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MODELLING SYSTEMS WITH ORDINARY DIFFERENTIAL EQUATIONS: DERIVATIVE ORDER REDUCTION OF INPUT SIGNALS

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Abstract. Many systems that process input signals can be described by ordinary differential equations (ODE). Common test input signals are impulse or step functions. Problems may occur, if the right hand side or source term of an ODE contains a derivative of an input signal or a sharp input transient, especially when solving the ODE numerically. It will be shown that such numerical problems can be circumvented by the application of ODE dependent forward and backward transformations. The proposed transformations are simple and perform a derivative order reduction of input signals present in source terms. However, they can be useful to extend performance and application range of numerical ODE solvers.

1 Introduction

A system can be characterised by how it responds to input signals [4]. To model and analyze systems ordinary differential equations (ODE) are often very useful. If the ODE model of a system contains derivatives of the input signal or a sharp input transient, then this may cause problems, especially when doing numerical simulations.

To illustrate the problem we consider an electrical circuit consisting of a resistor R, an inductor L, and a capacitor C, connected in series as shown in Figure 1. Using Kirchhoff's laws we find that the ODE for the current $I(t)$ of this RLC-circuit is given by

$$L\dot{I}(t) + RI(t) + \frac{1}{C}\int_{t_0}^{t} I(t)\,dt = E(t) - \frac{Q(t_0)}{C} \tag{1}$$

If the input signal or driving voltage $E(t)$ is differentiable for $t \geq t_0$, then we can write

$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = \dot{E} \tag{2}$$

with source term $\dot{E}$. If $E(t)$ is differentiable, then this ODE can be easily solved by analytical or numerical methods [2, 5, 6, 9], if not, we may run into some trouble. However, the exact numerical solution of ODE with transient or 'non-classically' differentiable input signals is still of great interest for electrical engineers [1, 10].

Important signals in order to test systems are (rectangular) impulse or step functions. Unfortunately, these signals are not classically differentiable, i.e. the derivative of an impulse or step function does not exist everywhere. However, this can often be cured for many good-natured source terms by using Fourier or Laplace transform methods. If these methods can not be applied, maybe due to the complexity of the ODE of interest, we have to use an ODE solver, which can induce numerical problems. A solution to such a

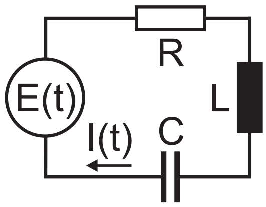

Figure 1: Driven RLC-circuit.

problem is to transform the ODE to a form without any disturbing input transient or first or higher order derivative component of the input signal.

2 Reducing the order of derivative components present in source signals

Assume that the ODE of interest has the following structure[1]:

$$\dot{\mathbf{x}} = A\mathbf{x} + B_0\mathbf{s} + B_1\dot{\mathbf{s}} + B_2\ddot{\mathbf{s}} + \cdots \tag{3}$$

where A and B_i are constant matrices[2], $\mathbf{x} = \mathbf{x}(t)$ is the variable or state vector to be computed and $\mathbf{s} = \mathbf{s}(t)$ denotes the input signal, $B_0\mathbf{s} + B_1\dot{\mathbf{s}} + B_2\ddot{\mathbf{s}} + \cdots$ is called source term. Since matrix A belongs to the homogenous part of the ODE it represents system's structure. Applying the transformation

$$\mathbf{y} = \mathbf{x} - B_1\mathbf{s} - B_2\dot{\mathbf{s}} - \cdots \tag{4}$$

leads to system of differential equations, where all orders of the time derivatives of the input signal are reduced by one:

$$\begin{aligned} \dot{\mathbf{y}} &= A\mathbf{y} + B_0\mathbf{s} + AB_1\mathbf{s} + AB_2\dot{\mathbf{s}} + \cdots \\ &= A\mathbf{y} + (B_0 + AB_1)\,\mathbf{s} + AB_2\dot{\mathbf{s}} + \cdots \end{aligned} \tag{5}$$

If a solution $\mathbf{y}$ of this system is known or can be computed by an ODE solver, we simply have to use the backtransformation

$$\mathbf{x} = \mathbf{y} + B_1\mathbf{s} + B_2\dot{\mathbf{s}} + \cdots \tag{6}$$

to get a solution of the ODE of interest.

One important characteristic of this approach is that if the highest order of a time derivative of the input signal $\mathbf{s}$ is n then the required differentiability of $\mathbf{s}$ is of order $n-1$. This order reduction probes that if the source term of an ODE obtains only first order derivatives of an rectangular impulse or a step function, then some numerical difficulties can be solved by a simple transformation pair. Before we show how to apply these transformations for the example given in the Introduction we indicate how to use them for derivative elimination.

To show examplarily how to use the transformation approach to eliminate completely derivative components of input signals present in an ODE source term we consider the ODE

$$\dot{\mathbf{x}} = A\mathbf{x} + B_0\mathbf{s} + B_1\dot{\mathbf{s}} + B_2\ddot{\mathbf{s}} \tag{7}$$

[1] It is sufficient to analyse a system of first order ordinary differential equations, since higher order ODE can be transformed to first order ODE.

[2] Without restriction to generality the matrices B_i can be diagonal. However, this requires more complex $\mathbf{s}, \dot{\mathbf{s}}, \ldots$.

Inserting the forward transformation

$$\mathbf{y} = \mathbf{x} - B_1\mathbf{s} - B_2\dot{\mathbf{s}} \tag{8}$$

leads to

$$\dot{\mathbf{y}} = A\mathbf{y} + (B_0 + AB_1)\,\mathbf{s} + AB_2\dot{\mathbf{s}} \tag{9}$$
$$= A\mathbf{y} + C_0\mathbf{s} + C_1\dot{\mathbf{s}} \tag{10}$$

We see that explicit second order derivative $\ddot{\mathbf{s}}$ has been eliminated. The first order derivative can be elimated by a second transformation. The transform

$$\mathbf{z} = \mathbf{y} - C_1\mathbf{s} \tag{11}$$

yields

$$\dot{\mathbf{z}} = A\,(\mathbf{z} + C_1\mathbf{s}) + C_0\mathbf{s} \tag{12}$$

Thus, by a repetitive or iterated application of the introduced transformation method all explicit derivative components of the input signal present in the source term of an ODE can be removed. The costs for this approach are more complicated ODE structures and some additional mathematical operations, e.g. matrix multiplications.

In the case of a time variant system we may have to solve

$$\dot{\mathbf{x}} = A(t)\mathbf{x} + B_0(t)\mathbf{s} + B_1(t)\dot{\mathbf{s}} + B_2(t)\ddot{\mathbf{s}} + \cdots \tag{13}$$

By using the transformation

$$\mathbf{y} = \mathbf{x} - B_1(t)\mathbf{s} - B_2(t)\dot{\mathbf{s}} - \cdots \tag{14}$$

we find after some basic manipulations

$$\dot{\mathbf{y}} = A\mathbf{y} + \left(B_0 + AB_1 - \dot{B}_1\right)\mathbf{s} + \left(AB_2 - \dot{B}_2\right)\dot{\mathbf{s}} + \cdots \tag{15}$$

As above we get the desired solution by using the backtransformation

$$\mathbf{x} = \mathbf{y} + B_1(t)\mathbf{s} + B_2(t)\dot{\mathbf{s}} - \cdots \tag{16}$$

It is interesting to observe that reducing the order of derivative components of the input signal $\mathbf{s}$ present in the source term requires the differentiation of the matrices $B_1,\ B_2,\ldots$, thereby changing the structure of the ODE. However, for many systems the differentiation of these matrices should be prefered instead of a differentiation approximation of the input signal. A repeated or iterated execution of the transformation procedure, as demonstrated, can be used to achieve an elimination of all derivative components of the input signal present in the source term.

Next we study a more general, nonlinear case for derivative order reduction of source components. Assume that the ODE of interest can be written as

$$\dot{\mathbf{x}} = \mathbf{f}\,(\mathbf{t},\mathbf{x},\mathbf{s}) + \mathbf{g}\,(\mathbf{t},\mathbf{s},\dot{\mathbf{s}},\ddot{\mathbf{s}},\ldots) \tag{17}$$

The goal is to find a transformation that reduces the order of derivative components of the input signal $\mathbf{s}$ present in the pure source term $\mathbf{g}$ and to give conditions for the structure of $\mathbf{g}$. According to the structure of (17) the ansatz

$$\mathbf{y} = \mathbf{x} - \mathbf{h}\,(\mathbf{t},\mathbf{s},\dot{\mathbf{s}},\ddot{\mathbf{s}},\ldots) \tag{18}$$

leads to

$$\dot{\mathbf{y}} = \mathbf{f}\,(\mathbf{t},\mathbf{y} + \mathbf{h}\,(\mathbf{t},\mathbf{s},\dot{\mathbf{s}},\ddot{\mathbf{s}},\ldots)\,,\mathbf{s}) + \mathbf{g}\,(\mathbf{t},\mathbf{s},\dot{\mathbf{s}},\ddot{\mathbf{s}},\ldots) - \dot{\mathbf{h}}\,(\mathbf{t},\mathbf{s},\dot{\mathbf{s}},\ddot{\mathbf{s}},\ldots) \tag{19}$$

and one conclude that

$$\dot{\mathbf{h}}(\mathbf{t},\mathbf{s},\dot{\mathbf{s}},\ddot{\mathbf{s}},\ldots) = \mathbf{g}(\mathbf{t},\mathbf{s},\dot{\mathbf{s}},\ddot{\mathbf{s}},\ldots) \tag{20}$$

or

$$\mathbf{h}(\mathbf{t},\mathbf{s},\dot{\mathbf{s}},\ddot{\mathbf{s}},\ldots) = \int_{\mathbf{t_0}}^{\mathbf{t}} \mathbf{g}(\tau,\mathbf{s},\dot{\mathbf{s}},\ddot{\mathbf{s}},\ldots)\,d\tau \tag{21}$$

has to be fulfilled. Thus, the source term part **g** has to be integrable. However, this and the required structure of **g** will be analyzed in more detail. Without restriction to generality we consider a single component and suppress, for simplicity, its subscript:

$$\dot{h}\left(t,s,\dot{s},\ddot{s},\ldots,\overset{(n)}{s}\right) = g\left(t,s,\dot{s},\ddot{s},\ldots,\overset{(n)}{s}\right) \tag{22}$$

where $\overset{(k)}{s} = \dfrac{d^k s}{dt^k}$, $k = 0,1,\ldots,n-1$, with $\overset{(0)}{s} = s$. To show that h is, as requested, not a function of $\overset{(n)}{s}$ we examine the time derivative of $h\left(t,s,\dot{s},\ldots,\overset{(n)}{s}\right)$:

$$\frac{d}{dt}h\left(t,s,\dot{s},\ldots,\overset{(n)}{s}\right) = \frac{\partial h}{\partial t}\dot{t} + \frac{\partial h}{\partial s}\dot{s} + \frac{\partial h}{\partial \dot{s}}\ddot{s} + \cdots + \frac{\partial h}{\partial \overset{(n)}{s}}\overset{(n+1)}{s} \tag{23}$$

Since g is not a function of $\overset{(n+1)}{s}$ we conclude that $h = h\left(t,s,\dot{s},\ldots,\overset{(n-1)}{s}\right)$ holds and get

$$h\left(s,\dot{s},\ldots,\overset{(n-1)}{s}\right) = \int_{t_0}^{t}\left[\frac{\partial h}{\partial \tau} + \frac{\partial h}{\partial s}\dot{s} + \frac{\partial h}{\partial \dot{s}}\ddot{s} + \cdots + \frac{\partial h}{\partial \overset{(n-1)}{s}}\overset{(n)}{s}\right]d\tau \tag{24}$$

To achieve the desired derivative order reduction by the above introduced transformations the source term part g must have the following structure

$$\begin{aligned} g\left(t,s,\dot{s},\ldots,\overset{(n)}{s}\right) &= \sum_{j=0}^{n} g_j\left(t,s,\dot{s},\ldots,\overset{(n)}{s}\right) \\ &= \frac{\partial h}{\partial t} + \frac{\partial h}{\partial s}\dot{s} + \frac{\partial h}{\partial \dot{s}}\ddot{s} + \cdots + \frac{\partial h}{\partial \overset{(n-1)}{s}}\overset{(n)}{s} \end{aligned} \tag{25}$$

By defining $g_0 = \tilde{g}_0 = \dfrac{\partial h}{\partial t}$ and $g_k = \tilde{g}_k \overset{(k)}{s}$ with $\tilde{g}_k = \dfrac{\partial h}{\partial \overset{(k-1)}{s}}$, $k = 1,2,\ldots,n$, we can write

$$\begin{aligned} h\left(t,s,\dot{s},\ldots,\overset{(n-1)}{s}\right) &= \int_{t_0}^{t}\sum_{j=0}^{n} g_j\left(\tau,s,\dot{s},\ldots,\overset{(n)}{s}\right)d\tau \\ &= \int_{t_0}^{t}\tilde{g}_0\left(\tau,s,\dot{s},\ldots,\overset{(n)}{s}\right)d\tau + \int_{t_0}^{t}\sum_{j=1}^{n}\tilde{g}_j\left(\tau,s,\dot{s},\ldots,\overset{(n)}{s}\right)d\overset{(j)}{s} \end{aligned} \tag{26}$$

At a first glance the substitution of differentiation by integration seems not to be advantageous. But many source term parts g can be integrated very easily, especially when $g = g\left(s,\dot{s},\ddot{s},\ldots,\overset{(n)}{s}\right)$. A well known example is the impulse function. In addition, the single or repeated application of the above introduced transformation can be useful in numerical ODE integration, this will be shown in the next section.

3 Examples

The second order ODE of the driven RLC-circuit given in the Introduction (2) can be transformed into a system of two first order ordinary differential equations. For this we set $x_1(t) = I(t)$ and $x_2(t) = \dot{x}_1(t)$ and find

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \frac{1}{L} \begin{pmatrix} 0 & 1 \\ -\frac{1}{C} & -R \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{1}{L} \begin{pmatrix} 0 \\ \dot{E} \end{pmatrix} \tag{27}$$

Using the transformation $\mathbf{y} = \mathbf{x} - B_1\mathbf{s}$ with matrix $B_1 = \frac{1}{L}\mathbf{1}$ and source term $\mathbf{s}=\frac{1}{L}\left(0,\dot{E}\right)^{\mathsf{T}}$ results in

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \frac{1}{L} \begin{pmatrix} 0 & 1 \\ -\frac{1}{C} & -R \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{L^2} \begin{pmatrix} 0 & 1 \\ -\frac{1}{C} & -R \end{pmatrix} \begin{pmatrix} 0 \\ E \end{pmatrix} \tag{28}$$

This ODE contains, as expected, only the input signal E and no derivative of it. Note that the matrix $A = \frac{1}{L}(\boxplus)$, which represents the structure of the RLC-circuit, is applied to the integral of the source term which is the input signal. However, this assures the elimination of the derivative $\dot{E}$ of the input signal E. After solving this ODE, the required solution can be esily obtained by using the backtransformation $\mathbf{x} = \mathbf{y} + B_1\mathbf{s}$.

In order to give an example of how to apply the transformation method we set $R = 1\,\Omega$, $L = 1\,\mathrm{H}$ and $C = 1\,\mathrm{F}$ and solve both ODE with the implicit trapezoidal rule and with the fourth order BDF-method [5, 6, 9] for initial values $I_0 = I(t<0) = 0$ and $\dot{I}_0 = \dot{I}(t<0) = 0$ and for the input signal

$$E(t) = H_a(t) = \begin{cases} 1, & t > 0 \\ a, & t = 0 \\ 0, & t < 0 \end{cases}, 0 \le a \le 1 \tag{29}$$

with $a = 1/2$. By using the well known differentiation properties of the unit or Heaviside step function $H(t)$ we obtain

$$\dot{E}(t) = \delta(t) \tag{30}$$

where $\delta(t)$ is Dirac's delta-function. For numerical simulation we approximated $\dot{E}(t)$ by second and fourth order symmetric numerical differentiations of the input signal sampled with sampling interval $T = h_s$ and subsequent ODE integration with time step h_s. In order to demonstrate the potential benefit of the herein introduced ODE transformation approach, we calculated, as a reference, the analytical solution by solving (2) by means of the Fourier transform method. Figure 2 shows the obtained analytical and numerical solutions for the current $I(t)$. The comparison of the graphs shows that the computational accuracies achieved for the transformed ODE are better than those for the non-transformed ODE, which still or explicitly contains the derivative of the input signal. Similar results were obtained for the explicit Euler method, the implicit Euler method, and for the second order BDF method.

In the second example we consider the Duffing-Holmes oscillator [8, 12], which can be written as a non-autonomous system of two first order ODE

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ F(x_1) - bx_2 + E(t) \end{pmatrix} \tag{31}$$

with internal force $F(x_1) = x_1 - x_1^3$, external driving force or input signal $E(t)$, and damping coeffcient b. Physically the Duffing-Holmes equation (31) describes an externally driven particle in an two-well non parabolic potential

$$W(x_1) = -\int F(x_1)\,dx_1 = -\frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 \tag{32}$$

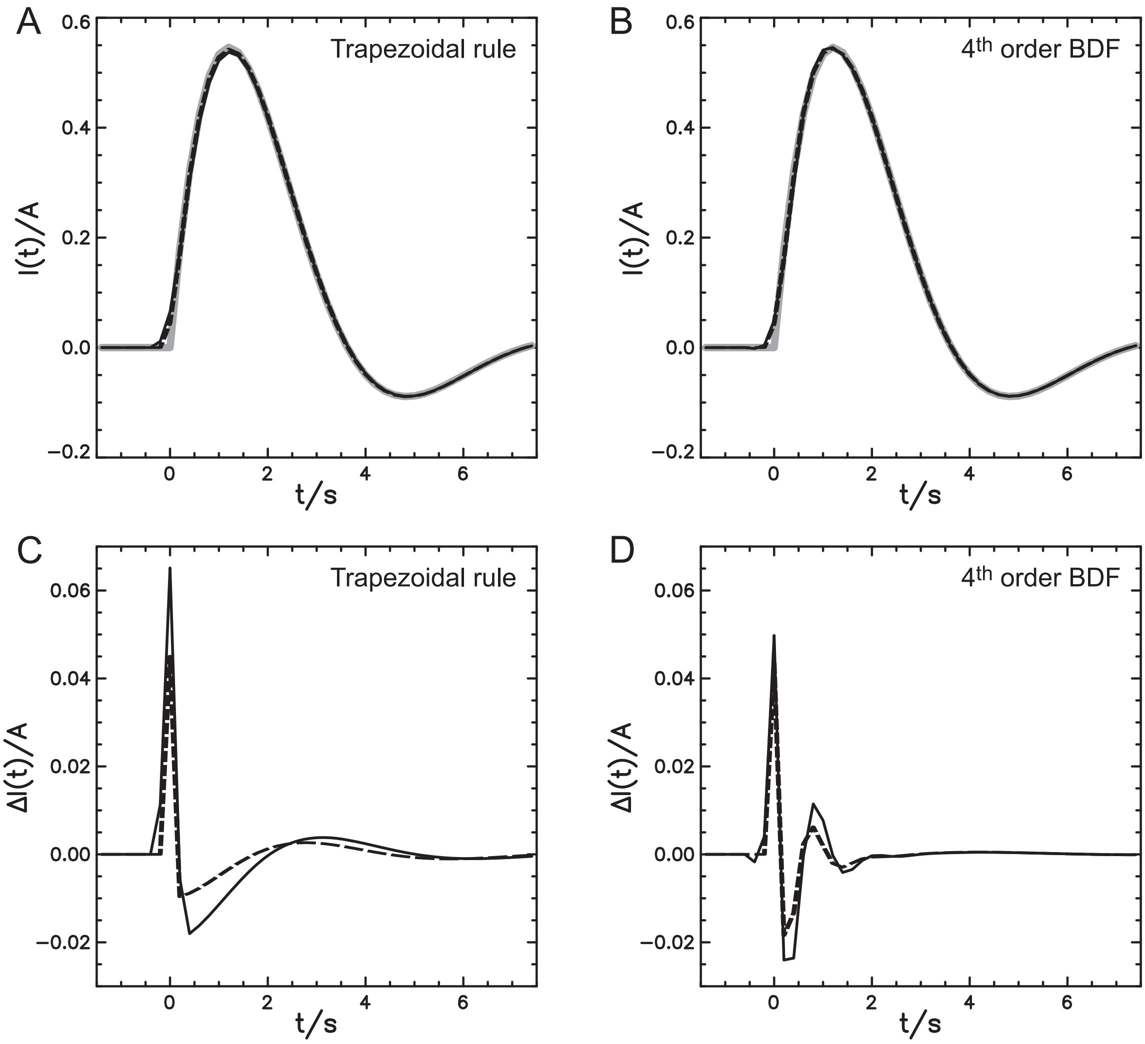

Figure 2: Numerical solutions for the current $I(t)$ for the driven RLC-circuit (Figure 1). The continuous black curves are the computational results for the non-transformed ODE (27) and the dashed curves those for the transformed ODE (28) with backtransformation: A) implicit trapezoidal rule, B) fourth order BDF method; sampling/integration time step: $h_s = 0.2\,s$. The continuous grey curves are the analytical solution. The differences between numerical and analytical solutions $\Delta I(t)$ are depicted in C) and D): the continuous curves are those for the direct and the dashed curves those for the transformation based approach. Note the input and solver related parasitic deviations at $t = 0$.

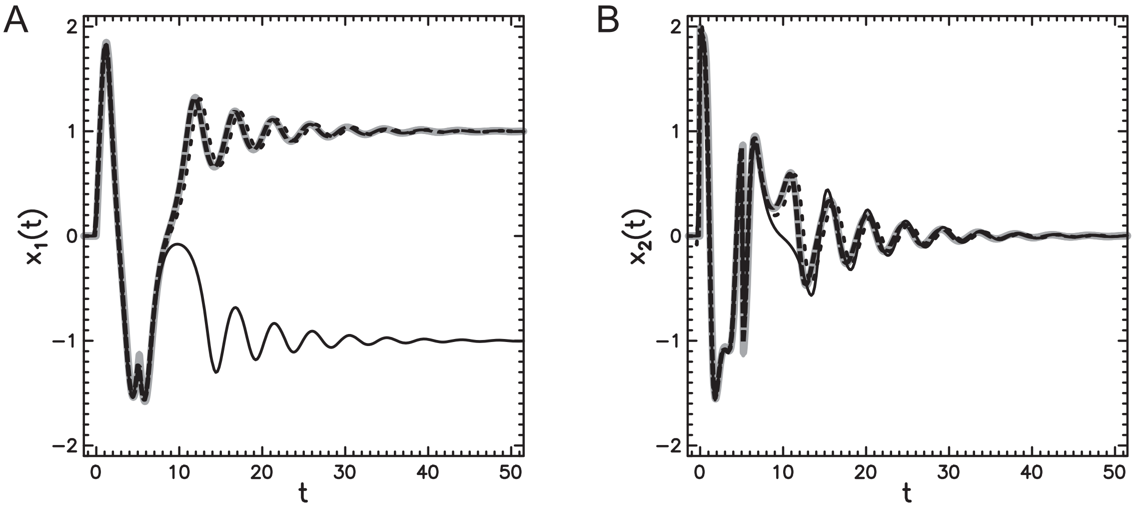

Figure 3: Numerical solutions for the rectangular impulse driven Duffing-Holmes oscillator: A) $x_1(t)$ and B) $x_2(t)$. The continuous black curves are the computational results for the non-transformed ODE (31) and the long-dashed curves are those for the transformed ODE (34) with backtransformation. The sampling/integration time step for the symmetrical second order numerical differentiation and the implicit trapezoidal rule was $h_s = 0.2$. The short-dashed curves show the results obtained for replacing second order by fourth order numerical differentiation. The results for solving the non-transformed ODE by second order numerical differentiation and trapezoidal rule for $h_s = 0.02$ are shown by thick grey curves. Note the differentiation method related results.

Depending on b and $E(t)$ one can observe various nonlinear responses e.g. chaotic oscillations. For transient input signals the Duffing-Holmes oscillator can show nonlinear oscillatory routes to the stable fixpoints $x_{1,fix} = \pm 1$. However, these routes can be influenced by the ODE solver. In order to demonstrate this and to give another performance example of the herein introduced ODE transformation approach we compare (31) with

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f_1(t;x_1,x_2;s_1,s_2) \\ f_2(t;x_1,x_2;s_1,s_2) \end{pmatrix} + \begin{pmatrix} g_1(t;s_1,s_2;\dot{s}_1,\dot{s}_2;\ldots) \\ g_2(t;s_1,s_2;\dot{s}_1,\dot{s}_2;\ldots) \end{pmatrix} \tag{33}$$

find $f_1(t;x_1,x_2;s_1,s_2) = x_2$, $f_2(t;x_1,x_2;s_1,s_2) = x_1 - x_1^3 - bx_2$, $g_1(t;s_1,s_2;\dot{s}_1,\dot{s}_2;\ldots) = 0$, and, finally, we have $g_2(t;s_1,s_2;\dot{s}_1,\dot{s}_2;\ldots) = E(t)$. Next we find $h_1(t;s_1,s_2;\dot{s}_1,\dot{s}_2;\ldots) = 0$ and $h_2(t;s_1,s_2;\dot{s}_1,\dot{s}_2;\ldots) = \int g_2(t;s_1,s_2;\dot{s}_1,\dot{s}_2;\ldots)\,dt$. Thus the transformed ODE is given by

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 + h_2 \\ y_1 - y_1^3 - b(y_2 + h_2) \end{pmatrix} \tag{34}$$

Figure 3 shows numerical solutions obtained by the implicit trapezoidal rule for the original Duffing-Holmes equation (31) and for (34) with backtransformation and parameter $b = 0.25$, $h_2(t) = 2(H_1(t) - H_1(t-5.2))$, and zero intial values. To compute E we applied symmetric second and, for comparison, fourth order numerical differentiation to h_2. The sampling/integration time step was $h_s = 0.2$ and, to compute a reference, $h_s = 0.02$. The comparison of the result graphs shows again that the transformation based ODE solving approach performes better than the direct method.

In the third and last example we consider the nonlinear initial value problem

$$\dot{x} = \varphi(t,x) - \varphi(t,s) + \dot{s} \quad \text{with} \quad x_0 = s_0 \tag{35}$$

which can be easily generalized to higher order or multidimensional ODE. It is obvious that this ODE has the solution $x(t) = s(t)$ [9]. Thus we have a device for generating nonlinear ODE problems with known solutions to test ODE solvers. By comparing (35) with $\dot{x} = f(t,x,s) + g(t,s,\dot{s}\ldots)$ we find $f(t,x,s) = \varphi(t,x) - \varphi(t,s)$ and $g(t,s,\dot{s}\ldots) = \dot{s}$. Hence the transformed ODE is given by

$$\dot{y} = \varphi(t,y+s) - \varphi(t,s) \tag{36}$$

with transformation $y = x - h = x - \int \dot{s}d\tau = x - s$. Note that (36) has obviously the solution $y = 0$. More interesting is how ODE solvers perform when solving (35) or (36).

To test this we define $\varphi(t,x) = t - x^2$. Hence we have

$$\dot{x} = -x^2 + s^2 + \dot{s} \tag{37}$$

and the transformed ODE

$$\dot{y} = -y^2 + 2ys \tag{38}$$

In order to get better understanding of the transformation, assume that s has a jump discontinuity. If so, we see that the problem to handle $\dot{s}$ when solving (37) is transformed to solve (38), which can be interpreted to solve two adjoining ODE separated by the jump discontinuity. To give a numerical example we set $s(t) = H_1(t) - H_1(t-3)$. The numerical methods used to solve these two ODE for zero intial values were the implicit Euler method, the implicit trapezoidal rule, the second order BDF method, and, the fourth order BDF method. The derivatives were obtained by symmetrical second order differentiation and by 'naive' differentiation, i.e. setting $\dot{s}(0) = 1/h_s$ and $\dot{s}(3) = -1/h_s$, where h_s is the sampling time step. Figure 4 shows the numerical results. Note the errors due to numerical differentiation. Finally, it is interestingly to observe that the results obtained for the transformed ODE (38) are independent of the ODE solver used. In addition, they are up to insignificant computational errors, due to solving of the ODE integration equations, identical to the theoretical solution $x = s$.

4 Discussion and Conclusions

An ODE dependent transformation approach has been presented to reduce the order of or to eliminate derivatives of input signals present in the source term of the ODE model of the system to be analysed. The order reduction or elimination of such derivatives is of interest when an input signal is not differentiable. Examples of important non-classically differentiable input signals are impulse or step functions that are very often used to analyse the transient behaviour of electrical or opto-electronical circuits. In the case of a linear ODE with constant coefficients a simple linear transformation is sufficient to remove these unwanted components by single or multiple transformation-based iterations. For time variant systems the introduced transformation replaces the time derivative of the input signal by a time derivative of matrices related to ODE's source term. Next a more general, nonlinear system was analyzed. It was shown, that, if the part of the source term containing the input signal's derivative can be written as an additive component of the ODE, a condition to the structure of this term can be stated. The related transformation is then given by a shift of the systems variable by the integral of this special but still unwanted part of the source term. Finally, examples were presented to demonstrate the usefulness of the approach.

The notion behind the presented paper relies on the observation that many ODE contain the solution and its derivative components in a mathematically convenient configuration. This fact was used for a transformation construction to reduce the order of or to eliminate a derivative of an input signal present in the ODE's source term. In other words, many ODE structures are intrinsically exploited to a transformation-based input signal differentiation or to transformation-based integration of the derivative of an input signal which is surely

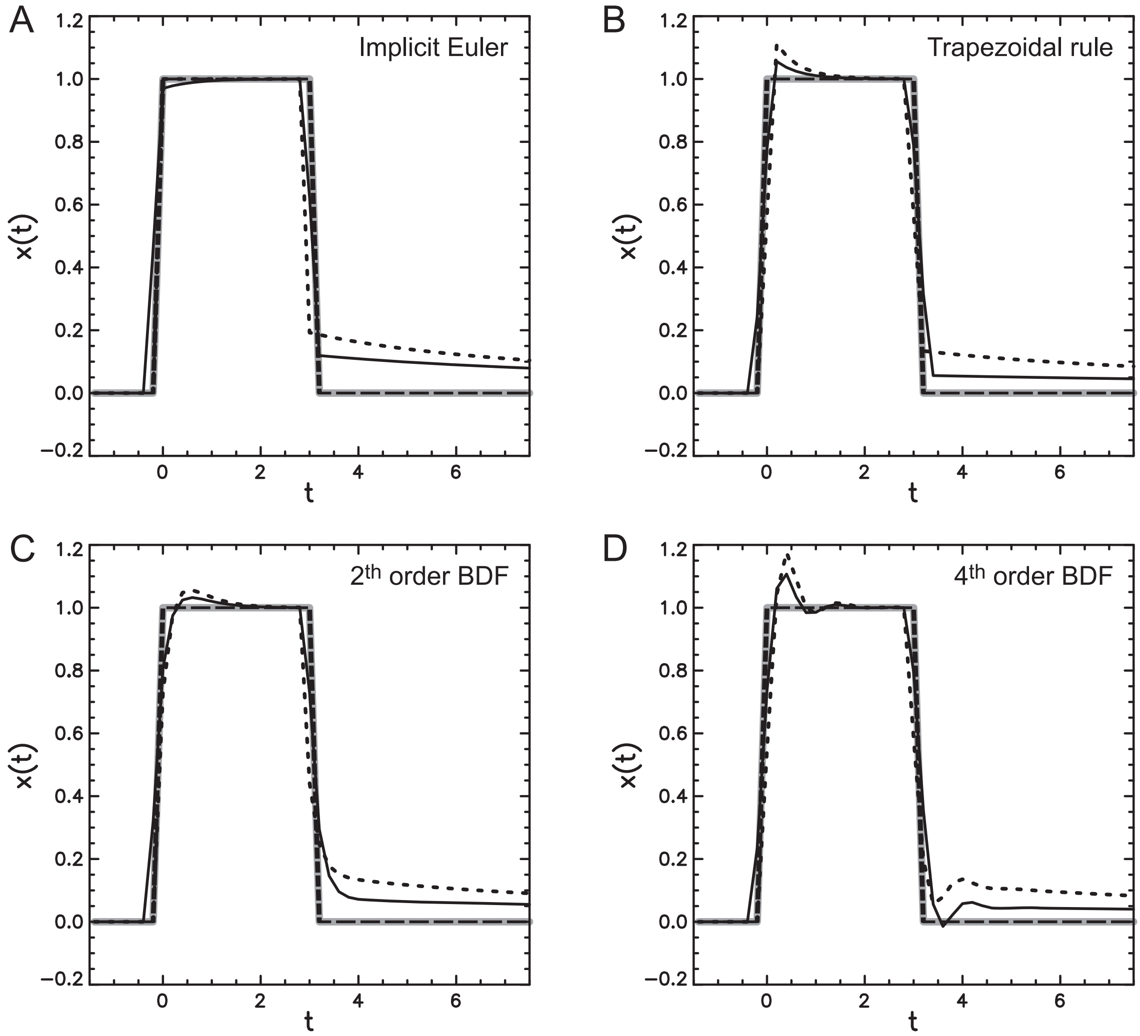

Figure 4: Numerical solutions of (37) and (38) for various ODE solvers: A) Implicit Euler method, B) implicit trapezoidal rule, C) second order BDF method, and, D) fourth order BDF method. The continuous black curves are the computational results for the non-transformed ODE (37) with second order numerical differentiation and the long-dashed curves are those for the transformed ODE (38) with backtransformation. The sampling/integration time step for numerical differentiation and integration was $h_s = 0.2$. The short-dashed curves show the results obtained for replacing second order by 'naive' differentiation. The continuous grey curves are the analytical solution. Note that the impulse responses of the ODE solvers can be different.

the input signal itself. A generalisation of this notion is the transformation-based order reduction of input signal derivatives present in ODE's source term or the integration of input signal components. Note that this transformation is not the same as the transformation of an ODE from a non-autonomous to an autonomous form. However, a derivative order increase can also be achieved.

Transformations, e.g. Fourier or Laplace transformation, are well known and powerful tools to solve ODE [2, 3, 5, 6, 7, 11]. The mathematical simplicity of the herein proposed transformations may help to use them as a standard procedure when analyzing a complex or nonlinear system by solving the related ODE whose source term contains unwanted or cumbrous transient input signals or derivative components of the system's input signal. However, more important is the fact that some additional mathematical operations can be useful to improve the performance and, certainly, the range of applications of numerical ODE solvers.

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