BOND GRAPH SIZING OF MECHATRONIC SYSTEMS: COUPLING OF INVERSE MODELLING WITH DYNAMIC OPTIMIZATION

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Abstract. For the last 15 years, a methodology has been developed for sizing mechatronic systems. Based on the use of inverse models described by bond graph, it has in particular the advantage of drastically decreasing the number of calculus iterations, compared to the trial and error procedure of the classical direct approach.
The aim of this article is to extend this sizing methodology to the case where only a part of the specifications can be translated in terms of functions of time (and so where only a part of the model can be inverted) and where the other part can be formulated as an optimal control problem. In particular, the illustration of this extension on an academic example of two masses in series will show how a partially inverted model can be coupled with the graphical construction of an *optimizing bond graph* established in recent articles.

1 Introduction

Inverse modelling consists of determining the unknown inputs of the model directly from the specified outputs. Taking advantage of this approach, a methodology has been developed for sizing mechatronic systems according to energy and dynamic criteria [13]. Based on the bond graph language (chosen for its multi-domain, physical and graphical aspects), this methodology consists of: graphically checking if the bond graph model is invertible and if the specifications can be reached by the given model structure; then graphically constructing the bond graph inverse model corresponding to the specific sizing problem under consideration; and then finding the unknown inputs by simulating the resulting inverse model from the given specified outputs. One of the original features of this methodology lies in its structural analysis step where graphical guidelines are given to the engineer to check if his problem is well-posed. Moreover the benefits of such a methodology have been shown in some industrial applications, especially in the automotive domain [8].
This methodology, however, can only be applied to the cases where the specifications can be translated in terms of functions of time. This requirement is not so easy to meet in practice [9]. Design constraints are most of the time expressed as: not crossing upper or lower limits, minimizing the weight or the energy consumption and so on. In order to handle this kind of design constraint, recent articles [6, 2, 7, 10] presented the bond graph formulation of an optimal control problem. The resulting procedure leads to graphically construct what it is called an *optimizing bond graph* and to couple it with the *initial bond graph* model of the system under study. The bond graph model obtained then mirrors a system of equations, identical to the *optimality conditions* given by the Pontryagin Maximum Principle [14]. Up to now this optimization procedure was considered only for direct models. The objective of this article is to substantiate the interest of such a procedure by coupling it with the sizing methodology and so by adapting it for inverse models.
After having recalled in Section 2 the procedure for constructing an *optimizing bond graph* in the case of direct models, Section 3 presents the coupling of inverse modelling with dynamic optimization through the academic example of two masses in series. The two manners of formulating the problem (inverting then optimizing or optimizing then inverting) are discussed in particular. Numerical results are then given in Section 4 to prove the feasibility of such a coupling. Finally Section 5 indicates some research directions that will be interesting to carry out.

2 Bond graph formulation of dynamic optimization

Coupling dynamic optimization with bond graph models has already been considered in [1, 5, 16]. However these works only present direct methods for solving the optimization problem under consideration: bond graph models are used only for modelling the system while the optimization is really performed by classical routines such as the genetic algorithm [5, 16]. Here, as the final objective is to couple a dynamic optimization procedure with a bond graph-based sizing methodology, constructing the solution of an optimization problem (or at least a system of equations whose solving leads to the exact solution of the problem) through a bond graph model appears to be relevant. This explains why indirect solving methods such as the Pontryagin Maximum Principle are considered here.

Now let us suppose that the optimization problem is to find $\mathbf{x}$ and $\mathbf{u}_{opt}$ such that:

- the performance index V is minimized:

$$V = \int_{t_0}^{t_f} \mathscr{L}(\mathbf{x},\mathbf{u},t)\,dt = \int_{t_0}^{t_f} \frac{1}{2}[\mathbf{u}_{opt}^T \cdot \mathbf{R}_{\mathbf{u}_{opt}} \cdot \mathbf{u}_{opt} + P_{diss}]\,dt \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector; $\mathbf{u} \in \mathbb{R}^m$ denotes the input vector, $\mathbf{u} = \begin{bmatrix} \mathbf{u}_u^T & \mathbf{u}_{opt}^T \end{bmatrix}^T$ with $\mathbf{u}_u \in \mathbb{R}^{m_u}$ (resp. $\mathbf{u}_{opt} \in \mathbb{R}^{m_{opt}}$) is the vector of fixed inputs (resp. of inputs to be optimized); $\mathbf{R}_{\mathbf{u}_{opt}} \in \mathbb{R}^{m_{opt} \times m_{opt}}$ is a weighting matrix, supposed to be diagonal, $\mathbf{R}_{\mathbf{u}_{opt}} = \begin{bmatrix} \mathbf{R}_{\mathbf{u}_{e,opt}}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{\mathbf{u}_{f,opt}} \end{bmatrix}$ with $\mathbf{R}_{\mathbf{u}_{e,opt}}$ and $\mathbf{R}_{\mathbf{u}_{f,opt}}$ the weighting matrices respectively associated with the effort and flow sources to be optimized; P_{diss} is all or a part of the energy dissipated by the system;

- the initial and final conditions (2) on state and time are fixed;

$$\begin{cases} \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{x}(t_f) = \mathbf{x}_f \end{cases} \tag{2}$$

- the state-space equations (3) of the model are taken as constraints.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u},t) \tag{3}$$

The Pontryagin Maximum Principle [14, 15] states then that the exact solution $(\mathbf{x}_{opt}, \boldsymbol{\lambda}_{opt}, \mathbf{u}_{opt})$ of such an optimization problem is given by solving analytically the following system:

$$\begin{cases} \dot{\mathbf{x}} = \frac{\partial H_p(\mathbf{x},\boldsymbol{\lambda},\mathbf{u})}{\partial \boldsymbol{\lambda}} \\ \dot{\boldsymbol{\lambda}} = -\frac{\partial H_p(\mathbf{x},\boldsymbol{\lambda},\mathbf{u})}{\partial \mathbf{x}} \\ \frac{\partial H_p(\mathbf{x},\boldsymbol{\lambda},\mathbf{u})}{\partial \mathbf{u}_{opt}} = \mathbf{0} \end{cases} \tag{4}$$

where $\boldsymbol{\lambda} \in \mathbb{R}^n$ is the co-state vector and $H_p = \mathscr{L}(\mathbf{x},\mathbf{u},t) + \boldsymbol{\lambda}^T \cdot \mathbf{f}(\mathbf{x},\mathbf{u},t)$ is called the *Pontryagin function*.

As already written, a procedure has been established for constructing, in a systematical manner, an *augmented bond graph* which mirrors the equations of these *optimality conditions* (4). This procedure can be summed up in the four steps below [6, 7]:

Step 1: Duplication. Duplicate the bond graph model of the system under study. The original part of the bond graph model will hereafter be called *initial bond graph* while the duplicated part will be called *optimizing bond graph*.

Step 2: Characteristics of the *optimizing bond graph*. In the *optimizing bond graph*, replace the characteristic matrices $\mathbf{R}$ of the $\mathbb{R}$-elements by their corresponding transposed and reversed sign matrices $-\mathbf{R}^T$.

Step 3: Take into account the performance index.

A - For the dissipative phenomena involved in the performance index, couple the $\mathbb{R}$-elements present simultaneously in the *initial bond graph* and in the *optimizing bond graph* and which both correspond to the same phenomena. Then, add the matrix $\frac{1}{2}[\mathbf{R}_{opt} + \mathbf{T}]$ as the lower extra diagonal submatrix where $\mathbf{R}_{opt} = \begin{bmatrix} \mathbf{R}_{rr} & \mathbf{R}_{cr} \\ \mathbf{R}_{rc} & \mathbf{R}_{cc} \end{bmatrix}$ is the characteristic matrix of the corresponding $\mathbb{R}$-elements in the *initial bond graph* and $\mathbf{T} = \begin{bmatrix} \mathbf{R}_{rr}^T & -\mathbf{R}_{cr}^T \\ -\mathbf{R}_{rc}^T & \mathbf{R}_{cc}^T \end{bmatrix}$ (the subscripts $_r$ and $_c$ correspond to the ports in a resistance and in a conductance causality respectively when the bond graph model is in preferential integral causality). Note that, in the specific case of 1-port R-element, the term $\frac{1}{2}[R_{opt} + T]$ reduces itself to R_{opt}.

B - For the inputs to be determined, insert a **0**-junction (resp. **1**-junction) between each effort source (resp. flow source) and the structure junction and then insert one multiport $\mathbb{R}$-element per pair of corresponding sources into the *initial* and *optimizing bond graphs*. For effort sources (resp. flow sources), set the characteristic matrices to $\begin{bmatrix} \mathbf{R}_{\mathbf{u}_{e,opt}} & \mathbf{0} \\ \mathbf{R}_{\mathbf{u}_{e,opt}} & -\mathbf{R}_{\mathbf{u}_{e,opt}} \end{bmatrix}$ (resp. $\begin{bmatrix} \mathbf{R}_{\mathbf{u}_{f,opt}} & \mathbf{0} \\ \mathbf{R}_{\mathbf{u}_{f,opt}} & -\mathbf{R}_{\mathbf{u}_{f,opt}} \end{bmatrix}$).

C - For the inputs not to be optimized (*i.e.* which are supposed to be known), replace the corresponding $\mathbb{MS}e$ and $\mathbb{MS}f$-elements respectively by $\mathbb{D}f$ and $\mathbb{D}e$-elements in the *optimizing bond graph*.

Step 4: Bicausality assignment. Replace the source elements involved in the performance index by double detectors $\mathbb{D}e\mathbb{D}f$ in the *initial bond graph* and by double sources $\mathbb{S}e\mathbb{S}f$ in the *optimizing bond graph*. These double sources impose both null efforts and flows. Then propagate bicausality [3] from the double sources to the double detectors through the $\mathbb{R}$-elements associated with the inputs to be determined. Finally assign the preferential integral causality to the rest of the model in order to obtain the *augmented bond graph*, mirror of the *optimality conditions* (4).

3 Coupling of inverse modelling with dynamic optimization

The first advantage of the optimization procedure described in the previous section lies in the fact that the *optimality conditions* are obtained in a systematical and graphical manner contrary to the Pontryagin Maximum Principle which requires analytical developments. Now even if this argument is appealing, the main interest of this procedure is to couple dynamic optimization with the sizing methodology and so with inverse modelling.

3.1 Framework

To initiate the coupling between inverse modelling and dynamic optimization, only linear and time-invariant systems are addressed in this article. The sizing problem is considered to be the determination of the open-loop controls in the case where the specifications can be divided into two parts: a first one where the specification can be expressed as a function of time and another one where the design constraint formulates itself as a problem of dissipative energy minimization.

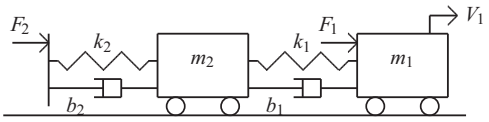


Figure 1: Example of two masses in series: technological diagram.

To illustrate this case, let us consider the example of two masses in series, joined by springs and dampers in parallel and where the controls are the efforts F_1 and F_2 (Fig. 1). Now suppose that the sizing problem is to find F_1 and F_2 so that:

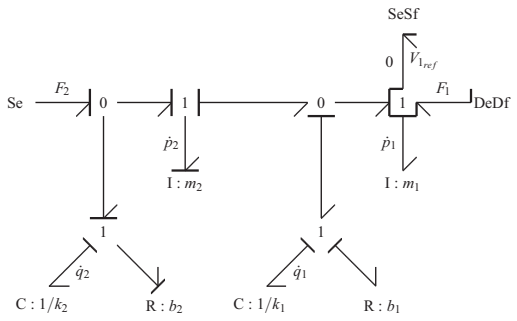
- the speed V_1 of the first mass m_1 follows a given trajectory $V_{1_{ref}}$ (specification n°1);
- the dissipative energy P_{diss,b_2} due to the second damper is minimized (specification n°2).

The key idea here is to simultaneously meet specification n°1 by inversion and specification n°2 by dynamic optimization. The following sections will present the two possible approaches for treating such a coupling: the first one where the model is partially inverted before being optimized and the second one where, on the contrary, the optimization problem is formulated before the inversion.

3.2 Approach 1: inverse modelling before optimization

Let us proceed to the first approach consisting of previously inverting the model before formulating the optimization problem.

Bond graph model:



State-space model:

$$p_1 = m_1 V_{1_{ref}} \tag{5}$$

$$\begin{cases} \dot{p}_2 = b_1 V_{1_{ref}} - \frac{b_1}{m_2} p_2 - k_1 q_1 + F_2 \\ \dot{q}_1 = -V_{1_{ref}} + \frac{1}{m_2} p_2 \\ \dot{q}_2 = -\frac{k_2}{b_2} q_2 + \frac{1}{b_2} F_2 \end{cases} \tag{6}$$

Output equation:

$$F_1 = m_1 \dot{V}_{1_{ref}} + b_1 V_{1_{ref}} - \frac{b_1}{m_2} p_2 - k_1 q_1 \tag{7}$$

Table 1: Inverse model with respect to (F_1, V_1).

In the case of the example, meeting specification n°1 by inversion amounts to partially inverting the bond graph model with respect to the couple (F_1, V_1) by using the bicausality concept (Tab. 1). It can then be deduced by the exploitation of the causality assignment that the remaining dynamic part of this inverse model is governed by the state-space equations (6) and that, once the optimal state vector is known, the unknown F_1 will be computed such as in (7).

Once this first step has be carried out, specification n°2 can be met by considering only the remaining dynamic part of the inverse model and formulating the following optimal control problem: *'find the input F_2 such that the performance index $V = \frac{1}{2}\int_{t_0}^{t_f} (\frac{F_2^2}{R_{F_2}} + P_{diss,b_2})dt$ is minimized when initial and final conditions on state and time are fixed and the state-space equations* (6) *are taken as constraints'*. The unknowns of this optimization problem are

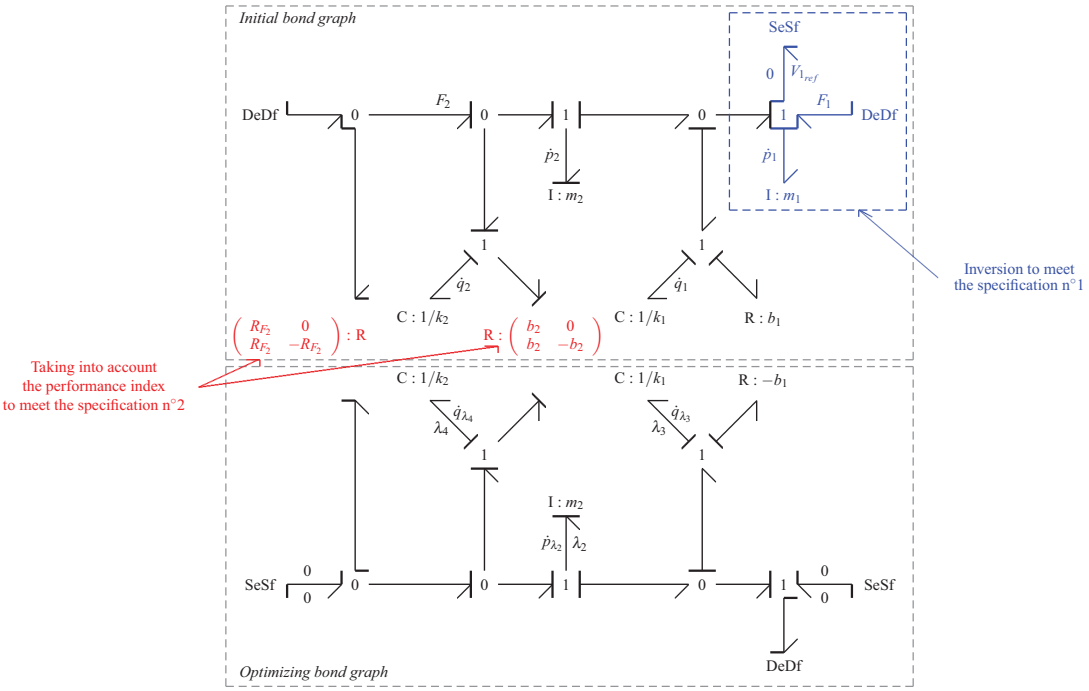


Figure 2: *Augmented bond graph model* coupling both inversion and optimization.

$(p_2, q_1, q_2, \lambda_2, \lambda_3, \lambda_4, F_2)$. After having constructed the corresponding *Pontryagin function*, applying the Pontryagin Maximum Principle in an analytical manner leads then to the *optimality conditions* (8).

$$\begin{cases} \dot{p}_2 = \frac{\partial H_p}{\partial \lambda_2} \\ \dot{q}_1 = \frac{\partial H_p}{\partial \lambda_3} \\ \dot{q}_2 = \frac{\partial H_p}{\partial \lambda_4} \\ \dot{\lambda}_2 = -\frac{\partial H_p}{\partial p_2} \\ \dot{\lambda}_3 = -\frac{\partial H_p}{\partial q_1} \\ \dot{\lambda}_4 = -\frac{\partial H_p}{\partial q_2} \\ \frac{\partial H_p}{\partial F_2} = 0 \end{cases} \Rightarrow \begin{cases} \dot{p}_2 = b_1 V_{1_{ref}} - \frac{b_1}{m_2} p_2 - k_1 q_1 + F_2 \\ \dot{q}_1 = -V_{1_{ref}} + \frac{1}{m_2} p_2 \\ \dot{q}_2 = -\frac{k_2}{b_2} q_2 + \frac{1}{b_2} F_2 \\ \dot{\lambda}_2 = \frac{b_1}{m_2} \lambda_2 - \frac{1}{m_2} \lambda_3 \\ \dot{\lambda}_3 = k_1 \lambda_2 \\ \dot{\lambda}_4 = -\frac{k_2^2}{b_2} q_2 + \frac{k_2}{b_2} \lambda_4 + \frac{k_2}{b_2} F_2 \\ (\frac{1}{R_{F_2}} + \frac{1}{b_2}) F_2 = \frac{k_2}{b_2} q_2 - \lambda_2 - \frac{1}{b_2} \lambda_4 \end{cases} \tag{8}$$

As a consequence, according to approach 1, finding the solution $(p_1, p_2, q_1, q_2, \lambda_2, \lambda_3, \lambda_4, F_1, F_2)$ which meets specifications n°1 and n°2 on V_1 and P_{diss,b_2} amounts to solving the following system, made of equations (5), (7) and (8):

$$\begin{cases} p_1 = m_1 V_{1_{ref}} \\ \dot{p}_2 = b_1 V_{1_{ref}} - \frac{b_1}{m_2} p_2 - k_1 q_1 + F_2 \\ \dot{q}_1 = -V_{1_{ref}} + \frac{1}{m_2} p_2 \\ \dot{q}_2 = -\frac{k_2}{b_2} q_2 + \frac{1}{b_2} F_2 \\ \dot{\lambda}_2 = \frac{b_1}{m_2} \lambda_2 - \frac{1}{m_2} \lambda_3 \\ \dot{\lambda}_3 = k_1 \lambda_2 \\ \dot{\lambda}_4 = -\frac{k_2^2}{b_2} q_2 + \frac{k_2}{b_2} \lambda_4 + \frac{k_2}{b_2} F_2 \\ F_1 = m_1 \dot{V}_{1_{ref}} + b_1 V_{1_{ref}} - \frac{b_1}{m_2} p_2 - k_1 q_1 \\ (\frac{1}{R_{F_2}} + \frac{1}{b_2}) F_2 = \frac{k_2}{b_2} q_2 - \lambda_2 - \frac{1}{b_2} \lambda_4 \end{cases} \tag{9}$$

Now, from a bond graph point of view, as the remaining dynamic part of the inverse model can be viewed, to some extent as a direct model with $\mathbf{x} = \begin{bmatrix} p_2 & q_1 & q_2 \end{bmatrix}^T$, $\mathbf{u}_u = \begin{bmatrix} V_{1_{ref}} \end{bmatrix}$ and $\mathbf{u}_{opt} = [F_2]$, the previous procedure for constructing *optimizing bond graphs* can still be applied to this case. The only differences lie in the fact that:

- only the remaining dynamic part (the part of the bond graph model whose causality has remained the same even after the partial inversion) is duplicated to form the *optimizing bond graph*;

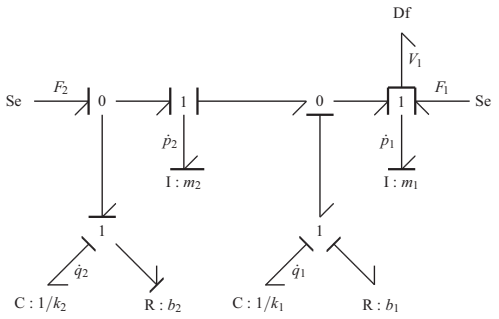
- each $\mathbb{S}e\mathbb{S}f$-element representing a fixed input in the *initial bond graph* (*i.e.* $V_{1_{ref}}$ in the case of the example) must be replaced by a $\mathbb{D}e\mathbb{D}f$-element in the *optimizing bond graph*;
- each $\mathbb{D}e\mathbb{D}f$-element in the *initial bond graph* has to be replaced by a $\mathbb{S}e\mathbb{S}f$-element imposing both null flow and effort in the *optimizing bond graph*.

By following these modifications and applying the four steps previously described in Section 2, we obtain the *augmented bond graph* model in Fig. 2. The analytical exploitation of this causal *augmented bond graph* will show that it leads to the same result as (9).

3.3 Approach 2: optimization before inverse modelling

Now let us proceed to the second approach where the optimization problem is formulated before the inversion.

Bond graph model:



State-space model:

$$\begin{cases} \dot{p}_1 = -\frac{b_1}{m_1}p_1 + \frac{b_1}{m_2}p_2 + k_1 q_1 + F_1 \\ \dot{p}_2 = \frac{b_1}{m_1}p_1 - \frac{b_1}{m_2}p_2 - k_1 q_1 + F_2 \\ \dot{q}_1 = -\frac{1}{m_1}p_1 + \frac{1}{m_2}p_2 \\ \dot{q}_2 = -\frac{k_2}{b_2}q_2 + \frac{1}{b_2}F_2 \end{cases} \tag{10}$$

Output equation:

$$V_1 = \frac{1}{m_1}p_1 \tag{11}$$

Table 2: Direct model.

In the case of the example, this is the same as starting with the direct bond graph model (Tab. 2) governed by the state-space equations (10) and the output equation (11). On the one hand, meeting specification n°1 can be reformulated as satisfaying the following constraint:

$$V_{1_{ref}} = \frac{1}{m_1}p_1 \tag{12}$$

On the other hand, meeting specification n°2 can be obtained by solving the following dissipative energy minimization problem: *'find the inputs F_1 and F_2 such that the performance index $V = \frac{1}{2}\int_{t_0}^{t_f}(\frac{F_2^2}{R_{F_2}} + P_{diss,b_2})dt$ is minimized when initial and final conditions on state and time are fixed and the state-space equations* (10) *and the output equation* (12) *are taken as constraints'*. In fact, as the unknown F_1 is determined by both inversion and optimization, the space of solutions for F_1 has to be restricted to the space where the inversion constraint (12) is satisfied. However, as this constraint (12) is algebraic, the Pontryagin Maximum Principle is no longer available. One has to consider the *optimality conditions* given by the Euler-Lagrange conditions (13) to solve the problem [15].

$$\begin{cases} \frac{\partial F_a}{\partial \mathbf{x}} - \frac{d}{dt}\frac{\partial F_a}{\partial \dot{\mathbf{x}}} = 0 \\ \frac{\partial F_a}{\partial \boldsymbol{\lambda}} - \frac{d}{dt}\frac{\partial F_a}{\partial \dot{\boldsymbol{\lambda}}} = 0 \\ \frac{\partial F_a}{\partial \mathbf{u}_{opt}} - \frac{d}{dt}\frac{\partial F_a}{\partial \dot{\mathbf{u}}_{opt}} = 0 \end{cases} \tag{13}$$

where $\boldsymbol{\lambda} \in \mathbb{R}^q$ is the co-state vector, q is the number of constraints, $\phi_i(\mathbf{x},\mathbf{u},t) = 0$ is the i^{th} constraint and $F_a = \mathscr{L}(\mathbf{x},\mathbf{u},t) + \sum_{i=1}^{q}\lambda_i\phi_i(\mathbf{x},\mathbf{u},t)$ is called the *augmented function*.

In the case of the example, applying the Euler-Lagrange conditions leads to the system (14).

$$\begin{cases} \frac{\partial F_a}{\partial \lambda_1} - \frac{d}{dt}\frac{\partial F_a}{\partial \dot{\lambda}_1} = 0 \\ \frac{\partial F_a}{\partial \lambda_2} - \frac{d}{dt}\frac{\partial F_a}{\partial \dot{\lambda}_2} = 0 \\ \frac{\partial F_a}{\partial \lambda_3} - \frac{d}{dt}\frac{\partial F_a}{\partial \dot{\lambda}_3} = 0 \\ \frac{\partial F_a}{\partial \lambda_4} - \frac{d}{dt}\frac{\partial F_a}{\partial \dot{\lambda}_4} = 0 \\ \frac{\partial F_a}{\partial \lambda_5} - \frac{d}{dt}\frac{\partial F_a}{\partial \dot{\lambda}_5} = 0 \\ \frac{\partial F_a}{\partial p_1} - \frac{d}{dt}\frac{\partial F_a}{\partial \dot{p}_1} = 0 \\ \frac{\partial F_a}{\partial p_2} - \frac{d}{dt}\frac{\partial F_a}{\partial \dot{p}_2} = 0 \\ \frac{\partial F_a}{\partial q_1} - \frac{d}{dt}\frac{\partial F_a}{\partial \dot{q}_1} = 0 \\ \frac{\partial F_a}{\partial q_2} - \frac{d}{dt}\frac{\partial F_a}{\partial \dot{q}_2} = 0 \\ \frac{\partial F_a}{\partial F_1} - \frac{d}{dt}\frac{\partial F_a}{\partial \dot{F}_1} = 0 \\ \frac{\partial F_a}{\partial F_2} - \frac{d}{dt}\frac{\partial F_a}{\partial \dot{F}_2} = 0 \end{cases} \Rightarrow \begin{cases} \dot{p}_1 = -\frac{b_1}{m_1}p_1 + \frac{b_1}{m_2}p_2 + k_1 q_1 + F_1 \\ \dot{p}_2 = \frac{b_1}{m_1}p_1 - \frac{b_1}{m_2}p_2 - k_1 q_1 + F_2 \\ \dot{q}_1 = -\frac{1}{m_1}p_1 + \frac{1}{m_2}p_2 \\ \dot{q}_2 = -\frac{k_2}{b_2}q_2 + \frac{1}{b_2}F_2 \\ V_{1_{ref}} = \frac{1}{m_1}p_1 \\ \dot{\lambda}_1 = \frac{b_1}{m_1}\lambda_1 - \frac{b_1}{m_1}\lambda_2 + \frac{1}{m_1}\lambda_3 - \frac{1}{m_1}\lambda_5 \\ \dot{\lambda}_2 = -\frac{b_1}{m_2}\lambda_1 + \frac{b_1}{m_2}\lambda_2 - \frac{1}{m_2}\lambda_3 \\ \dot{\lambda}_3 = -k_1\lambda_1 + k_1\lambda_2 \\ \dot{\lambda}_4 = \frac{k_2^2}{b_2}q_2 + \frac{k_2}{b_2}\lambda_4 - \frac{k_2}{b_2}F_2 \\ \lambda_1 = 0 \\ (\frac{1}{R_{F_2}} + \frac{1}{b_2})F_2 = \frac{k_2}{b_2}q_2 + \lambda_2 + \frac{1}{b_2}\lambda_4 \end{cases} \tag{14}$$

Then, once this first step has been carried out, specification n°1 can be met by inversion with respect to the couple (F_1, V_1). Proceeding to such an inversion and substituting λ_1 by its value, then enables us to conclude that, according to approach 2, finding the solution $(p_1, p_2, q_1, q_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, F_1, F_2)$ amounts to solving the following systems:

$$\begin{cases} p_1 = m_1 V_{1_{ref}} \\ \dot{p}_2 = b_1 V_{1_{ref}} - \frac{b_1}{m_2}p_2 - k_1 q_1 + F_2 \\ \dot{q}_1 = -V_{1_{ref}} + \frac{1}{m_2}p_2 \\ \dot{q}_2 = -\frac{k_2}{b_2}q_2 + \frac{1}{b_2}F_2 \\ \dot{\lambda}_2 = \frac{b_1}{m_2}\lambda_2 - \frac{1}{m_2}\lambda_3 \\ \dot{\lambda}_3 = k_1\lambda_2 \\ \dot{\lambda}_4 = \frac{k_2^2}{b_2}q_2 + \frac{k_2}{b_2}\lambda_4 - \frac{k_2}{b_2}F_2 \\ F_1 = m_1\dot{V}_{1_{ref}} + b_1 V_{1_{ref}} - \frac{b_1}{m_2}p_2 - k_1 q_1 \\ (\frac{1}{R_{F_2}} + \frac{1}{b_2})F_2 = \frac{k_2}{b_2}q_2 + \lambda_2 + \frac{1}{b_2}\lambda_4 \end{cases} \tag{15}$$

$$\begin{cases} \lambda_1 = 0 \\ \lambda_5 = -b_1\lambda_2 + \frac{m_1}{m_3}\lambda_3 \end{cases} \tag{16}$$

It can be noticed that, since λ_1 and λ_5 each appear in only one equation, solving them is totally independent from solving the other variables. In fact, only solving the system (15) is sufficient to give the answer to the sizing problem under consideration and to determine the optimal values of the variables p_1, p_2, q_1, q_2, F_1 and F_2.

Finally, one can note that this system (15) is identical to the system (9) of *optimality conditions* obtained by approach 1. In fact, if some differences between the signs of the equations are noticed, this is only due to the fact that the variables λ_i are defined to within a $+$ or $-$ sign between the Pontryagin Maximum Principle and the Euler-Lagrange method. As a consequence, the translation of the system (15) into the bond graph language will be the same as in Fig. 2. An equivalent but simplified *augmented bond graph* is possible too and is shown in Fig. 3.

3.4 Discussion

The previous subsections have shown the two possible approaches for coupling inversion and dynamic optimization on the specific example of two masses in series. Let us discuss here some features of each approach and draw a brief comparison.

At first sight, approach 2 consisting of optimizing the direct model and then inverting it seems to be more complicated than approach 1. In fact, this approach requires to *a priori* anticipate the inversion by considering an additional constraint (such as one of the output equations of the direct model) for the optimization problem. Without such an anticipation, there is no guarantee that the solutions found will satisfy the inversion specifications and thus meet the requirements of the sizing problem. Even if this task is relatively easy for the example dealt with in this article, this can reveal itself not so simple in the general case. Moreover, taking into account additional constraints implies the introduction of additional variables (*e.g.* λ_1 and λ_5 in the example) and so of additional *optimality conditions* (*e.g.* the system (16) in the example) to solve the problem. For sizing problems with numerous specifications and where it is necessary to consider many constraints, this approach can thus appear as less efficient, causing the solving of a non-simplified system of *optimality conditions*. The number of unknowns will *a priori* be greater than with approach 1 and will then require more calculation time. Here, the example of two masses in series is a very specific case where solving the additional variables is totally independent of solving the

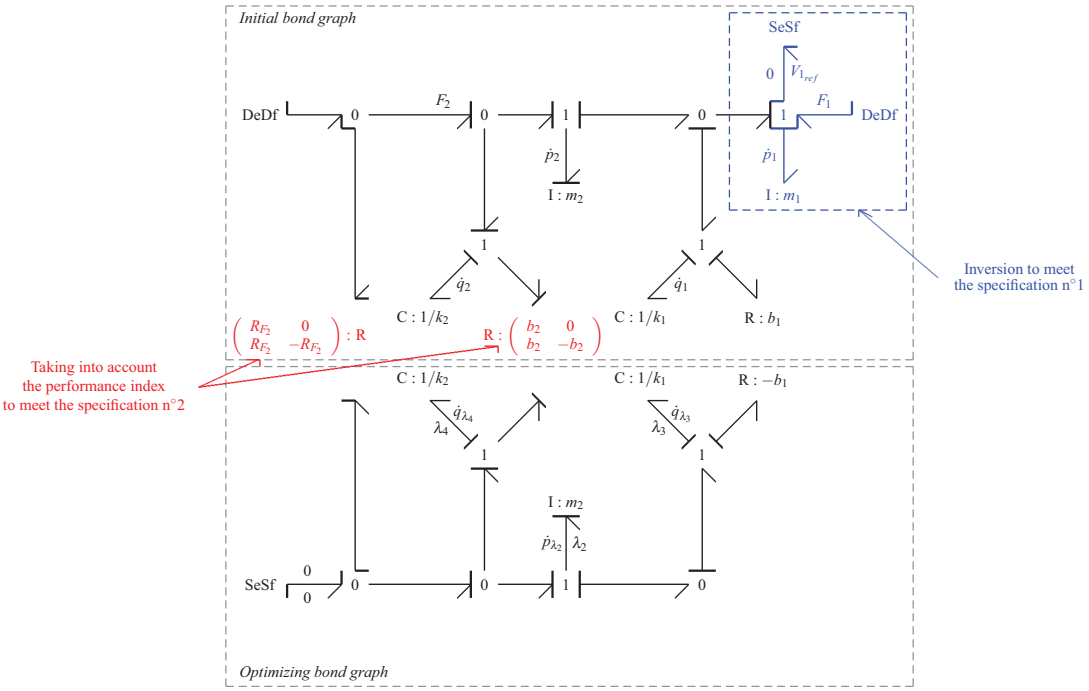


Figure 3: Simplified *augmented bond graph model* coupling both inversion and optimization.

other variables and where the simplication of the *optimality conditions* can be carried out manually. In the general case, this simplication will not be so explicit. In particular, further studies have to be made on other examples to conclude if these two approaches are really equivalent in the general case or not.

Now, compared to approach 2, approach 1 consisting of inverting the model and then optimizing it, appears not only to be easier but more intuitive too: once the model is inverted, one can proceed to optimization without anticipating anything. If the optimization problem is formulated on the remaining dynamic part of the model, it automatically takes into consideration the constraints due to the inversion without introducing any additional variable λ_i. Moreover it enables us to check, in one sense, if the sizing problem is well-posed or not:

- if all of the inputs are entirely determined by the specified outputs, it means that the model has been totally inverted. Adding a sizing specification expressed in an optimization problem form will amount to obtaining an over-constrained problem with no solution.
- if some inputs remain undetermined on the inverted model, it means that the problem is under-constrained and that it is possible to add some sizing specifications.

Finally, it is worth noting that, even if such a coupling is appealing, the two approaches present the same ambiguity: no indication is given to determine according to which couple of variables the inversion has to be made to give *a posteriori* the smallest performance index value.

4 Numerical results

In the specific example of two masses in series, the previous sections have proved that approaches 1 and 2 lead to the same system (9) of *optimality conditions*. This section will handle more specifically the numerical solving of such equations.

In fact, if the initial and final conditions on state and time are fixed by the initial sizing problem, one can note that no *a priori* information is given on the initial and final values of the co-state vector $\boldsymbol{\lambda}$. Unfortunately, since the system of *optimality conditions* is a set of differential-algebraic equations, its solving requires at least the initial value of the co-state vector.

If the inputs are replaced by their expressions which are functions of the state and the co-state vectors, the system of the *optimality conditions* can, however, be written in the following form:

$$\begin{pmatrix} \dot{\mathbf{x}}(t) \\ \dot{\boldsymbol{\lambda}}(t) \end{pmatrix} = \mathbf{A} \begin{pmatrix} \mathbf{x}(t) \\ \boldsymbol{\lambda}(t) \end{pmatrix} + \mathbf{B}\mathbf{u}_{ref}(t) \tag{17}$$

As this system is linear and time-invariant, its analytical solving results in [4]:

$$\begin{aligned}\begin{pmatrix}\mathbf{x}(t)\\ \boldsymbol{\lambda}(t)\end{pmatrix} &= exp^{\mathbf{A}(t-t_0)}\begin{pmatrix}\mathbf{x}(t_0)\\ \boldsymbol{\lambda}(t_0)\end{pmatrix} + \int_{t_0}^{t} exp^{\mathbf{A}(\tau-t_0)}\mathbf{B}\mathbf{u}_{ref}(\tau)\,d\tau \\ &= \begin{pmatrix}\mathbf{M}_1(t) & \mathbf{M}_2(t)\\ \mathbf{M}_3(t) & \mathbf{M}_4(t)\end{pmatrix}\begin{pmatrix}\mathbf{x}(t_0)\\ \boldsymbol{\lambda}(t_0)\end{pmatrix} + \begin{pmatrix}\mathbf{N}_1(t)\\ \mathbf{N}_2(t)\end{pmatrix}\end{aligned} \tag{18}$$

Then, taking into account the fixed conditions on the initial and final state leads to the following equation:

$$\mathbf{x}(t_f) = \begin{pmatrix}\mathbf{M}_1(t_0) & \mathbf{M}_2(t_0)\end{pmatrix}\begin{pmatrix}\mathbf{x}(t_0)\\ \boldsymbol{\lambda}(t_0)\end{pmatrix} + \mathbf{N}_1(t_f) \tag{19}$$

Thus, as already shown in [2], it can be deduced that the initial value of the co-state vector is given in an analytical manner by:

$$\boldsymbol{\lambda}(t_0) = \mathbf{M}_2^{-1}(t_0)(\mathbf{x}(t_f) - \mathbf{M}_1(t_0)\mathbf{x}(t_0) - \mathbf{N}_1(t_f)) \tag{20}$$

This method has in particular been applied to the case of the example of two masses in series. Tab. 3 sums up the several numerical values fixed by the sizing problem and used to parametrize the model. Note on that subject that the sizing problem can not impose arbitrary initial and final conditions on state: these values have to be coherent with the given specificied outputs (here only the values of p_1 have to match with $V_{1_{ref}}$, the initial and final values of the other state variables can be chosen arbitrarily).

Model parameters:	**Optimization problem parameters:**
$\begin{cases} m_1 = 10.0 \text{ kg} \\ m_2 = 5.0 \text{ kg} \\ k_1 = 1.0 \text{ N/m} \\ k_2 = 2.0 \text{ N/m} \\ b_1 = 0.1 \text{ N/(m.s}^{-1}\text{)} \\ b_2 = 0.4 \text{ N/m} \end{cases}$	$\begin{cases} t_0 = 0 \text{ s} \\ t_f = 5 \text{ s} \\ R_{F_2} = 10.0 \\ V_{1_{ref}}(t) = \sin(t) \text{ m.s}^{-1} \end{cases}$
State initial conditions:	**State final conditions:**
$\begin{cases} p_{1_0} = m_1 V_{1_{ref}}(t_0) = 0 \text{ kg.m.s}^{-1} \\ p_{2_0} = 0.5 \text{ kg.m.s}^{-1} \\ q_{1_0} = 0.4 \text{ m} \\ q_{2_0} = 0.3 \text{ m} \end{cases}$	$\begin{cases} p_{1_f} = m_1 V_{1_{ref}}(t_f) \approx -9.58924 \text{ kg.m.s}^{-1} \\ p_{2_f} = 1.0 \text{ kg.m.s}^{-1} \\ q_{1_f} = 2.5 \text{ m} \\ q_{2_f} = 5.0 \text{ m} \end{cases}$

Table 3: Parameters for the example of two masses in series.

Because of the difficulties announced in [11] to calculate exponentials of matrices, it has been preferred to determine the values of the matrices $\mathbf{M}_1(t_0)$, $\mathbf{M}_2(t_0)$ and $\mathbf{N}_1(t_f)$ by simulation as detailed in [10]. Thanks to the MS1 software (a software which supports, among another things, the bond graph language) [12], these simulations have enabled us to obtain:

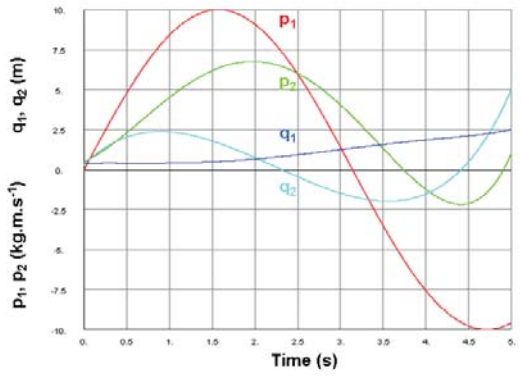
$$\begin{cases} \lambda_{2_0} = -1.75168 \text{ m.s}^{-1} \\ \lambda_{3_0} = -3.02466 \text{ N} \\ \lambda_{4_0} = -1.55337 \text{ N} \end{cases} \tag{21}$$

After having injected these initial values, the simulation of the *augmented bond graph* shown in Fig. 3 (and then of the system (9)) has been carried out on MS1. Fig. 4 thus presents the evolution of the state variables, the inputs and the two variables representing the initial specifications: V_1 and P_{diss,b_2}.

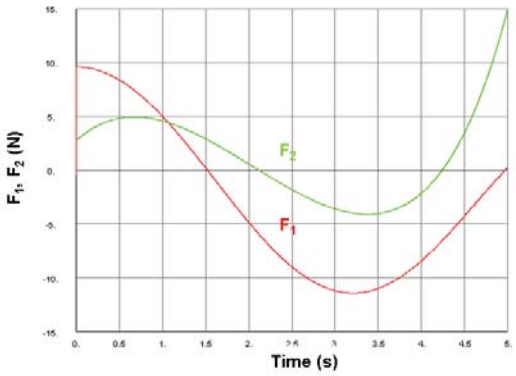
It can in particular be checked that the state variables reach the specified final values at the time t_f (Fig. 4(a)) and that specification n°1 is well satisfied since the speed V_1 of the first mass perfectly follows the sinusoidal form of $V_{1_{ref}}$ (Fig. 4(c)).

5 Conclusion

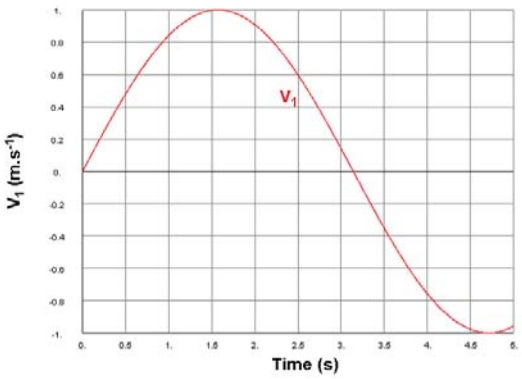
Up to now the optimization procedure has only been carried out on direct models. The performance index was minimized by taking into account the state-space equations of the model and then by solving the *optimality conditions* given by the Pontryagin Maximum Principle. Nothing *a priori* forbids considering constraints representing the state-space equations of an inverse model. Mathematically, in both cases (inverse models or direct models), it amounts to taking into account a set of differential-algebraic equations. The coupling of dynamic optimization with inverse modelling is then legitimate. This is particularly relevant in the sizing context. Coupling the sizing



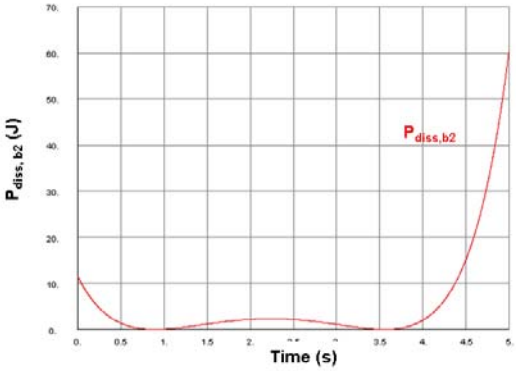
(a) State variables.



(b) Inputs.



(c) Verification of the specification n°1.



(d) Verification of the specification n°2.

Figure 4: Numerical results for the approach 1.

methodology based on the use of inverse models to the optimization procedure has the advantages of the two approaches by limiting their drawbacks. In fact the optimization procedure enables the handling of design constraints not expressed as functions of time, while the inversion gives a stronger constraint in term of trajectory tracking (compared to a problem minimizing the error between the output and the specified trajectory on this output). It then enables us to handle some very general sizing problems where the specifications are of a heterogeneous kind.

This article presents the first research carried out on treating this type of coupling. The first challenge was in particular to adapt the optimizing procedure to the case of inverse models. The adaptation has been made on the example of two masses in series. This has allowed some conjectural modifications on the construction of *optimizing bond graphs* for inverse models. In fact, the only difference with the optimization procedure for direct models lies in the fact that:

- the *optimizing bond graph* is only formed by the remaining dynamic part of the initial inverse model;
- each $\mathbb{S}e\mathbb{S}f$-element (resp. $\mathbb{D}e\mathbb{D}f$-element) in the *initial bond graph* is replaced by a $\mathbb{D}e\mathbb{D}f$-element (resp. a $\mathbb{S}e\mathbb{S}f$ imposing both null flow and effort) in the *optimizing bond graph*.

Of course, the extension of this optimization procedure has to be demonstrated theoretically to check if these modifications are still available for the general case.

Besides, not only the *optimality conditions* have been successfully translated in term of an *augmented bond graph*, but some numerical results have been given to prove the feasibility of such a coupling. The simulation of the resulting *augmented bond graph* has enabled the determination of the optimal controls which match the specifications given by the sizing problem.

From a theoretical point of view, the *optimality conditions* have been obtained according to two different approaches: the first one where the model is inverted before being optimized and the second one where, on the contrary, the optimization problem is formulated before the inversion. The need to separate the inversion from the formulation of the optimization problem is obviously due to a human point of view where the analytical calculus has to be carried out in a chronological order. In reality, the numerical solvings of the equations resulting from the

inversion and of those issued from the optimization are performed simultaneously. In the example of two masses in series, the two approaches lead to the same system of *optimality conditions* even if the second approach (consisting of optimizing and then inverting) seems to be more complicated to carry out. However, this example can reveal itself as a very specific problem and further cases have to be studied to conclude if these two approaches are really equivalent in the general case. Anyway, the first approach appears as more intuitive and more pertinent in the sense that it allows us to check if the sizing problem is well-posed or not: if the model is entirely inverted (*i.e.* the totality of its inputs are determined by the specified outputs), no sizing specification can be added whereas if the model is partially inverted (*i.e.* some inputs remain undetermined after the inversion), sizing specifications such as optimization problems can be added.

Finally, it is worth underlying that this article constitutes the first step towards the coupling between dynamic optimization and inverse modelling. In this way it deserves further development concerning the comparison between the theoretical approaches (are they really equivalent ?), the extended optimization procedure (is it still valid in the general case ?), the numerical solving (how to solve the case of non-linear systems where no analytical solution is given to determine the initial co-state ?) and the applications (the example dealt with in this article concerns an optimal control problem but the problem of generating specifications can be envisaged too).

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