(3q+2g)\xi] + \dots}{c_4 \exp[5q\xi] + \dots} = \frac{c_3 \exp[(6q+2g)\xi] + \dots}{c_4 \exp[8q\xi] + \dots}, \tag{11}$$

where c_i are coefficients for simplicity. By balancing highest order of Exp-function in Eqs. (10) and (11), we have $7q+g=6q+2g$,

which leads to the limit $q=g$.

Proceeding the same manner as illustrated above, we can determine values of d and q. Balancing the linear term of lowest order in Eq.(9)

$$p''' = \frac{\dots + d_1 \exp[-(7p+f)\xi]}{\dots + d_2 \exp[-8p\xi]}$$

and

$$p''' = \frac{\dots + d_3 \exp[-(3p+2f)\xi]}{\dots + d_4 \exp[-5p\xi]} + \frac{\dots + d_3 \exp[-(6p+2f)\xi]}{\dots + d_4 \exp[-8p\xi]}$$

Therefore, we can obtain the following relation $-(7p+f)=-(6p+2f)$, resulting in $p=f$.

4 Solution

We For simplicity, we set $q=g=1$ and $p=f=1$, thus Eq.(5) becomes:

$$p(\xi) = \frac{a_1 \exp(\xi) + a_0 \dots + a_{-1} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)} \tag{12}$$

Substituting Eq.(12) into Eq.(9), we have:

$$\frac{1}{A}[C_{-3}\exp(-3\xi) + C_{-2}\exp(-2\xi) + C_{-1}\exp(-\xi) + C_0 + C_1\exp(\xi) + C_2\exp(2\xi) + C_3\exp(3\xi)] = 0$$

Setting the coefficients of $\exp(n\xi)$ to zero and solving them, we obtain the following sets of results:

4.1 Case 1

$$\begin{cases} a_0 = \dfrac{(\gamma b^2 + \beta\omega^3 - \gamma a^2 + \omega c)b_0}{\omega^2} \\ a_{-1} = \dfrac{b_{-1}(\gamma a^2 - \omega c + \beta\omega^3 + \gamma b^2)}{\omega^2} \\ a_2 = 0 \quad b_2 = 0 \end{cases}$$

Where $b, \omega, a, b_{-1}, b_0, c$ and φ are free parameters

Substituting these results into (12), we obtain the following exact solution

$$u = \frac{-b_{-1}(\gamma a^2 - \omega c + \beta\omega^3 + \gamma b^2)e^{-\xi} + (-\gamma b^2 + \beta\omega^3 - \gamma a^2 + \omega c)b_0}{(b_{-1}e^{-\xi} + b_0)\omega^2} \tag{13}$$

4.2 Case 2

$$\begin{cases} a_{-1} = -\dfrac{(\gamma b^2 + 2\beta\omega^3 + \gamma a^2 - \omega c)b_{-1}}{\omega^2} \\ a_0 = \dfrac{b_1(-\gamma a^2 + \omega c + 2\beta\omega^3 - \gamma b^2)}{\omega^2} \\ a_0 = 0 \quad b_0 = 0 \end{cases}$$

Where $b, \omega, a, b_{-1}, b_1, c$ and φ are free parameters. Substituting these results into (12), we obtain the following exact solution:

$$u = \frac{-(\gamma b^2 + 2\beta\omega^3 + \gamma a^2 - \omega c) b_{-1} e^{-\xi} + b_2(-\gamma a^2 + \omega c + 2\beta\omega^3 - \gamma b^2) e^{\xi}}{(b_{-1} e^{-\xi} + b_1 e^{\xi})\omega^2} \tag{14}$$

4.3 Case 3

$$\begin{cases} a = ib \\ \omega = 0 \end{cases}$$

Where $b, b_{-1}, b_0, b_1, , a_{-1}, a_0, a_1, c$ and φ are free parameters.
Substituting these results into (12), we obtain the following exact solution:

$$u = \frac{a_{-1} e^{-\eta} + a_0 + a_1 e^{\eta}}{b_{-1} e^{-\eta} + b_0 + b_1 e^{\eta}} \qquad \eta = ibx + by + cz + \varphi \tag{15}$$

4.4 Case 4

$$\begin{cases} b_0 = -\dfrac{1}{4}\dfrac{(a_0{}^2 b_1{}^2 - 2\beta\omega a_1 b_0{}^2 b_1 - 2a_1 b_0 a_0 b_1 + a_1{}^2 b_0{}^2 + 2\beta\omega a_1 b_1{}^2 b_0)}{\beta^2\omega^2 b_1{}^3} \\ a_{-1} = \dfrac{1}{4}\dfrac{(-a_1 a_0{}^2 b_1{}^2 + 4\beta\omega a_1{}^2 b_0{}^2 b_1 + 2a_1{}^2 b_0 a_0 b_1 - a_1{}^3 b_0{}^2 - 6\beta\omega a_1 a_0 b_1{}^2 b_0)}{\beta^2\omega^2 b_1{}^4} \\ + \dfrac{1}{4}\dfrac{(2\beta\omega a_0{}^2 b_1{}^3 b_0 - 4\beta^2\omega^2 a_1 b_0{}^2 b_1{}^2 + 4\beta^2\omega^2 a_0 b_0 b_1{}^3)}{\beta^2\omega^2 b_1{}^4} \\ c = -\dfrac{(-\omega^2 a_1 + \beta\omega^3 b_1 - \gamma b_1 a^2 - \gamma b_1 b^2)}{\omega b_1} \end{cases}$$

Where $b, \omega, a, b_0, a_0, a_1, b_1$, and φ are free parameters.ubstituting these results into (12), we obtain the following exact solution:

$$\begin{aligned} u = {} & \frac{b_1(-a_1 a_0{}^2 b_1{}^2 + 4a_1{}^2 b_0{}^2 \beta\omega b_1 + 2a_1{}^2 b_0 a_0 b_1 - a_1{}^3 b_0 a_0 b_1 - a_1{}^3 b_0{}^2 - 6a_1 b_0 a_0 b_1{}^2 \beta\omega) e^{-\eta}}{-(a_0{}^2 b_1{}^2 - 2\beta\omega a_1 b_0{}^2 b_1 - 2a_1 b_0 a_0 b_1 + a_1{}^2 b_0{}^2 + 2\beta\omega a_0 b_1{}^2 b_0) e^{-\eta} + 4(b_0 + b_1 e^{\eta})\beta^2\omega^2 b_1{}^3} \\ & + \frac{b_1(2a_0{}^2 b_1{}^2 \beta\omega - 4\beta^2\omega^2 a_1 b_0{}^2 b_1{}^2 + 4\beta^2\omega^2 a_0 b_1{}^3 b_0) e^{-\eta} + 4(a_0 + a_1 e^{\eta})\beta^2\omega^2 b_1{}^4}{-(a_0{}^2 b_1{}^2 - 2\beta\omega a_1 b_0{}^2 b_1 - 2a_1 b_0 a_0 b_1 + a_1{}^2 b_0{}^2 + 2\beta\omega a_0 b_1{}^2 b_0) e^{-\eta} + 4(b_0 + b_1 e^{\eta})\beta^2\omega^2 b_1{}^3} \end{aligned} \tag{16}$$

$$\eta = ax + by - \frac{(-\omega^2 a_1 + \beta\omega^3 b_1 - \gamma b_1 a^2 - \gamma b_1 b^2)}{\omega b_1} z + \omega t + \varphi,$$

Here, we only discuss the more general solution (Eq.(14)). Consequently , the solution is given, at $b_{-1} = b_1$, by:

$$u = \frac{\gamma a^2 + 2\beta\omega^3 \tanh(\xi) - \gamma b^2 + \omega c}{\omega^2} \tag{17}$$

If q=g=n, p=f=m and m, n>1, we obtain new cases that some of them are like previous cases and others are new, providing more boundary conditions.

5 Conclusion

We were able to obtain an exact solution in KZK equation for a specific boundary condition; i.e. for conditions in which the Eq. (17) is satisfied. This can be used to represent the behavior of step shock, which is a known phenomenon in the propagation of nonlinear ultrasound wave.

We could not find exact solutions for Eq. (9) with boundary condition (6) using exp-function method. It seems that continuing our research to find these solutions will aid in different scientific fields such as ultrasound, fluid dynamics, etc.

The performance of the Exp-function method is reliable, effective, and gives more solutions compared to other analytical methods which were already used. An increase in the cases of results in future works can provide wider boundary conditions and more applications.

Our results showed that based on the Exp-function method, some nonlinear evolution equations can be solved exactly. The validity of this method has been tested by applying it successfully to the KZK equation. The main advantage of this method is that it can be applied to a wide variety of nonlinear evolution equations. It may be concluded that the Exp-function method can be easily extended to many kinds of nonlinear equations.

6 References

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