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BLOCK BASED PHYSICAL MODELS DERIVED FROM BOUNDARY VALUE PROBLEMS

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Abstract. Block based modeling of physical systems with distributed parameters requires to specify the interconnection of component blocks by the boundary conditions of partial differential equations. This contribution shows how to build a model of a distributed parameter system that can be connected to similar blocks in a variable way. The description is based on a behavioral model without predefined inputs and outputs. The specific boundary conditions which are imposed by a connection to other systems are realized by appropriate interconnection structures without changing the distributed parameter model.

1 Introduction

Practical simulation programs for complex systems usually consist of a library of component blocks and of a software environment for the construction of system models from these components. This software environment may be a simulation language or – more comfortable – a graphical user interface for the arrangement and connection of various components. The construction of component blocks requires considerable expertise in science, engineering, and mathematics. This knowledge is "canned" in a component block and is at the service of the user who can rely on the correct implementation whithout knowing about its inner life. Thus the task of simulation of complex systems is separated between two groups of experts, here called the *block designers* who build all the component blocks in the library and the *model builders* who select and connect the appropriate components to form a complex system.

This approach has been successfully applied mostly for component blocks with clearly defined inputs and outputs and for systems described by "pipe models" of different kinds (hydraulic, electric, logistic, etc.). The situation is different when the single components are described by differential equations for potential and flow variables. Since there are no predefined inputs and outputs, the connections between different components are defined in terms of port variables for e.g. voltage and current or pressure and flow. For distributed parameter systems described by partial differential equations defined on a finite domain, the connection to other system components is given by boundary conditions.

Here arises a fundamental problem for the construction of flexible simulation environments: Mathematical rigor requires that the description of a distributed parameter system is given by a properly posed problem, i.e. that the boundary conditions are specified. However in a simulation environment the block designer has to implement a numerical model for unspecified boundary conditions. These are given later by the model builder through connection to other blocks.

Possible solutions to this problem have been discussed lately in specific applications fields like acoustic simulation [1, 2, 3] and virtual musical instruments [4, 5, 6, 7]. More formal approaches from the perspective of control theory are port-Hamiltonian systems [8] and the behavioral approach [9]. The interconnection of transmission lines and electrical circuits has also been a topic of classical circuit theory, where descriptions by impedances, admittances, etc. for voltage and flow representations or by scattering matrices for wave variable representations have been introduced [10, 11]. The wave variable representation has been carried to the discrete-time domain by the wave digital principle [12].

It seems that the relations between these different theoretical approaches and other practical solutions for special cases have not yet been fully explored. This contribution presents a small step in this direction by linking the general idea of the behavioral approach to some well-known tools from circuit theory and digital signal processing. The following section introduces the idea of block based physical modeling. It shows the connection between the behavior of a distributed parameter system and possible input output assignments induced by the boundary conditions. Furthermore the relations to the well known two-port parameters from circuit theory are shown. Section 3 presents the implications between block design and model building, i.e. the design of a component block for later use in a larger model. It is shown that a block for a distributed parameter system can be designed according to standard boundary conditions and that the boundary conditions can be modified subsequently without opening up the component block. This method is extended to include not only effort and flow variables but also wave variables. It is explained how component blocks designed for effort and flow variables can be connected to those designed with wave variables. Finally Section 4 shows how to obtain a multiport description from the formulation of a distributed parameter system as a boundary value problem.

2 Block based physical modeling

2.1 Transfer Function Model

As an example for a physical system consider a spatially 1D object (a pipe, a transmission line, a string, etc.) with a pair of port variables at each end (Fig. 1 left). The variables Q_{nm} are pairs of effort/flow quantities (across and through variables, intensive and extensive quantities), i.e. deflection/force, pressure/velocity, voltage/current etc. A physical analysis involves typically the determination of two of these quantities (the outputs) in dependence on the other two (the inputs). The selection of the inputs and outputs is not unique and depends on the problem. If the quantities Q_{11} and Q_{12} are arbitrarily chosen as inputs and Q_{21} and Q_{22} as outputs then description by Laplace transfer functions $H(s)$ (for linear systems) takes the form

$$\begin{aligned} Q_{21}(s) &= H_{11}(s)Q_{11}(s) + H_{12}(s)Q_{12}(s), \\ Q_{22}(s) &= H_{21}(s)Q_{11}(s) + H_{22}(s)Q_{12}(s). \end{aligned} \tag{1}$$

To avoid this arbitrary assignment of inputs and outputs, an implicit formulation may be chosen as (Fig. 1 right)

$$\mathbf{MQ} = \mathbf{0}, \quad \text{with} \quad \mathbf{M} = \begin{bmatrix} H_{11} & H_{12} & -1 & 0 \\ H_{21} & H_{22} & 0 & -1 \end{bmatrix}, \quad \mathbf{Q} = [Q_{11}(s) \;\; Q_{12}(s) \;\; Q_{21}(s) \;\; Q_{22}(s)]^{\mathrm{T}} \tag{2}$$

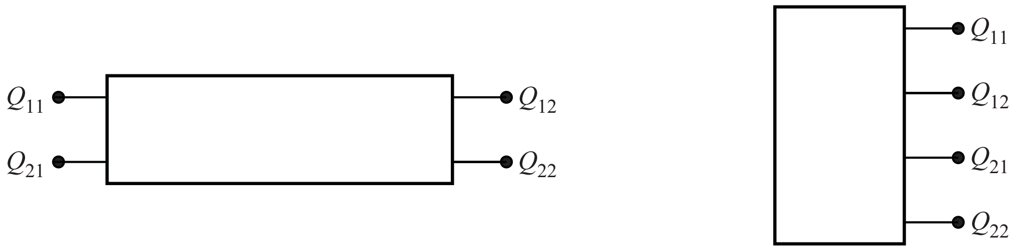

Figure 1: Two-port description (left) and implicit multi-port representation (right). Each pair of variables Q_{1m} and Q_{2m} is a pair of effort/flow quantities at port m for $m = 1, 2$. The nature of each variable (effort or flow) and the role as input or output is not specified at this point.

In (2) the elements of the vector $\mathbf{Q}$ represent the physical variables that are connected to the outside world. They are neither inputs nor outputs. The set of all possible values for $\mathbf{Q}$ is restricted by $\mathbf{M}$ to the set of values that the system admits (behavior in the sense of [9]).

For spatially distributed systems in arbitrary domains D, the matrix $\mathbf{M}$ is typically derived from a partial differential equation of the general form

$$L\{\mathbf{q}(\mathbf{x},t)\} + \frac{\partial}{\partial t}\mathbf{q}(\mathbf{x},t) = \mathbf{0} \qquad \mathbf{x} \in D\,. \tag{3}$$

L is a matrix operator consisting of constants and spatial derivatives and $\mathbf{q}(\mathbf{x},t)$ is a vector of physical variables. For simplicity it is assumed that no physical variables have been eliminated in the process of the physical analysis, such that the highest oder of the spatial derivatives in L is one.

For linear systems the analysis is simplified by considering the one-sided Laplace transforms $\mathbf{Q}(\mathbf{x},s)$ of the variables $\mathbf{q}(\mathbf{x},t)$. The initial conditions are assumed to be zero, since the connection of modeling blocks is ony determined by the boundary conditions.

The example shown in (1) describes a system with two ports each with two port variables. However, the matrix notation

$$\mathbf{MQ} = \mathbf{0} \tag{4}$$

represents also the general case of m ports with two port variables each. Then the vector $\mathbf{Q}$ of port variables has length $2m$ and the matrix $\mathbf{M}$ is of the size $m \times 2m$. Further generalizations to ports with more than two port variables are possible, e.g. for the three-phase transmission of electrical power.

2.2 Input-Output Assignment

A computable model requires to obtain output data from input data. Therefore, the algorithmic design of the component blocks starts with assigning input and output signals to the physical variables in $\mathbf{Q}$. The concept of a *well-defined n-port* from classical circuit theory [10] requires that the variables for each port are equally divided into inputs and outputs. As a trivial consequence, a port with two variables has one input and one output.

Thus for a system according to (4) there are m input variables $V(\mathbf{x},s)$ and m output variables $Y(\mathbf{x},s)$, where $V(\mathbf{x},s)$ and $Y(\mathbf{x},s)$ are vectors of length m. They are defined on the boundary ∂D of the domain D for the distributed parameter system from (3). For spatially one-dimensional systems, D is an interval on the spatial axis and ∂D are its two end points.

The assignment of $V(\mathbf{x},s)$ and $Y(\mathbf{x},s)$ to $Q(\mathbf{x},s)$ is defined by certain linear combinations of the elements of $Q(\mathbf{x},s)$ as inputs and other linear combinations as outputs. These linear combinations are given by two matrices of size $m \times 2m$, the boundary matrix $\mathbf{f}_\mathrm{b}$ and the output matrix $\mathbf{f}_\mathrm{o}$ (see Fig. 2 left)

$$\mathbf{f}_\mathrm{b}^\mathrm{T}\mathbf{Q}(\mathbf{x},s) = \mathbf{V}(\mathbf{x},s), \qquad \mathbf{x} \in \partial D, \tag{5}$$

$$\mathbf{f}_\mathrm{o}^\mathrm{T}\mathbf{Q}(\mathbf{x},s) = \mathbf{Y}(\mathbf{x},s), \qquad \mathbf{x} \in \partial D. \tag{6}$$

Although these equations look very similar, there is a fundamental difference between $\mathbf{f}_\mathrm{b}$ and $\mathbf{f}_\mathrm{o}$: The output matrix $\mathbf{f}_\mathrm{o}$ simply computes the output $\mathbf{Y}$ from $\mathbf{Q}$ once $\mathbf{Q}$ is available. This is not the case for the boundary matrix $\mathbf{f}_\mathrm{b}$, since $\mathbf{V}$ is an input and needs not to be computed. Instead $\mathbf{f}_\mathrm{b}$ describes the boundary conditions which restrict the behaviour $\mathbf{Q}$ to those values that satisfy (5). The fullfillment of the boundary conditions is part of the numerical method for solving the partial differential equation (3).

The simplest choice for $\mathbf{f}_\mathrm{b}$ and $\mathbf{f}_\mathrm{o}$ is the selection of a subset of the port variables $\mathbf{Q}$ as inputs and the remainder as outputs (see Fig. 2 right). The corresponding boundary and output operators are in the standard form

$$\mathbf{f}_\mathrm{b} = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix}, \qquad \mathbf{f}_\mathrm{o} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}, \tag{7}$$

where $\mathbf{I}$ and $\mathbf{0}$ are identity and zero matrices of size $m \times m$.

Fig. 2 (left) shows general boundary and output operators for the two-port representation of Fig. 1. In Fig. 2 (right) the boundary and output operators have the standard form according to (7).

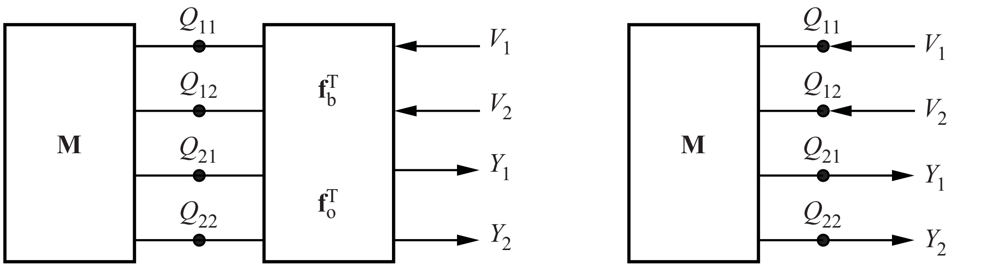

Figure 2: General form of the boundary and output operators (left) and standard form from (7) (right).

2.3 Two-Port Parameters

For electrical systems of with two ports, the matrix $\mathbf{M}$ describing its behaviour is closely related to the two-port parameters from classical network theory [10, 11]. As an example, chose the following assignment of voltage U_n (effort) and current J_n (flow) variables

$$\begin{aligned} Q_{11} &= J_1, & Q_{12} &= J_2, \\ Q_{21} &= U_1, & Q_{22} &= U_2, \end{aligned} \qquad \mathbf{U} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}. \tag{8}$$

Partitioning $\mathbf{M}$ accordingly yields from (2)

$$\mathbf{MQ} = \begin{bmatrix} \mathbf{M}_J & -\mathbf{M}_U \end{bmatrix} \begin{bmatrix} \mathbf{J} \\ \mathbf{U} \end{bmatrix} = \mathbf{0}, \tag{9}$$

where the matrices $\mathbf{M}_U$ and $\mathbf{M}_J$ correspond to $\mathbf{U}$ and $\mathbf{J}$, respectively.

Left multiplication with the inverse matrix $\mathbf{M}_U^{-1}$ gives

$$\begin{bmatrix} \mathbf{Z} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{J} \\ \mathbf{U} \end{bmatrix} = \mathbf{0}, \quad \text{or} \quad \mathbf{U} = \mathbf{ZJ} \quad \text{with} \quad \mathbf{Z} = \mathbf{M}_U^{-1}\mathbf{M}_J \tag{10}$$

with the unit matrix $\mathbf{I}$. According to the physical dimensions of the vector of voltages $\mathbf{U}$ and the vector of currents $\mathbf{J}$, the matrix $\mathbf{Z}$ turns out to be a matrix of impedances. Note that (10) holds also for multiport systems with appropriate definition of the vectors $\mathbf{U}$ and $\mathbf{J}$.

The exemplary assignment of two kinds of quantities (voltage or currents) to four different variables according to (8) is only one of six possibilites. In total they result in a description by impedance (as above), admittance, transmission, and hybrid parameters and the inverses of the latter two [10, 11].

3 Block design and model building

This section shows how the framework presented above allows to separate the tasks of the block designer and the model builder. Designing a component block according to some partial differential equation (3) requires to adopt a specific set of boundary and output matrices such that the component block is well defined. However, the model builders who use this component might have different requirements for the boundary conditions.

The first subsection presents a method to modify the boundary conditions for which a component block is designed. The following subsections extend this method from effort and flow variables to wave variables.

3.1 Modification of the boundary conditions

This conflict can be solved when the component block is designed according to Fig. 2. The shaded area in Fig. 3 represents the input-output description from Fig. 2 either in general or in standard form (7). This component block is accessible to the model builder only via the vectors of input and output signals $\mathbf{V}$ and $\mathbf{Y}$. Now assume that the model builder wants to realize a different set of input and output signals $\mathbf{V}_1$ and $\mathbf{Y}_1$ described by

$$\mathbf{f}_{b1}^T \mathbf{Q}(\mathbf{x},s) = \mathbf{V}_1(\mathbf{x},s), \qquad \mathbf{x} \in \partial V, \tag{11}$$
$$\mathbf{f}_{o1}^T \mathbf{Q}(\mathbf{x},s) = \mathbf{Y}_1(\mathbf{x},s), \qquad \mathbf{x} \in \partial V. \tag{12}$$

For easier notation, define the matrices of size $2m \times 2m$

$$\mathbf{F} = \begin{bmatrix} \mathbf{f}_b^T \\ \mathbf{f}_o^T \end{bmatrix} \quad \text{and} \quad \mathbf{F}_1 = \begin{bmatrix} \mathbf{f}_{b1}^T \\ \mathbf{f}_{o1}^T \end{bmatrix}, \tag{13}$$

such that the input-output assignments (5,6) and (11,12) can be written as

$$\mathbf{F}\,\mathbf{Q}(\mathbf{x},s) = \begin{bmatrix} \mathbf{V} \\ \mathbf{Y} \end{bmatrix} \quad \text{and} \quad \mathbf{F}_1\,\mathbf{Q}(\mathbf{x},s) = \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{Y}_1 \end{bmatrix}. \tag{14}$$

Then the inputs and outputs $\mathbf{V}$ and $\mathbf{Y}$ of the component block can be accessed through the input and output signals $\mathbf{V}_1$ and $\mathbf{Y}_1$ required by the model builder via a connection block described by the matrices $\mathbf{F}$ and $\mathbf{F}_1$

$$\mathbf{F}_1 \mathbf{F}^{-1} \begin{bmatrix} \mathbf{V} \\ \mathbf{Y} \end{bmatrix} = \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{Y}_1 \end{bmatrix}. \tag{15}$$

The connection block defined by $\mathbf{F}_1\mathbf{F}^{-1}$ exists if the component block is designed such that matrix $\mathbf{F}$ is invertible. This condition is guaranteed by the standard form (7) which turns $\mathbf{F}$ into a $2m \times 2m$ identity matrix. Then the input and output signals $\mathbf{V}$ and $\mathbf{Y}$ can be directly expressed by $\mathbf{Q}$ such that the new boundary conditions (11,12) apply directly to the accessible port signals $\mathbf{V}$ and $\mathbf{Y}$ (see Fig. 2, right)

$$\mathbf{Q} = \begin{bmatrix} \mathbf{V} \\ \mathbf{Y} \end{bmatrix}, \qquad \begin{bmatrix} \mathbf{f}_{b1}^T \\ \mathbf{f}_{o1}^T \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{Y} \end{bmatrix} = \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{Y}_1 \end{bmatrix}. \tag{16}$$

As shown in Fig. 3 the desired boundary conditions can be realized without changing the existing component block (shaded area).

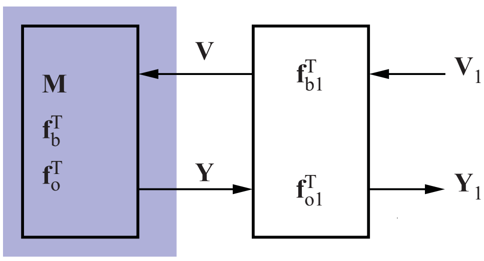

Figure 3: Realization of arbitrary boundary conditions based on a given component block defined by $\mathbf{f}_b$ and $\mathbf{f}_o$ (shaded area) and a connection block defined by $\mathbf{f}_{b1}$ and $\mathbf{f}_{o1}$ (not shaded).

3.2 Changing the Port Impedances

As an example consider a component block where the port variables are linked by a certain impedance (admittance, etc.), as shown in Sec. 2.3 for a simple two-port. Here the situation is more general, since more than two ports are permitted ($m \geq 2$) and the input and output signals are not restricted to a certain assignment of effort and flow variables.

With reference to Fig. 3, a component block is given by the relation

$$\begin{bmatrix} \mathbf{Z} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{Y} \end{bmatrix} = \mathbf{0} \tag{17}$$

When **V** and **Y** represent flow and effort quantities, respectively, then **Z** is an impedance matrix. For other assignments of **V** and **Y**, **Z** is an admittance, transmission, etc. matrix. Since all these matrix representations can be converted to each other, **Z** is called here an impedance matrix without special assignments of **V** and **Y**. Further, assume that the assignment of inputs and outputs to the physical variables in the component block follows the standard form (7), i.e. **F** according to (14) is an identity matrix.

Now this component block with the fixed impedance **Z** shall be used in a larger model in lieu of component with the requested impedance $\mathbf{Z}_1$. Therefore a connection block is required which converts the impedance **Z** on the side of the given component block to the requested impedance $\mathbf{Z}_1$ on the other side (see Fig. 3). Such a connection block can be designed according to Sec. 3.1.

The boundary matrix $\mathbf{f}_{\mathrm{b1}}$ and the output matrix $\mathbf{f}_{\mathrm{o1}}$ are chosen such that the matrix $\mathbf{F}_1$ takes the form

$$\mathbf{F}_1 = \begin{bmatrix} \mathbf{f}_{\mathrm{b1}}^{\mathrm{T}} \\ \mathbf{f}_{\mathrm{o1}}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{G} \\ \mathbf{R} & \mathbf{I} \end{bmatrix} . \tag{18}$$

The physical dimensions of the matrix elements in **R** and **G** have to cancel in such a way that the matrix products **RG** and **GR** have the dimension unity.

The inverse of the matrix $\mathbf{F}_1$ follows from the block matrix inversion lemma as

$$\mathbf{F}_1^{-1} = \begin{bmatrix} \mathbf{I} & -\mathbf{G} \\ -\mathbf{R} & \mathbf{I} \end{bmatrix} \begin{bmatrix} (\mathbf{I}-\mathbf{GR})^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I}-\mathbf{RG})^{-1} \end{bmatrix} . \tag{19}$$

Then

$$\begin{bmatrix} \mathbf{V} \\ \mathbf{Y} \end{bmatrix} = \mathbf{F}_1^{-1} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{Y}_1 \end{bmatrix} \tag{20}$$

can be substituted into (17) to obtain

$$\begin{bmatrix} \mathbf{Z} & -\mathbf{I} \end{bmatrix} \mathbf{F}_1^{-1} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{Y}_1 \end{bmatrix} = \mathbf{0} . \tag{21}$$

Performing all the matrix multiplications in (19) and (21) results in

$$\begin{bmatrix} \mathbf{Z}_1 & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{Y}_1 \end{bmatrix} = \mathbf{0} \qquad \text{with} \qquad \mathbf{Z}_1 = (\mathbf{I}-\mathbf{RG})(\mathbf{ZG}+\mathbf{I})^{-1}(\mathbf{Z}+\mathbf{R})(\mathbf{I}-\mathbf{GR})^{-1} . \tag{22}$$

$\mathbf{Z}_1$ is the impedance matrix corresponding to the input and output variables $\mathbf{V}_1$ and $\mathbf{Y}_1$. It depends on the impedance **Z** corresponding to the component block and on the matrices **R** and **G** defining the connection block. By proper choice of **R** and **G**, the fixed impedance **Z** can be turned into the desired impedance $\mathbf{Z}_1$.

For a single port with scalar input and output V and Y, these relations take an especially simple form shown in Fig. 4. The given component block exhibits a fixed impedance Z at the port (V,Y). To change its impedance to a desired value Z_1 at the port (V_1,Y_1), the boundary and output matrices are

$$\mathbf{f}_{b1} = \begin{bmatrix} 1 \\ G \end{bmatrix} , \quad \mathbf{f}_{o1} = \begin{bmatrix} R \\ 1 \end{bmatrix} , \quad \mathbf{F}_1 = \begin{bmatrix} 1 & G \\ R & 1 \end{bmatrix} . \tag{23}$$

resulting in the relations

$$\begin{bmatrix} 1 & G \\ R & 1 \end{bmatrix} \begin{bmatrix} V \\ Y \end{bmatrix} = \begin{bmatrix} V_1 \\ Y_1 \end{bmatrix} \qquad \text{or} \qquad \begin{array}{rcl} V & = & V_1 - GY , \\ Y_1 & = & RV + Y . \end{array} \tag{24}$$

Thus the connection block is realized by the lattice structure in shown in Fig. 4.

Finally, the relation between the impedances takes the form

$$Z_1 = \frac{Z+R}{ZG+1} , \tag{25}$$

where a wide range of desired values for Z_1 can be obtained by adjusting R and G properly.

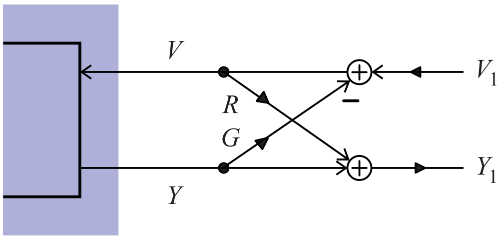

Figure 4: A simple example for changing the port impedances at the ports (V,Y) and (V_1,Y_1).

3.3 Conversion Between a Pair of Effort and Flow Variables and a Pair of Wave Variables

Apart from the representation of physical quantities by effort and flow variables, also the representation by incident and reflected (ingoing and outgoing) waves is frequently used. Well-known discrete-time representations are wave digital filters [12, 13, 6] and digital waveguides in one and more spatial dimensions [14, 15, 3, 4, 5].

Wave variables **A** and **B** are a superposition of effort and flow variables **Y** and **V** at a port with port resistance **R**. This superposition can also be expressed by a matrix $\mathbf{F}_2$ similar to (18)

$$\begin{aligned}\mathbf{A} &= \mathbf{Y}+\mathbf{R}\mathbf{V}\,,\\ \mathbf{B} &= \mathbf{Y}-\mathbf{R}\mathbf{V}\,,\end{aligned} \qquad \begin{bmatrix}\mathbf{A}\\ \mathbf{B}\end{bmatrix} = \begin{bmatrix}\mathbf{R} & \mathbf{I}\\ -\mathbf{R} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{V}\\ \mathbf{Y}\end{bmatrix}, \qquad \mathbf{F}_2 = \begin{bmatrix}\mathbf{R} & \mathbf{I}\\ -\mathbf{R} & \mathbf{I}\end{bmatrix} \tag{26}$$

When **Y** and **V** represent electrical voltage and current (or other pairs of effort and flow variables), then (26) defines so-called voltage waves. For other assignments of the physical quantities, also current waves and power waves have been defined [12]

This matrix notation in (26) is not only a concise notation for the wave variables. It describes also how to connect a component block with effort and flow variables to other blocks with wave variables as inputs and outputs. The corresponding connection block follows from rearranging (26) such that the input signals **V** and **B** on either side are expressed in terms of the output signals **Y** and **A**

$$\begin{aligned}\mathbf{V} &= \mathbf{R}^{-1}(\mathbf{A}-\mathbf{Y})\,,\\ \mathbf{B} &= \mathbf{Y}-\mathbf{R}\mathbf{V}\,,\end{aligned} \tag{27}$$

Fig. 5 shows one port of a given component block with scalar effort and flow variables Y and V (shaded area). It is connected to another port with wave variables A and B. The connection block is realized in the lattice structure corresponding to (27).

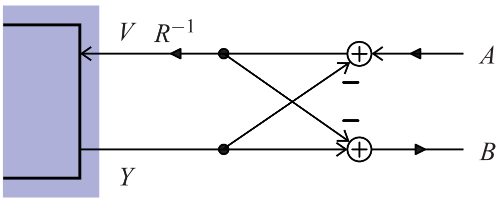

Figure 5: Conversion between a pair of effort and flow variables (V,Y) and a pair of wave variables (A,B).

The connection block from Fig. 5 follows directly from (27) by requiring the component block to interface with wave variables. Thus the formalism with the boundary and output matrix introduced in Sec. 3.1 works not only for effort and flow variables but equally well also for wave variables.

The structure of the connection block had been derived in a different way under the name *KW-converter type I* in [16]. KW stands for Kirchhoff and wave variables, where Kirchoff variables are another name for effort and flow variables.

3.4 Connection of Two Ports with Wave Variables and Different Port Resistances.

The connection of two wave ports with different port resistances requires so-called adaptors to ensure the continuity of efforts and flows also in the wave representation [12, 15]. The framework introduced above allows also to derive the adaptor equations directly from the boundary and output matrices for the definition of the wave variables.

The idea for connecting to different wave ports uses two KW-converters from Sec. 3.3. Instead of connecting one converter according to Fig. 5 to an existing component block, two converters with different port resistances are connected back-to-back. The wave variables at both ends are denoted by $\mathbf{A}_l$ and $\mathbf{B}_l$ for $l = 1,2$ and the respective port restistances are $\mathbf{R}_l$. The corresponding wave definitions are similar to (26)

$$\begin{bmatrix} \mathbf{A}_l \\ \mathbf{B}_l \end{bmatrix} = \begin{bmatrix} \mathbf{R}_l & \mathbf{I} \\ -\mathbf{R}_l & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{V}_l \\ \mathbf{Y}_l \end{bmatrix}, \qquad \mathbf{W}_l = \begin{bmatrix} \mathbf{R}_l & \mathbf{I} \\ -\mathbf{R}_l & \mathbf{I} \end{bmatrix} \qquad \text{for} \qquad l = 1,2 \tag{28}$$

Now the effort and flow variables $\mathbf{V}_l$ and $\mathbf{Y}_l$ are eliminated from (28). The back-to-back connection forces the efforts to be equal and the flows to add up to zero. For electrical circuits, these relations are known as the Kirchhoff laws. They can be expressed in matrix form as

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{Y}_1 \end{bmatrix} = \mathbf{P} \begin{bmatrix} \mathbf{V}_2 \\ \mathbf{Y}_2 \end{bmatrix} \qquad \text{with} \qquad \mathbf{P} = \begin{bmatrix} -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{29}$$

The elimination of $\mathbf{V}_1$, $\mathbf{Y}_1$ and $\mathbf{V}_2$, $\mathbf{Y}_2$ from (28) and (29) results in

$$\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{B}_1 \end{bmatrix} = \mathbf{W}_0 \begin{bmatrix} \mathbf{A}_2 \\ \mathbf{B}_2 \end{bmatrix}, \qquad \text{where} \qquad \mathbf{W}_0 = \mathbf{W}_1 \mathbf{P} \mathbf{W}_2^{-1} = \frac{1}{2} \begin{bmatrix} \mathbf{I} - \mathbf{R}_0 & \mathbf{I} + \mathbf{R}_0 \\ \mathbf{I} + \mathbf{R}_0 & \mathbf{I} - \mathbf{R}_0 \end{bmatrix} \qquad \text{with} \quad \mathbf{R}_0 = \mathbf{R}_1 \mathbf{R}_2^{-1}. \tag{30}$$

This relation between the waves $\mathbf{A}_1$, $\mathbf{B}_1$ and $\mathbf{A}_2$, $\mathbf{B}_2$ can be rearranged in various ways, yielding different classical adaptor structures [12].

Adding and subtracting the two rows in (30) recovers the relations between the efforts $\mathbf{Y}_1$ and $\mathbf{Y}_2$ (equality) and the flows $\mathbf{V}_1$ and $\mathbf{V}_2$ (add up to zero). In [8, Sec. 4.2] these relations have been interpreted as power balance and loss balance, respectively.

4 Formulation of the Boundary Conditions

So far the boundary conditions for the input-output assignment according to (5,6) have been assumed to be known. This section shows how to obtain the boundary conditions and the physical variables $\mathbf{Q}(\mathbf{x},s)$ at the boundary ∂D from the governing partial differential equation. A fairly general case is considered first and then the acoustic wave equation is used as an example.

4.1 General Distributed Parameter Systems

Consider a partial differential equation for the time variable t and the vector of space variables $\mathbf{x} = [x_1\ x_2\ x_3]^T$. It is defined on a spatial domain $\mathbf{x} \in D$. The dependent variables form the vector of physical quantities $\mathbf{q}(\mathbf{x},t)$. It contains a sufficient number of variables such that the partial differential equation contains only first order time and spcae derivatives. The derivatives are combined by coefficient matrices $\mathbf{B}_0 \ldots \mathbf{B}_3$. The physical nature of this equation is determined by the kind of variables in $\mathbf{q}(\mathbf{x},t)$ and the matrix entries in $\mathbf{B}_0 \ldots \mathbf{B}_3$. With this notation the partial differential equation is of the general form

$$\left[\mathbf{B}_0 \frac{\partial}{\partial t} + \mathbf{B}_1 \frac{\partial}{\partial x_1} + \mathbf{B}_2 \frac{\partial}{\partial x_2} + \mathbf{B}_3 \frac{\partial}{\partial x_3}\right] \mathbf{q}(\mathbf{x},t) = \mathbf{0}\,, \qquad L = \mathbf{B}_1 \frac{\partial}{\partial x_1} + \mathbf{B}_2 \frac{\partial}{\partial x_2} + \mathbf{B}_3 \frac{\partial}{\partial x_3}\,. \tag{31}$$

where the operator of spatial derivatives is denoted by L.

Now define the normal component of the spatial differential operator as [17]

$$\mathbf{L}_\mathrm{n} = n_1 \mathbf{B}_1 + n_2 \mathbf{B}_2 + n_3 \mathbf{B}_3\,, \qquad \mathbf{n} = [n_1\ n_2\ n_3]^T \tag{32}$$

with $\mathbf{n}$ is the normal vector on the boundary ∂D. The normal component of the solution on the boundary is then given by

$$\mathbf{q}_\mathrm{n}(\mathbf{x},t) = \mathbf{L}_\mathrm{n} \mathbf{q}(\mathbf{x},t)\,. \tag{33}$$

The boundary conditions are now specified by selecting suitable components from $\mathbf{q}_\mathrm{n}$ or linear combinations thereof through

$$\mathbf{f}_\mathrm{b}^T \mathbf{q}_\mathrm{n}(\mathbf{x},t) = v(\mathbf{x},t) \qquad \mathbf{x} \in \partial D\,. \tag{34}$$

The excitation function $v(\mathbf{x},t)$ determines the values of $\mathbf{q}_\mathrm{n}(\mathbf{x},t)$ on the boundary and acts as an input signal.

The output values $y(\mathbf{x},t)$ at the boundary are given in the same way as

$$\mathbf{f}_\mathrm{o}^T \mathbf{q}_\mathrm{n}(\mathbf{x},t) = y(\mathbf{x},t) \qquad \mathbf{x} \in \partial D\,. \tag{35}$$

The vectors $\mathbf{f}_\mathrm{b}$ and $\mathbf{f}_\mathrm{o}$ must not be collinear.

4.2 Acoustic Wave Equation

The acoustic wave equation on a two-dimensional spatial domain is selected as an example. The physical variables are the the sound pressure $p(\mathbf{x},t)$ and the particle velocity $\mathbf{v}(\mathbf{x},t) = [v_1(\mathbf{x},t) \quad v_2(\mathbf{x},t)]^T$ with its components in x_1 and x_2 direction. The density of the propagation medium is denoted by ρ and c is the propagation speed (speed of sound). The relations between sound pressure (effort) and particle velocity (flow) are governed by the equation of continuity and the equation of motion

$$\begin{aligned} -\frac{\partial}{\partial t}p(\mathbf{x},t) &= \rho c^2 \nabla \mathbf{v}(\mathbf{x},t)\,, \\ -\nabla p(\mathbf{x},t) &= \rho \frac{\partial}{\partial t}\mathbf{v}(\mathbf{x},t)\,, \end{aligned} \tag{36}$$

with the nabla operator ∇. It would be possible to eliminate either the particle velocity or the sound pressure from these two equations to obtain one second order partial differential equation. However, no elimination is performed here to retain the general form of (31). Instead a vector-matrix notation is chosen which represents (36) in the general form of (31). Such a representation is not unique. One possible form is obtained with the following vector of three scalar physical variables and the corresponding coefficient matrices

$$\mathbf{q} = \begin{bmatrix} v_1(\mathbf{x},t) \\ v_2(\mathbf{x},t) \\ -\frac{1}{\rho}p(\mathbf{x},t) \end{bmatrix}, \quad \mathbf{B}_0 = -\begin{bmatrix} 0 & 0 & c^{-2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{37}$$

To evaluate the normal components of the differential operator L on the boundary, a rectangular region with $[0,l_1] \times [0,l_2]$ is chosen as spatial domain D (see Fig. 6). On each side of the rectangle, the normal vector n from (32) points either in the direction of $\pm x_1$ or $\pm x_2$, such that $\mathbf{L}_\mathrm{n}$ is either

$$\mathbf{L}_\mathrm{n} = \pm\mathbf{B}_1 \quad \text{or} \quad \mathbf{L}_\mathrm{n} = \pm\mathbf{B}_2. \tag{38}$$

The corresponding normal component of the solution is then e.g

$$\mathbf{q}_\mathrm{n}(\mathbf{x},t) = \begin{bmatrix} v_1(\mathbf{x},t) \\ -p(\mathbf{x},t)/\rho \\ 0 \end{bmatrix} \quad \text{for} \quad \mathbf{x} = \begin{bmatrix} l_1 \\ x_2 \end{bmatrix} \quad \text{(right vertical boundary)} \quad \text{or} \tag{39}$$

$$\mathbf{q}_\mathrm{n}(\mathbf{x},t) = \begin{bmatrix} v_2(\mathbf{x},t) \\ 0 \\ -p(\mathbf{x},t)/\rho \end{bmatrix} \quad \text{for} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ l_2 \end{bmatrix}. \quad \text{(upper horizontal boundary)} \tag{40}$$

The input-output assignment is now shown for the right vertical boundary (39). The appearance of a zero in the vector $\mathbf{q}_\mathrm{n}$ reflects the reduction by one dimension when restricting the area D to its boundary ∂D. The rectangular shape of D and the alignment with the axes makes this fact explicit here. Therefore $\mathbf{q}_\mathrm{n}$ may also be represented by a vector $\tilde{\mathbf{q}}_\mathrm{n}$ with two non-zero elements, omitting the zero at the bottom of $\mathbf{q}_\mathrm{n}$. Then the conditions

$$\mathbf{f}_\mathrm{b}^T \tilde{\mathbf{q}}_\mathrm{n}(\mathbf{x},t) = v(x_2,t), \qquad \mathbf{f}_\mathrm{o}^T \tilde{\mathbf{q}}_\mathrm{n}(\mathbf{x},t) = y(x_2,t) \tag{41}$$

define the input and output signals. E.g. the choice

$$\mathbf{f}_\mathrm{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{f}_\mathrm{o} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{42}$$

sets the sound pressure on the boundary equal to the input signal $v(x_2,t)$ and selects the component v_1 of the particle velocity as output signal.

In computer implementations of distributed parameter systems it is desirable to prescribe boundary values not on a spatially continuous boundary (e.g. $v(x_2,t)$ for $0 \geq x_2 \geq l_2$) but on a discrete set of points along the boundary. A possible selection of points has been marked in Fig. 6. The input and output values from (41) are then no more parametrized by the spatially continuous x_2 variable, but by an discrete index (say m) of spatial locations. The input and output values at location m constitute a simple port with two port variables. The input and output definition (41) for all spatial samples turns in the description by multiple ports with two port variables each. This is the description that as been adopted in Sec. 2.2. By this example it has been shown how to turn a boundary value problem of a distributed parameter system into a multiport description.

5 Conclusion

In conclusion, it has been shown that the boundary conditions and output definitions of an existing component block of a spatially distributed system can be adapted without changing the interior of the component block. A *block*

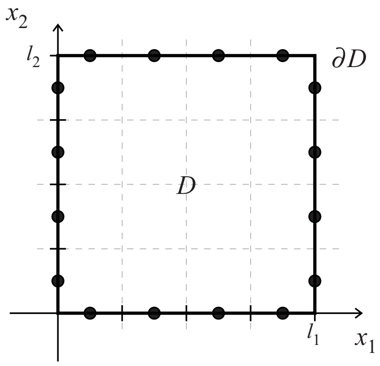

Figure 6: Rectangular region as spatial domain for the acoustic wave equation. The spatial sampling points at the boundary ∂D are denoted by black dots (•).

designer can implement in an early development state pre-defined model descriptions in terms of partial diferential equations, which are well-posed via standard sets of boundary conditions. The *model builder* on the other hand is free to adjust the boundary conditions and model connections in a later stage of the model development using this pre-defined component block models. This feature is very useful in hierarchically structured design processes of advanced simulation environments. A further benefit is the possiblity to pre-optimize the realization of the component block model.

The block design process proposed in this paper is given in a quite general and fundamental form with several links to well known literature in physical modeling techniques. Nonetheless, there are several practical applications of this design process.

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