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AN EFFECTIVE NUMERICAL ALGORITHM FOR SOLVING THREE-DIMENSIONAL REACHABILITY PROBLEMS[1]

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Abstract. A simple numerical algorithm for solving the reachability problem is discussed that is aimed at a particular class of nonlinear control systems rather than those of general kind. The algorithm is based on using the *Cauchy characteristics* to construct the boundary of reachable set numerically.

Numerical examples are also given, which apply the algorithm to several *three-dimensional control systems* of a selected class that are connected to robotics. These examples illustrate some possible situations that may occur when the algorithm works successfully. Here all the considered control systems are bilinear with singularities caused by nonsmoothness of the boundary of reachable set. These consist either of several distinct points, or of several cuts diffeomorphic to closed segments which do not divide the set of smooth points of the mentioned boundary into separate components, or of closed curves dividing the set of smooth points into separate disconnected components.

1 Introduction

The construction of *reachable sets* which consist of all states reachable through some of the available controls is a key problem in the mathematical theory of controlled processes ([1, 2, 3, 4, 5, 6]). They allow to solve the *problem of control synthesis* (see, for instance, [2])and are well studied for linear control systems. Many numerical methods were developed for their approximation through unions or intersections of simpler standard-shape domains, such for example, as ellipsoids ([2, 3, 7]) or parallelotopes (e.g. [8]). For *nonlinear* control systems the reachability problem is much more complicated. It is known however to be reducable to appropriate problems of dynamic optimization ([9]). Here we introduce the value function $V(t,x)$ as being a generalized solution to a corresponding Hamilton-Jacobi equation, such that the reachable set $X[t] = X(t,t_0,X^0)$ from initial states $x(t_0) = x^0 \in X^0$ is a level set of $V(t,x)$ ([9]). Hence the question under consideration is how to find the value function. This may be done by some numerical methods for the Hamilton-Jacobi equation, for example, the *level set methods* ([10, 11]). But these methods may firstly be very sensitive to the discretization rate in time and state space and even to the choice of the corresponding grids (especially those around the nonsmooth points of $V(t,x)$). Secondly, we may have too many mesh points in these grids in large spatial dimensions.

It is thus reasonable to concentrate on the reachability problem for special classes of nonlinear control systems when it may be possible to construct more regular and simple numerical methods or even to construct some estimates of the exact solution in analytical form. As for the second route, it may be possible to apply a *comparison principle* for the Hamilton-Jacobi equation ([12]) that leads to internal and external estimates for the reachable sets. But it may be difficult to decide what form of these estimates to seek.

In this paper we discuss a simple numerical algorithm for solving the exact reachability problem that is aimed at a particular class of nonlinear control systems rather than those of general kind. Besides that, we present numerical examples, applying the algorithm to several 3-dimensional systems of a selected class.

2 General assumptions

Consider the next system *linear* in the control parameters:

$$\dot{x}(t) = G(x(t))u(t), \qquad x(t) \in \mathbb{R}^n,\ u(t) \in \mathscr{P} \subset \mathbb{R}^m,\ t \in [t_0,T]. \tag{1}$$

Here we assume that the matrix-valued function $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is smooth enough and that $\mathscr{P}$ is a strongly convex compact with smooth boundary and zero in its interior, for instance, an ellipsoid. We assume also that $m > 1$, hence,that the boundary of $\mathscr{P}$ is connected. We then consider the Hamiltonian $H(x,s) = \max\{(G^T(x)s,u) \mid u \in \mathscr{P}\}$, so that the value function $V(t,x)$ solves the Cauchy problem for Hamilton-Jacobi-Bellman equation

$$\partial V(t,x)/\partial t + H(x, \partial V(t,x)/\partial x) = 0, \quad V(t_0,x) = \|x-x^0\|^2 \tag{2}$$

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determining further the reachable set $X[t] = X(t,t_0,x^0)$ as $X[t] = \{x : V(t,x) \leq 0\}$ ([9]). We assume also that $X[t]$ is of full dimension, i.e. its interior points are dense in $X[t]$. This property may be established through methods of differential geometry, see [13]. However, if for the given control system this property is not fulfilled, then the methods of [13] allow in some cases a transformation of the phase coordinates that decomposes the state into "controllable" and "non-controllable" parts. The reachability problem may thus be reduced to the treatment of the "controllable" part, so that the reduced problem would meet the assumption of the above.

Further on, write down formally the Hamiltonian system of ODEs determining the *Cauchy characteristics* for (2):

$$\dot{x}(t) = \partial H(x(t),s(t))/\partial s, \quad \dot{s}(t) = -\partial H(x(t),s(t))/\partial x, \qquad x(t_0) = x^0, \quad s(t_0) = s^0 \neq 0. \tag{3}$$

If $G^T(x^0)s^0 \neq 0$, hence, $H(x^0,s^0) > 0$, then due to the above assumptions $H(x,s)$ is differentiable along the unique trajectory $x[t] = x(t,t_0,x^0,s^0)$, $s[t] = s(t,t_0,x^0,s^0)$ of (3), determining, moreover, the corresponding open-loop control $u[t] = \arg\max\{(G^T(x(t))s(t),u) \mid u \in \mathscr{P}\}$. In this case the equations (3) coincide with those of the *L.S.Pontryagin's maximum principle* under condition $x[t] \in \partial X[t]$ ([14]) when $x[t]$ is attained within the *non-singular* mode. On the other hand, if $G^T(x^0)s^0 = 0$, hence, $H(x^0,s^0) = 0$, then for any pair $\{x(\cdot),u(\cdot)\}$ that satisfies Pontryagin's maximum principle with some $s(\cdot)$, $s \not\equiv 0$, we have $H(x(t),s(t)) \equiv 0$ for all $t \in [t_0,T]$, moreover, $H(x,s)$ is non-differentiable at $(x(t),s(t))$ and the equations (3) are invalid. This case corresponds to the *singular* mode of the maximum principle, when characteristics are not applicable. We further denote the set of states $x(t)$ attained by all singular modes as $Z[t] = Z(t,t_0,x^0)$.

Our additional assumption on the control system (1) under consideration is that $Z[t]$ lies in some manifold of dimension less than $n-1$. Then, since $X[t]$ is assumed to be of full dimension, the dimension of $\partial X[t]$ will be $n-1$. Hence, from the numerical point of view, we may ignore set $Z[t]$ when calculating $\partial X[t]$. The fulfillment of this assumption may be checked directly by analogy with [15]. But even if the accepted assumption is not satisfied, it is may be possible to regularize the control system (for instance, by introducing additional control parameters with small absolute values), so that the regularized system would meet the assumption.

Further on we consider only non-singular modes of the maximum principle and the corresponding trajectories $x[t] = x(t,t_0,x^0,s^0)$ and $s[t] = s(t,t_0,x^0,s^0)$. Due to the positive homogeneity of $H(x,s)$ with respect to s it is convenient to take $s^0 \in S^0 = \{s : H(x^0,s) = 1\}$. It is known however, that for nonlinear systems the inclusion $x[t] \in \partial X[t]$ is true not for all $s^0 \in S^0$. This is so since here the maximum principle of L.S. Pontryagin is just a *necessary*, but not a sufficient condition of optimality. Thus, the problem of finding $\partial X[t]$ may be reduced to the selection of set $\bar{S}^0[t] \subseteq S^0$ such that $\cup\{x(t,t_0,x^0,s^0) \mid s^0 \in \bar{S}^0[t]\} = \partial X[t]$ for given $t \in [t_0,T]$. The algorithm discussed in this paper produces set $\bar{S}^0[t]$ numerically.

Describe the additional properties necessary for this algorithm. For all the given examples these properties are checked directly. First, it is necessary that $\partial X[t]$ would be connected. This means there are no cavities inside the reachable set. Second, $\bar{S}^0[t]$ should contain a finite number of connected components, moreover, we have a finite set of "start points" $\hat{S}^0[t]$ containing at least one point from each of the components. All these "start points" are figured out for each of the control systems under consideration through some specific properties of these.

3 Description of the algorithm

Now let us describe the idea of the algorithm. From the smoothness of $x(t,t_0,x^0,s^0)$ in s^0 it is reasonable to expect that $\partial X[t]$ is a locally smooth manifold of the dimension $n-1$. But the issue is that $\partial X[t]$ may contain *singularities* (see, for instance, [16]). These singularities may appear in two ways. The first is that the rank of $\partial x(t,t_0,x^0,s^0)/\partial s^0$ (which in in regular points is equal to $n-1$) may become less than $n-1$. This is an analogy of the notion of *conjugate locus* known in variational calculus. Secondly, there may be two points $s^{0,1} \neq s^{0,2}$, $s^{0,1}, s^{0,2} \in S^0$, such that $x(t,t_0,x^0,s^{0,1}) = x(t,t_0,x^0,s^{0,2})$. This is analogous to the *cut locus* from variational calculus. Both conditions are *stopping conditions* for the numerical algorithm. For simplicity we further consider only the case $n = 3$, though the algorithm may be extended to other cases. Suppose we have the set $A \subseteq \mathbb{R}^2$ and the smooth mapping $\sigma : A \to \mathbb{R}^3$ such that $\mathrm{rank}(\partial\sigma(\alpha)/\partial\alpha) = 2$ and $\sigma(A) = S^0$. Suppose also that the finite set $\hat{A} = \{\hat{\alpha}_1,\ldots,\hat{\alpha}_K\}$ is given such that $\sigma(\hat{A}) = \hat{S}^0[t]$. Then the algorithm needs to find the set $\bar{A} \subseteq A$, $\hat{A} \subseteq \bar{A}$, such that $\sigma(\bar{A}) = \bar{S}^0[t]$. Consider the functions $\bar{\alpha}_k(\xi,\eta) : [0,2\pi] \times [0,\eta_{\max}] \to A$, $k = 1,\ldots,K$, continuous in (ξ,η) such that

- $\bar{\alpha}_k(\xi,0) = \hat{\alpha}_k$;
- $\bar{A}_k(\eta) = \cup_{\xi\in[0,2\pi]}\{\bar{\alpha}_k(\xi,\eta)\}$ is a closed Jordan curve, in particular, $\bar{\alpha}_k(0,\eta) = \bar{\alpha}_k(2\pi,\eta)$;
- $\mathrm{cl}(\mathrm{int}\bar{A}_k(\eta_1)) \subseteq \mathrm{cl}(\mathrm{int}\bar{A}_k(\eta_2))$ for each $0 \leq \eta_1 \leq \eta_2 \leq \eta_{\max}$;
- if one of stopping conditions is fulfilled at $\bar{\alpha}_k(\bar{\xi},\bar{\eta})$ for given $\bar{\xi} \in [0,2\pi]$ and $\bar{\eta} \in [0,\eta_{\max}]$, then $\bar{\alpha}_k(\bar{\xi},\eta) = \bar{\alpha}_k(\bar{\xi},\bar{\eta})$ for all $\eta \in [\bar{\eta},\eta_{\max}]$.

Besides, $\mathrm{int}\bar{A}_k(\eta_1) \cap \mathrm{int}\bar{A}_l(\eta_2) = \emptyset$ for any $k,l = 1,\ldots,K$, $k \neq l$, and for any $\eta_1,\eta_2 \in [0,\eta_{\max}]$. We name the union $\bar{A}(\eta)$ of closed Jordan curves $\bar{A}_k(\eta)$ as *iteration in A* and η as *iteration parameter*. Thus, the desired set $\bar{A}$ would be represented as $\bar{A} = \cup\{\bar{A}(\eta) \mid \eta \in [0,\eta_{\max}]\}$ while the boundary of the reachable set $\partial X[t] = \cup\{\Xi(\eta) \mid \eta \in$

$[0,\eta_{\max}]\}$ where $\Xi(\eta) = x(t,t_0,x^0,\sigma(\overline{A}(\eta)))$ is an *iteration in the state space*. Stopping conditions are fulfilled at $\alpha = \overline{\alpha}_k(\overline{\xi},\overline{\eta})$ if and only if either $\operatorname{rank}(\partial x(t,t_0,x^0,\sigma(\alpha))/\partial\alpha) < 2$ or there exists a point $(l,\xi,\eta) \neq (k,\overline{\xi},\overline{\eta})$, $l = 1,\dots,K$, $0 < \eta \leq \overline{\eta}$, such that $x(t,t_0,x^0,\sigma(\overline{\alpha}_k(\overline{\xi},\overline{\eta}))) = x(t,t_0,x^0,\sigma(\overline{\alpha}_l(\xi,\eta)))$.

In reality, the algorithm approximates the iterations mentioned above by unions of polygonal curves depending on parameter η with values from some discrete grid, such that both nodes within iterations in the state space and nodes in consecutive iterations in the state space are not closer than some fixed *nominal step* $\Delta x > 0$ trying to make the grid on $\partial X[t]$ as *uniform* as possible. This is important not only to make the grid have less number of points, but also for the stability of the algorithm.

4 Numerical examples

In this section we consider examples for several *three-dimensional control systems* to which the algorithm was applied. The equations of these control systems are connected to robotics and were taken from [17].

4.1 Example 1

This control system may be treated as projection of three-input second-order controllable system from [17, p. 362]:

$$\dot{x}_1(t) = u_1(t), \quad \dot{x}_2(t) = x_1(t)\cdot u_2(t), \quad \dot{x}_3(t) = u_3(t), \qquad u_1^2(t)+u_2^2(t)+u_3^2(t) \leq 1, \quad t \in [0,T]. \tag{4}$$

We take $\sigma_1(\alpha) = \cos(\alpha_1)\cdot\sin(\alpha_2)$, $\sigma_2(\alpha) = \sin(\alpha_1)\cdot\sin(\alpha_2)/x_1^0$, $\sigma_3(\alpha) = \cos(\alpha_2)$, $\hat{A} = \{(0,-\pi/2)^T,(0,\pi/2)^T\}$.

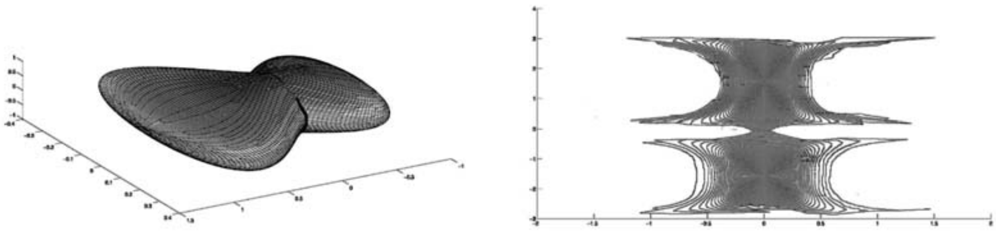

Figure 1: The reachability set for (4) (left) and the set $\overline{A}$ of parameters α generating $\partial X[t]$ for (4), lines correspond to separate iterations $\overline{A}(\eta)$ (right); both the set $\overline{A}$ and the set of smooth points of $\partial X[t]$ are connected, the singularities detached by the bold lines are diffeomorphic to closed segments

4.2 Example 2

The equations of control system are are obtained by projection of two-input two-chained system from [17, p. 392]:

$$\dot{x}_1(t) = u_1(t), \quad \dot{x}_2(t) = x_1(t)\cdot u_2(t), \quad \dot{x}_3(t) = x_2(t)\cdot u_2(t), \qquad u_1^2(t)+u_2^2(t) \leq 1, \quad t \in [0,T]. \tag{5}$$

We determine $\sigma_1(\alpha) = \sin(\alpha_1)$, $\sigma_2(\alpha) = \alpha_2$, $\sigma_3(\alpha) = (\cos(\alpha_1) - x_1^0\cdot\sin(\alpha_2))/x_2^0$, $\hat{A} = \{(-\pi/2,0)^T,(\pi/2,0)^T\}$.

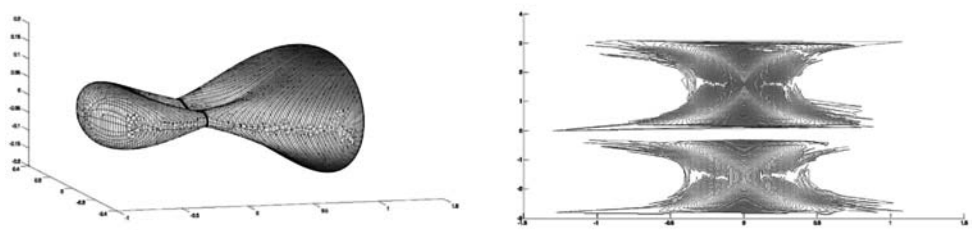

Figure 2: The reachability set for (5) (left) and the set $\overline{A}$ of parameters α generating $\partial X[t]$ for (5), lines correspond to separate iterations $\overline{A}(\eta)$ (right); the set $\overline{A}$ consists of two components, hence, the set of smooth points of $\partial X[t]$ is divided into separate disconnected components by the closed curve containing singularities (see the bold line)

4.3 Example 3

Consider the following three-dimensional *cascade control system* ([16, 17]):

$$\dot{x}_1(t) = u_1(t), \quad \dot{x}_2(t) = x_1(t)\cdot u_2(t), \quad \dot{x}_3(t) = u_2(t), \qquad u_1^2(t)+u_2^2(t) \leq 1, \quad t \in [0,T]. \tag{6}$$

For this system the reachability set is constructed in [16] in *analytical form*, but it may be interesting to observe how the numerical algorithm discussed in this paper deals with this system without regard for its specific properties and the parametrization of characteristics inherent to it obtained in [16]. We define $\sigma_1(\alpha) = \sin(\alpha_1)$, $\sigma_2(\alpha) = \alpha_2$, $\sigma_3(\alpha) = \cos(\alpha_2) - x_1^0 \cdot \alpha_2$, $\hat{A} = \{(\pi/2, 0)^T\}$.

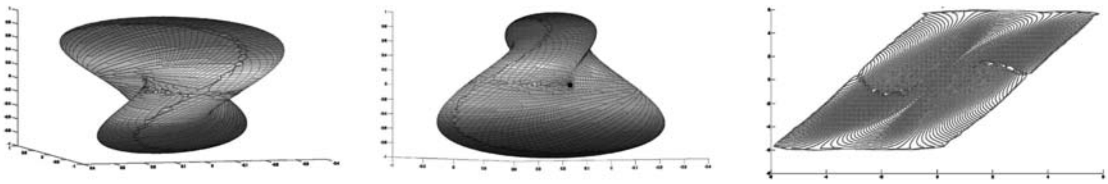

Figure 3: The reachability set for (6) and the set $\overline{A}$ of parameters α generating $\partial X[t]$ for (6), lines correspond to separate iterations $\overline{A}(\eta)$ (right); the set $\overline{A}$ is connected, the singularities of $\partial X[t]$ consist of two distinct points (the second point is invisible because it is on the reverse side of $\partial X[t]$)

5 Conclusion

The perspectives for the suggested simple numerical algorithm are as follows. Firstly, in contrast with numerical methods which require values of $V(t,x)$ for preceding instnts of time, here it is not necessary to know the approximations of $\partial X[t]$ for such instants. In fact, the Hamiltonian system (3) may be integrated numerically immediately up to the necessary time. Secondly, $\partial X[t]$ is parameterized by $n-1$ parameters while $V(t,x)$ is by $n+1$ parameters (this is crucial for large values of n). Thirdly, if $\partial X[\overline{t}]$ is already calculated and we need to calculate $\partial X[t]$ for values of t close to $\overline{t}$ (as it is for the *backward reachable set* used in the control synthesis problem), then we need not calculate this set afresh, but may use the already calculated information. Fourthly, the algorithm may be extended to calculation of only the projection of the reachable set on a linear manifold of dimension less than n rather than the whole reachable set. The difficulty of dimensionality may thus be softened by this move. The obvious defect of the suggested algorithm is the abundance of properties that must be checked in advance which may be rather difficult. But in practice it turns out that this defect may be overcome when dealing with specific examples, while “universal” algorithms may often implicitly presume some of the mentioned properties to be true.

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