DIFFERENCE EQUATIONS FOR SIZING INTERMEDIATE STORAGES IN DISCRETE STOCHASTIC MODELS AND THEIR MATHEMATICAL GENERALIZATION

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Abstract. In this paper difference equations arising from process engineering and financial applications are presented and investigated. Starting out of a stochastic model we introduce the *z* transform of the sequence of the appropriate probabilities. We present the difference equation satisfying by the function, we prove the existence and uniqueness of the solution, and we show some properties of the solution. In special cases we give the explicit solution as well. Finally computational results will be presented.

1 Introduction

Intermediate storages play important role in process engineering systems. They connect process subsystems with different operational characteristics. It is an important question how large the intermediate storage should be in order to avoid overflow. In order to find the appropriate size of storage it is worth investigating the change of amount of material in the function of the time.
In most of practical problems the process is rather stochastic than deterministic. Hence the appropriate size can be determine to a given reliability level. The change of amount of material is stochastic process and we would like determine the distribution of its maximum value. In batch/continuous systems the distribution function of the maximum value of material in the storage satisfies such integral equations which can be transformed into integro-differential or differential equation with delay or advances in their arguments [3,4,5]. In this paper we investigate a problem in discrete stochastic model and we deal with the probabilities of that the maximum value of the change of amount of material exceed a level in the function of the level. As a generalization, we introduce the *z* transform of an appropriate sequence, we prove the difference equation for it, we analyze the equation, we investigate the limit properties of the solution, and in some cases we give the explicit solution. Substituting the value 1 into the solution and into its derivative we get the probabilities of overflowing and the expected time of overflow, respectively. This model can be applied as a discrete version of a risk process, as we will refer to it on the appropriate points of our paper.
The structure of the paper will be the following: first we present the problem and the model which will be investigated, we introduce notations. Then using the methods of probability theory we prove the difference equation satisfied by the z transform of the overflow probabilities. We prove the existence and the uniqueness of the solution, we prove that the solution tends to zero and we give some explicit solutions. Using explicit solutions we present an example in which we determine the size of storage for a given reliability level.

2 The model

Let consider the following processing system. Some of batch units produce material and some of other units use them at different time. The amount of material produced is filled into the intermediate storage which stores it and the material is withdrawn from it when it is needed for the output subsystems. The filling time points are supposed to be random.

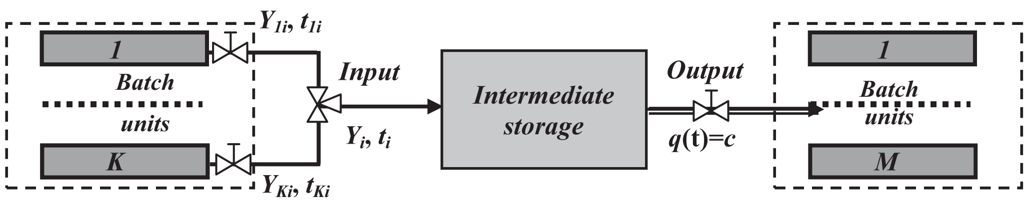

Figure 1. Intermediate storage connecting two batch subsystems of a processing system

Let $t_0 = 0$, and let us denote the times between the consecutive fillings by t_i (*i=1,2,3,…*), which are nonnegative, independent, identically distributed random variables. The counting process $\{N(t) : t \geq 0\}$ denotes the number of fillings up to time *t*, and is defined as

$$N(t) = \begin{cases} 0, & \text{if } t_1 > t \\ \max\left\{l : \sum_{i=1}^{l} t_i \leq t\right\}, & \text{if } t_1 \leq t \end{cases}.$$

The amount of the *k*-th filling is denoted by Y_k, $k = 1,2...$, and variables Y_k, $k = 1,2...$ are also nonnegative, independent and identically distributed random variables. Also, we assume that $N(t)$ and $\{Y_k\}_{k=1}^{\infty}$ are independent.

At this point we note that similar model is used in insurance mathematics to investigate the ruin probability. In insurance mathematics surplus corresponds to the material, the payment for the insurance company is described by a deterministic process and the claims arising from damages are random.

The amount of material being in the intermediate storage can be expressed as

$$V(t) = z_0 + \sum_{k=1}^{N(t)} Y_k - ct \text{ , } t \geq 0 \tag{2.1}$$

where $z_0 \geq 0$ is the initial amount of material.
If the size of the storage is z_S, then we avoid the overflow if $V(t) \leq z_S$ for any $t \geq 0$, which means that $0 \leq x - \sum_{k=1}^{N(t)} Y_k + ct$ for any $t \geq 0$ with $x = z_S - z_0$.
If we investigate probability of overflow, we investigate the probability

$$\psi_1(x) = P(x + ct - \sum_{k=1}^{N(t)} Y_k < 0 \quad \text{for} \quad \text{some} \quad t \geq 0)$$

This probability is often investigated in insurance mathematics as ruin probability. $N(t)$ can be interpreted as a claim process, Y_k are interpreted as claims, *c* is the rate, and *x* is the initial surplus.
If distributions of random variables t_k and Y_k are continuous, the integral equation for $\psi_1(x)$ in special cases is proved in [3,4] and concerning $R_1(x) = 1 - \psi_1(x)$ are presented and analyzed.
But often random variables Y_k, $k = 1,2...$ and t_k, $k = 1,2...$ have discrete distributions. In these cases the appropriate probabilities and they generalizations satisfy difference equations instead of integro-differential equations. These equations are quite complicated, they can have advances and also delays in the arguments. In this publication we set up the appropriate difference equations, we analyze them, we give analytical solutions for them.

In this paper we suppose both the time intervals between consecutive fillings times and the amount of material have discrete distributions with notation $P(t_k = j) = f(j)$ $j = 0,1,2...$ and $P(Y_k = i) = g(i)$, $i = 0,1,2,...$, furthermore $c = 1$. Now $f(j) \geq 0$, $g(i) \geq 0$, $\sum_{j=0}^{\infty} f(j) = 1$, $\sum_{i=1}^{\infty} g(i) = 1$. We assume that expectations of the random variables are finite, that is $\mu_f = \sum_{j=0}^{\infty} jf(j) < \infty$, $\mu_g = \sum_{i=0}^{\infty} ig(i) < \infty$. We draw the attention that the value of j can be zero too, which means that at the same time points more than one fillings can happen.

Let $u(n)$ be defined as the probability of overflow supposing the size for changing of amount of material to be n, namely

$$u(n)=P(m-\sum_{k=1}^{N(m)}Y_k+n<0 \quad \text{for} \quad \text{some} \quad m=0,1,...) \tag{2.2}$$

Furthermore let us introduce the notation for the probability of the time point of the first overflow

$$p_m^{(n)}=P(m-\sum_{k=1}^{N(m)}Y_k+n<0 \quad \text{and} \quad s-\sum_{k=1}^{N(s)}Y_k+n\geq 0 \quad 0\leq s<m)$$

The event $\left\{m-\sum_{k=1}^{N(m)}Y_k+n<0 \quad \text{and} \quad s-\sum_{k=1}^{N(s)}Y_k+n\geq 0 \quad 0\leq s<m\right\}$ expresses that the first overflow happens at time point *m*, if the size for change of material is *n*. We note that $p_0^{(0)}=0$.

The above defined probability $u(n)$ expresses the probability that having n initial surplus ruin will happen and $(p_m^{(n)})_{m\geq 0}$ is the probability that this ruin will happen at time point *m*. In the continuous case investigating ruin probability, the Gerber-Shiu discounted penalty function is usually applied and it is the Laplace-transform of the density function of the time of ruin [1,2]. Actually we introduce the *z* transform of the above defined sequence $p_m^{(n)}$ for all fixed values of *n*. The *z* transform of the sequence $(p_m^{(n)})_{m\geq 0}$ is a function

$$\varphi^{(n)}(\mathrm{z})=\sum_{m=0}^{\infty}p_m^{(n)}\,\mathrm{z}^{-m} \tag{2.3}$$

provided z is a complex number for which the series converges absolutely. We will restrict ourselves for real values of z and $z\geq 1$. For fixed values of $\mathrm{z}\geq 1$ $\left(\varphi^{(n)}(\mathrm{z})\right)_{n=0}^{\infty}$ is sequence, but for fixed value of $n\geq 0$ $\varphi^{(n)}(\mathrm{z})$ is a function mapping from $[1,\infty)$ to the set of real numbers. Moreover, it is clear, that $0\leq\varphi^{(n)}(\mathrm{z})\leq 1$ for any $\mathrm{z}\geq 1$ and $n\in N$, $\varphi^{(n)}(1)=\sum_{m=0}^{\infty}p_m^{(n)}=u(n)$ for any $n\in N$. Furthermore $-\frac{d\varphi^{(n)}(\mathrm{z})}{dz}|_{\mathrm{z}=1}$ equals to the expectation of the time of the first overflow.

3 Difference equations for sequence $u(n)$ and function $\varphi^{(n)}(\mathrm{z})$

Theorem 3.1

Sequence $u(n)=P(n-\sum_{k=1}^{N(m)}Y_k+m<0 \quad \textit{for} \quad \textit{some} \quad m\in N)$, $n\in N$ satisfies the following difference equation:

$$u(n)=\sum_{j=0}^{\infty}\sum_{i=1}^{n+j}u(n+j-i)f(j)g(i)+\sum_{j=0}^{\infty}\sum_{i=n+j+1}^{\infty}f(j)g(i). \tag{3.1}$$

Proof:
Let $Y_1=j$ and $t_1=i$. We apply the theorem of total probability with conditions $Y_1=j$ and $t_1=i$ $j=0,1,...$, $i=1,2,...$

$$u(n)=P(n-\sum_{k=1}^{N(m)}Y_k+m<0 \quad \text{for} \quad \text{any} \quad m\in N)=$$

$$=\sum_{j=0}^{\infty}\sum_{i=1}^{\infty}P(n-\sum_{k=1}^{N(m)}Y_k+m<0 \quad \text{for} \quad \text{any} \quad m\in N\,|\,Y_1=i \quad \text{and} \quad t_1=j)f(j)g(i). \tag{3.2}$$

If $i>n+j$, then overflow happens at time point *j*. If $i\leq n+j$, then overflow will not happen to the time point *j*. At this time point the size for change of amount of material is *n+j-i* as the filled material is *i* and the amount of withdrawn material is *j*. Hence using the renewal technique one can see that (3.2) equals

$$\sum_{j=0}^{\infty}\sum_{i=1}^{n+j}P(n-\sum_{k=1}^{N(m)}Y_k+m<0 \quad \text{for} \quad \text{any} \quad m\in N\,|\,Y_1=i \quad \text{and} \quad t_1=j)f(j)g(i)+\sum_{j=0}^{\infty}\sum_{i=n+j+1}^{\infty}f(j)g(i)=$$

$$= \sum_{j=0}^{\infty}\sum_{i=1}^{n+j} u(n+j-i) f(j) g(i) + \sum_{j=0}^{\infty}\sum_{i=n+j+1}^{\infty} f(j) g(i)$$

Now let us turn to the function $\varphi^{(n)}(z)$. We proved that the series of functions $\varphi^{(n)}(z)$ (n=0,1,2,…) satisfies the following difference equation for any fixed values of $z \geq 1$:

Theorem 3.2

For any $z \geq 1$ $\varphi^{(n)}(z) = \sum_{j=0}^{\infty}\sum_{i=1}^{n+j} \varphi^{(n+i-j)}(z) f(j) g(i) z^{-j} + \sum_{j=0}^{\infty}\sum_{i=n+j+1}^{\infty} f(j) g(i) z^{-j}$ (3.3)

Proof:

Let the symbol $t^{(n)}$ denote the time of the first overflow, that is

$$t^{(n)} = \begin{cases} \infty \text{ if } n+l-\sum_{k=1}^{N(l)} Y_k \geq 0 \quad \text{for all } l = 0,1,.. \\ \min\left\{ l : n+l-\sum_{k=1}^{N(l)} Y_k < 0 \right\} \text{ if } n+l-\sum_{k=1}^{N(l)} Y_k < 0 \text{ for some } l = 0,1,.. \end{cases}.$$

One can check that $\varphi^{(n)}(\mathrm{z}) = E(\mathrm{z}^{-t^{(n)}} 1_{t^{(n)}<\infty}) = E\left(E\left(\mathrm{z}^{-t^{(n)}} 1_{t^{(n)}<\infty} \mid t_1, Y_1\right)\right)$

Two cases can be distinguished. If $Y_1 \leq n + t_1$, then the first overflow can not happen to t_1, hence the process will be renewed but the expectation of overflow time has to be increased by the value of t_1 and the initial amount of material decreases by i as well. If $Y_1 > n + t_1$ then the process falls below zero at t_1.
Consequently

$$\varphi^{(n)}(z) = \sum_{j=0}^{\infty}\sum_{i=1}^{n+j} E\left(z^{-(t^{(n)}+j)} 1_{t^{(n)}<\infty} \mid t_1 = j, Y_1 = i\right) P(t_1 = j) P(Y_1 = i) + \sum_{i=n+j+1}^{\infty} E\left(z^{-t_1} \mid Y_1 = i\right)$$
$$= \sum_{j=0}^{\infty}\sum_{i=1}^{n+j} \varphi^{(n+j-i)}(z) f(j) g(i) z^{-j} + \sum_{j=0}^{\infty}\sum_{i=n+j+1}^{\infty} f(j) g(i) z^{-j}$$

which coincides with (3.3).

We note that *Theorem 3.2* is a generalization of *Theorem 3.1.* Although the technique of the proof is different, one can check that if we substitute $z = 1$ into *Theorem 3.3* we get *Theorem 3.1.*

4 Existence and uniqueness of the solution of the difference equation (3.3)

Now we turn our attention to the Eq.(3.3). We investigate what can be state about the existence of the solution, and in the case when it is not unique we answer the question that which of the solutions of the Eq.(3.3) is the solution of the original problem, namely which of the solutions can be the function defined by (2.3).

Theorem 4.1

The Eq.(3.3) has a unique solution for any fixed value of $z \geq 1$ in the set of bounded sequences assuming

$$c(z) = \sum_{j=0}^{\infty} f(j) z^{-j} < 1 .$$

Proof:
Let us introduce the operator K_z by the following definition

$$K_z(\varphi)(n) := \sum_{j=0}^{\infty}\sum_{i=1}^{n+j}\varphi^{(n+j-i)}f(j)g(i)z^{-j} + \sum_{j=0}^{\infty}\sum_{i=n+j+1}^{\infty}f(j)g(i)z^{-j} .$$

K_z is an operator both of the argument and image of which are bounded sequences. Furthermore

$$\left|K_z(\varphi_1)(n) - K_z(\varphi_2)(n)\right| \le \sum_{j=0}^{\infty}\sum_{i=1}^{n+j}\left|\varphi_1^{(n+j-i)} - \varphi_2^{(n+j-i)}\right| f(j)g(i)z^{-j} \text{, hence}$$

$$\left\|K_z(\varphi_1) - K_z(\varphi_2)\right\|_\infty \le \left\|\varphi_1 - \varphi_2\right\|_\infty \cdot \sum_{j=0}^{\infty}\sum_{i=1}^{n+j+1} f(j)g(i)z^{-j} \le c(z)\left\|\varphi_1 - \varphi_2\right\|_\infty$$

Hence K_z is contraction and as the set of bounded sequences is complete hence there is a unique bounded sequence for which $K_z(\varphi) = \varphi$.

Note that condition $c(z) < 1$ holds if $z > 1$ and $f(0) < 1$. Consequently in this case the solution of Eq. (3.3) is unique in the set of bounded sequences. It is clear, that $\varphi^{(n)}(z)$ defined in (2.3) is bounded as $0 \le \varphi^{(n)}(z) \le 1$ for any $z \ge 1$. Hence if $z > 1$ and $f(0) < 1$ finding a bounded solution of (3.3), we have determined the function defined by (2.3) and it can be used for solving the original engineering problem.

We have to draw the attention that if $z = 1$, the assumptions of Theorem 4.1 are not satisfied as $c(1) = \sum_{j=0}^{\infty} f(j) = 1$. Hence the uniqueness of the solution of (3.1) is not the consequence of Theorem 4.1. Moreover the uniqueness of the solution in the set of bounded sequences does not hold. Example will be presented in section dealing with special cases. In this case we have to choose the function defined by (2.3) from solutions of (3.1) by taking $\lim_{z\to 1+}\varphi^{(n)}(z)$, or using further properties of $u(n) = \varphi^{(n)}(1)$.

5 Qualitative property of the solution of Eq.(3.3)

Now we prove that Eq.(3.3) implies a special property of the solution in some cases. More precisely, we show that under condition $c(z) < 1$ the bounded solution of Equation (3.3) tends to zero if $n \to \infty$.

Theorem 5.1
For any fixed $z \ge 1$ for which $c(z) < 1$, the bounded solution of Eq. (3.3) $\varphi^{(n)}(z)$ holds $\lim_{n\to\infty}\varphi^{(n)}(z) = 0$.

Proof: First prove the inequality

$$\limsup_{n\to\infty}\left|\varphi^{(n)}(z)\right| \le \frac{\limsup_{n\to\infty}\left|\sum_{j=0}^{\infty}\sum_{i=n+j+1}^{\infty} f(j)g(i)z^{-j}\right|}{1 - c(z)} . \tag{5.1}$$

Let $z \ge 1$ for which $c(z) < 1$. Let $N \ge 1, m \ge 2, n \ge mN - 1$. Denote $\overline{\varphi}_K(z) = \sup_{n\ge K}\left|\varphi^{(n)}(z)\right|$ and

$$\overline{a}_K(z) = \sum_{j=0}^{\infty}\sum_{i=K+j+1}^{\infty} f(j)g(i)z^{-j} .$$

Now $\left|\varphi^{(n)}(z)\right| \le \overline{\varphi}_{(m-1)N}(z)\sum_{j=0}^{\infty}\sum_{i=1}^{N-1} f(j)g(i)z^{-j} + \overline{\varphi}_1(z)\sum_{j=0}^{\infty}\sum_{i=N}^{n+j} f(j)g(i)z^{-j} + \sum_{j=0}^{\infty}\sum_{i=n+j+1}^{\infty} f(j)g(i)z^{-j}$

If $n \ge mN - 1$ then $\left|\overline{\varphi}_{mN-1}(z)\right| \le \overline{\varphi}_{(m-1)N}(z)\sum_{j=0}^{\infty}\sum_{i=1}^{N-1} f(j)g(i)z^{-j} + \overline{\varphi}_1(z)\sum_{j=0}^{\infty}\sum_{i=N}^{n+j} f(j)g(i)z^{-j} + \overline{a}_{mN-1}$

If $N \to \infty$ we get $\limsup_{n\to\infty}\left|\varphi^{(n)}(z)\right| \le \limsup_{n\to\infty}\left|\varphi^{(n)}(z)\right| c(z) + \limsup_{n\to\infty}\left|\sum_{j=0}^{\infty}\sum_{i=n+j+1}^{\infty} f(j)g(i)z^{-j}\right|$.

Recall that $\varphi^{(n)}(z)$ is bounded and $c(z)<1$, we get (5.1). Taking into account that $c(z)=\sum_{j=0}^{\infty} f(j)z^{-j} = \sum_{j=0}^{\infty}\sum_{i=1}^{\infty} f(j)g(i)z^{-j} < \infty$, hence $\sum_{j=0}^{\infty}\sum_{i=n+j+1}^{\infty} f(j)g(i)z^{-j} \to 0$ if $n\to\infty$, and $f(j)g(i)z^{-j}$ are nonnegative, hence $\limsup_{n\to\infty}\left|\sum_{j=0}^{\infty}\sum_{i=n+j+1}^{\infty} f(j)g(i)z^{-j}\right| = 0$, the limit of $\lim_{n\to\infty}\varphi^{(n)}(z)=0$.

Corollary If $z>1$ and $f(0)<1$, then the (unique) bounded solution of Eq. (3.3) satisfies $\lim_{n\to\infty}\varphi^{(n)}(z)=0$, but we can not state this property for the solutions of Eq. (3.1).

6 Analytical solutions in special cases

In this section we give analytical solutions for Equation (3.3) in the special case when the time intervals between the consecutive fillings have geometric distribution.

In insurance mathematics in the continuous case, exponential distribution for the consecutive filling times is very important and often investigated, this is called compound Poisson risk process [1,6]. As the discrete analogue of the exponential distribution is the discrete geometrical distribution, we solve the difference equation in this special case.

Theorem 6.1

Let $f(j)=(1-\overline{f})(\overline{f})^{j}$ $j=0,1,2,...$ with $0<\overline{f}<1$. Now the Equation (3.3) can be transformed into the following equation:

$$\varphi^{(n)}(z)-\frac{z}{\overline{f}}\varphi^{(n-1)}(z) = -\frac{1-\overline{f}}{\overline{f}}z\left(\sum_{i=1}^{n-1}\varphi^{(n-i-1)}(z)g(i)+\sum_{i=n}^{\infty}g(i)\right) \tag{6.1}$$

The proof requires a lot of elementary transformations hence we do not detail it.

We note that this equation has only delays in the indices. This phenomenon is analogous to the Poisson risk process, in which the integral equation is a Volterra-type integral equation, which means that it has delay in the argument [1,2].

Now we simplify the Equation (6.1) in special case of $g(i)$ $i=1,2,...$.

Let $f(j)=(1-\overline{f})(\overline{f})^{j}$ $j=0,1,2,...$ with $0<\overline{f}<1$ and $g(i)=\begin{cases}1 & \text{if} \quad i=1\\ 0 & \text{otherwise}\end{cases}$.

Now equation (6.1) can be written in the following form:

$$\varphi^{(n)}(z)=\begin{cases}\frac{z}{\overline{f}}\varphi^{(0)}(z)-\frac{1-\overline{f}}{\overline{f}}z \quad \text{if} \quad n=1 & (6.2.a)\\ \frac{z}{\overline{f}}\varphi^{(n-1)}(z)-\frac{1-\overline{f}}{\overline{f}}z\varphi^{(n-2)}(z) \quad \text{if} \quad n=2,3,... & (6.2.b)\end{cases} \tag{6.2}$$

Try to find the solution in special form. As we know that the solution is unique assuming $z>1$ this is a possible way for solving the equation. As in some special case in the Poisson risk process the solution is exponential function, we use the form $\varphi^{(n)}(z)=\alpha(z)(\mu(z))^{n}$ $n=0,1,2,...$. Substituting this form into the formula concerning $n=2,3,...$ (6.2.b) and simplifying by $\alpha(z)(\mu(z))^{n-2}$ we get the following equation for $\mu(z)$:

$$(\mu(z))^2 - \frac{z}{\overline{f}}\mu(z) + \frac{1-\overline{f}}{\overline{f}}z = 0 \tag{6.3}$$

Solving the Equation (6.3) we get

$$\mu_{1,2}(z) = \frac{z \mp \sqrt{z^2 - 4z(1-\overline{f})\overline{f}}}{2\overline{f}} \tag{6.4}$$

Substituting this into (6.2.a) we can see that $\alpha_i(z) = \mu_i(z)$. If we use notation $\mu_1(z) \le \mu_2(z)$, one can easily check that $0 < \mu_1(z) \le 1$, and $1 \le \mu_2(z)$, and in case of $z > 1$ equality can not hold. As $(\mu_2(z))^n \to \infty$, if $z > 1$ hence the bounded solution of Equation (3.3) is

$$\varphi^{(n)}(z) = (\mu_1(z))^{n+1} = \left(\frac{z - \sqrt{z^2 - 4z(1-\overline{f})\overline{f}}}{2\overline{f}}\right)^{n+1}.$$

As we know that the solution of Equation (3.3) is unique in the set of bounded sequences we can state that we have found the solution.

Let us turn to the case $z = 1$. Now $\mu_1(1) = \frac{1 - |2\overline{f} - 1|}{2\overline{f}}$, $\mu_2(1) = \frac{1 + |2\overline{f} - 1|}{2\overline{f}}$.

If $\frac{1}{2} < \overline{f} < 1$, then $\mu_1(1) < 1$ and $\mu_2(1) = 1$. In this case two bounded fundamental solutions of (6.2.b) exist, one of them is $\left(\frac{1 - |2\overline{f} - 1|}{2\overline{f}}\right)^n$, and the other one is constantly 1. It is well known from the theory of difference equation that all of solutions of (6.2.b) can be expressed by the linear combination of the fundamental solutions, therefore

$\varphi^{(n)}(1) = c_1\left(\frac{1 - |2\overline{f} - 1|}{2\overline{f}}\right)^n + c_2 \cdot 1$, c_1, c_2 are appropriate real numbers. As (6.2.a) has to be satisfied as well, the relation between c_1 and c_2 looks like $c_1\mu_1(1) + c_2\mu_2(1) = \frac{1}{\overline{f}}(c_1 + c_2) - \frac{1-\overline{f}}{\overline{f}}$ hence the uniqueness of the solution of (3.3) (that is the uniqueness of the solution of (3.1)) in the set of bounded sequences does not hold. Therefore we have to choose that solution of solutions of (3.1) which is the solution of the original physical problem as well.

Using the technique used in [4] by the help of probability theory one can prove that if $\frac{\mu_g}{\mu_f} < c$, then $u(n) \to 0$, when $n \to \infty$, which implies $c_2 = 0$ and $c_1 = \mu_1(1)$,

$$u(n) = \varphi^{(n)}(1) = \left(\frac{1 - |2\overline{f} - 1|}{2\overline{f}}\right)^{n+1}. \tag{6.5}$$

We note that as $\mu_f = \frac{1-\overline{f}}{\overline{f}}$ and $\mu_g = 1$, $c = 1$, condition $\frac{\mu_g}{\mu_f} < c$ holds if and only if $\frac{1}{2} < \overline{f}$. Hence in the case of $z = 1$, Eq.(3.1) does not imply that the solution tends to zero but if we know the limit of the solution from the physical process, the solution of (3.1) will be unique.

If $\overline{f} < \frac{1}{2}$, then $\mu_1(1) = 1$ and $\mu_2(1) > 1$ $\mu_2(z) > 1$, hence $(\mu_2(1))^n$ is not bounded, hence $u(n) = \varphi^{(n)}(1) \equiv 1$.

Finally if $\overline{f} = \frac{1}{2}$, then $\mu_1(1) = \mu_2(1) = 1$ and $u(n) = \varphi^{(n)}(1) \equiv 1, \quad n = 0,1,2,...$

Now we can summarize our result in Theorem 6.2 as follows:

Theorem 6.2

Let $f(j)=(1-\overline{f})(\overline{f})^{j}\quad j=0,1,2,...$ with $0<\overline{f}<1$ and $g(i)=\begin{cases}1 & if \quad i=1\\ 0 & otherwise\end{cases}$.

Assume $z>1$ and $f(0)<1$ holds. Now the unique bounded solution of Eq. (3.3) is

$$\varphi^{(n)}(z)=\left(\frac{z-\sqrt{z^{2}-4z(1-\overline{f})\overline{f}}}{2\overline{f}}\right)^{n+1}. \tag{6.6}$$

If $\frac{1}{2}<\overline{f}<1$, then the function defined by (2.3) is $u(n)=\varphi^{(n)}(1)=\left(\frac{1-\left|2\overline{f}-1\right|}{2\overline{f}}\right)^{n+1}$. If $0<\overline{f}\le\frac{1}{2}$, then the function at $z=1$ defined in (2.3) $u(n)=\varphi^{(n)}(1)\equiv 1$.

Let us turn to another special case when $g(i)=(1-\overline{g})(\overline{g})^{i-1}$, $i=1,2,3,...$

Let $f(j)=(1-\overline{f})(\overline{f})^{j}\quad j=0,1,2,...$ with $0<\overline{f}<1$ and $g(i)=(1-\overline{g})(\overline{g})^{i-1}$, $i=1,2,3,...$ with $0<\overline{g}<1$.

Now equation (6.1) can be written in the following form:

$$\varphi^{(n)}(z)=\frac{z}{\overline{f}}\varphi^{(n-1)}(z)-\frac{1-\overline{f}}{\overline{f}}z\left(\sum_{i=1}^{n-1}\varphi^{(n-i-1)}(z)(1-\overline{g})(\overline{g})^{i-1}+\sum_{i=n}^{\infty}(1-\overline{g})(\overline{g})^{i-1}\right). \tag{6.7}$$

If we try to find the solution in form $\varphi^{(n)}(z)=\alpha(z)(\mu(z))^{n}$, then we get

$$\mu_1(z)=\frac{z+\overline{g}\,\overline{f}-\sqrt{\left(z+\overline{g}\,\overline{f}\right)^{2}-4z(1-\overline{f})\overline{f}(1-\overline{g})-4\overline{g}\,\overline{f}z}}{2\overline{f}} \tag{6.8}$$

$$\mu_2(z)=\frac{z+\overline{g}\,\overline{f}+\sqrt{\left(z+\overline{g}\,\overline{f}\right)^{2}-4z(1-\overline{f})\overline{f}(1-\overline{g})-4\overline{g}\,\overline{f}z}}{2\overline{f}}$$

and $$\alpha_i(z)=\frac{\mu_i(z)-\overline{g}}{1-\overline{g}},\quad i=1,2 \tag{6.9}$$

Again, one can check that if $z>1$, then $\mu_1(z)<1<\mu_2(z)$, and the unique solution of (3.3) is $\varphi^{(n)}(z)=\alpha_1(z)(\mu_1(z))^{n}$.

If we turn to the case $z=1$, we get

$$\mu_1(1)=\frac{1+\overline{g}\,\overline{f}-\left|1-2\overline{f}+\overline{g}\,\overline{f}\right|}{2\overline{f}} \tag{6.10}$$

$$\mu_2(1)=\frac{1+\overline{g}\,\overline{f}+\left|1-2\overline{f}+\overline{g}\,\overline{f}\right|}{2\overline{f}}$$

In this case if $\frac{\mu_g}{\mu_f}=\frac{1}{1-\overline{g}}\cdot\frac{1-\overline{f}}{\overline{f}}<1$, then $\mu_1(1)<1$, and $\mu_2(1)=1$, which means that the bounded solution of (3.1) is not unique but the solution which tends to zero is $u(n)=\varphi^{(n)}(1)=\alpha_1(1)(\mu_1(1))^{n}$.

If $\frac{\mu_g}{\mu_f} = \frac{1}{1-\overline{g}} \cdot \frac{1-\overline{f}}{\overline{f}} \geq 1$, then $u(n) = \varphi^{(n)}(1) \equiv 1$ for any value of $n = 0,1,2,...$ **.**

Our results can be summarize in

Theorem 6.3

Let $f(j) = (1-\overline{f})(\overline{f})^j \quad j = 0,1,2,...$ with $0 < \overline{f} < 1$ and $g(i) = (1-\overline{g})(\overline{g})^{i-1}$, $i = 1,2,3,...$ with $0 < \overline{g} < 1$.

If $z > 1$, then the unique bounded solution of Eq. (3.3) is

$$\varphi^{(n)}(z) = \frac{\dfrac{z+\overline{g}\,\overline{f} - \sqrt{\left(z+\overline{g}\,\overline{f}\right)^2 - 4z(1-\overline{f})\overline{f}(1-\overline{g}) - 4\overline{g}\,\overline{f}z}}{2\overline{f}} - \overline{g}}{1-\overline{g}} \cdot \left(\frac{z+\overline{g}\,\overline{f} - \sqrt{\left(z+\overline{g}\,\overline{f}\right)^2 - 4z(1-\overline{f})\overline{f}(1-\overline{g}) - 4\overline{g}\,\overline{f}z}}{2\overline{f}} \right)^n$$

If $z = 1$, and $\frac{1}{1-\overline{g}} \cdot \frac{1-\overline{f}}{\overline{f}} < 1$, then the function at $z = 1$ defined in (2.3) is

$$\varphi^{(n)}(1) = \frac{\dfrac{1+\overline{g}\,\overline{f} - \left|1-2\overline{f}+\overline{g}\,\overline{f}\right|}{2\overline{f}} - \overline{g}}{1-\overline{g}} \cdot \left(\frac{1+\overline{g}\,\overline{f} - \left|1-2\overline{f}+\overline{g}\,\overline{f}\right|}{2\overline{f}} \right)^n . \qquad (6.11)$$

If $z = 1$ and $\frac{1}{1-\overline{g}} \cdot \frac{1-\overline{f}}{\overline{f}} \geq 1$, then the function at $z = 1$ defined in (2.3) is $\varphi^{(n)}(1) \equiv 1$.

7 Numerical examples

Finally we present figures which illustrate the results in previous section and we turn back the original physical problem.

First we have chosen $f(j) = (1-\overline{f})(\overline{f})^j \quad j = 0,1,2,...$ with $\overline{f} = 10/19$ and $g(i) = \begin{cases} 1 & if \quad i = 1 \\ 0 & otherwise \end{cases}$.

In this case the overflow probabilities $u(n)$ getting from (6.5) can be seen on Figure 6.1.

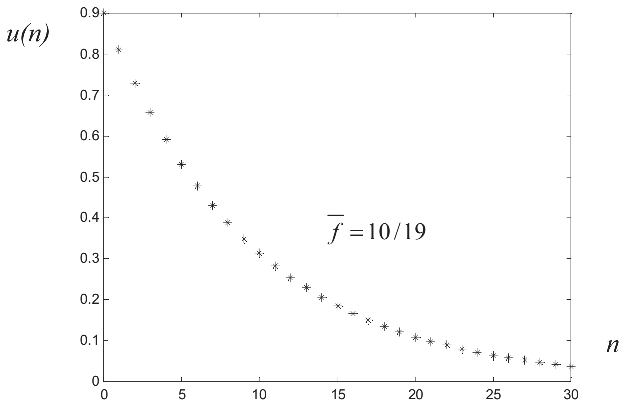

Figure 6.1. Overflow probabilities $u(n)$ in the function of the volume (n) in the intermediate storage for the change of a amount of material supposing discrete geometrical distribution for the distribution of time intervals

If we would like to use this analytical results to determine the appropriate size of the storage to the reliability level 0.95, we have to find the smallest n for which $1-u(n)\geq 0.95$. As in the presented case $\mu_1(1)=0.9$, applying (6.5) we get $n=28$. If we would like to know the expectation of the overflow if the volume for change of material is $n=28$, we have to derivate the function $\varphi_1^{(28)}(z)=(\mu_1(z))^{29}$ and substitute $z=1$ to the derivative and the expectation is 12.29.

On Figure 6.2. one can see the dependence of the roots of Eq.(6.3) of z. The parameter is the same as it was in the previous example. One can see that if $z>1$, then one of the roots is smaller than 1 and decreasing in z , and the second one is greater than one. If $z=1$, the larger root equals 1.

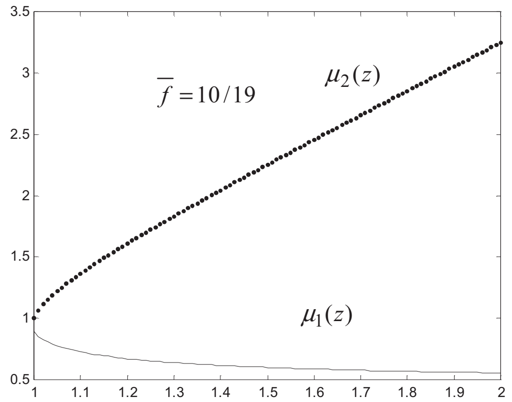

Figure 6.2. The roots of the Eq.(6.3) in the function of z
assuming discrete geometric distribution for the distribution of time intervals

Have a glance at the function $\varphi^{(n)}(z)$ with two variables z and n as well. Parameter is not changed.

On Figure 6.3. we can see that the function $\varphi^{(n)}(z)$ is close to zero even for small values of n, and as $\mu_1(z)$ is decreasing, hence the larger z the faster convergence.

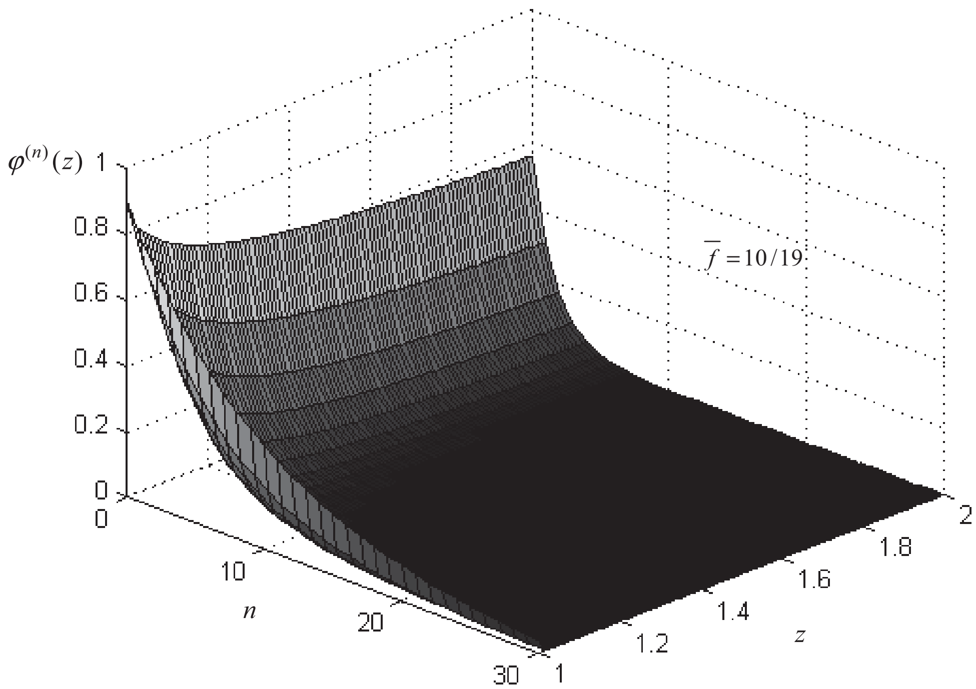

Figure 6.3. The function $\varphi^{(n)}(z)$ with two variables z and n
assuming discrete geometric distribution for the distribution of time intervals

In order to compare the results with the results we got from continuous case we recall the example given in [4] in case when t_k and Y_k have exponential distribution. We choose $\overline{f} = 20/23$, $\overline{g} = 0.8$ in order to have appropriate expectations with $c = 1$ to the parameters given in [4] on Figure 2. The analytical result for the reliability $R_1(x) = 1 - \psi_1(x) = 1 - \frac{\mu_g}{\mu_f c} e^{-(\frac{1}{\mu_f c} - \frac{1}{\mu_g})x}$ given by (14) in [4] and $1 - u(n)$ given by (6.11) can be seen together on Figure 6.4. We can see that the numerical results are quite close to each other which is useful if we would like to use discrete model instead of a continuous model.

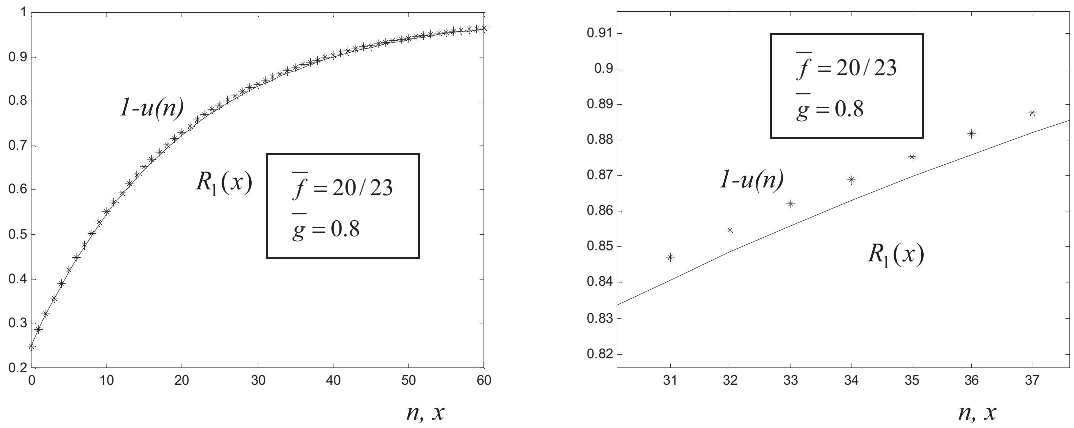

Figure 6.4. Analytical result for the reliability $R_1(x)$ (__) in the continuous model with exponential distributions and analytical results for $1 - u(n)$ (*) in the discrete model with discrete geometrical distributions

8 Summary

A discrete mathematical model is presented and analyzed for sizing intermediate storages to a given reliability level. First we introduced the probabilities of overflow in the function of volume as a sequence then we defined its z transformed as a generalization. We set up and proved the difference equations satisfying by the overflow probabilities and the z transform as well. We proved the existence and the uniqueness of the solution for $z > 1$ and presented example when the solution is not unique if $z = 1$. We analysed the limit of the solution. We solved the equation analytically in special cases and used the analytical solution for solving the original sizing problem.

9 References

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