I. Troch, F. Breitenecker, eds. ISBN 978-3-901608-35-3

λ-SCALE CONVERGENCE APPLIED TO THE STATIONARY HEAT CONDUCTION EQUATION WITH NON-PERIODIC THERMAL CONDUCTIVITY MATRIX

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Abstract. In this contribution we study the homogenization of non-periodic stationary heat conduction problems with homogeneous Dirichlet boundary data by applying the recently developed λ-scale convergence technique developed by Holmbom and Silfver. λ-scale convergence can be seen as either being a special case of scale convergence (developed by Mascarenhas and Toader) or of "generalized" two-scale convergence (developed by Holmbom, Silfver, Svanstedt and Wellander). From either viewpoint, it is a possibly powerful generalization of Nguetseng's classical, periodic two-scale convergence method. We give a definition of the concept of λ-scale convergence, which is then used to claim a main theorem on homogenization of certain non-periodic stationary heat conduction problems. The original part of the contribution starts by defining a two-dimensional "toy model". We show that the "toy model" satisfies the right conditions such that the aforementioned main theorem on the homogenization can be employed. In this way we derive the homogenized problem, i.e. the homogenized thermal conductivity matrix, and the local problem. The contribution is concluded by giving a numerical example where we explicitly compute the homogenized thermal conductivity matrix.

1 Introduction

The concept of homogenization theory, i.e. the theory of the convergence of sequences of partial differential equations, arises naturally from the study of microscale behaviour of systems which can not be examined in a numerically satisfactory manner. When homogenizing a sequence of partial differential equations describing periodical structures one can use a fairly recent technique referred to as two-scale convergence [4]. The two-scale convergence technique has been generalized to so called scale convergence which covers cases with possibly non-periodic structures and a general Young measure on the Cartesian product of the spaces associated with the two different scales [3]. By specifically choosing the Lebesgue measure and employing test functions periodic in the second-scale variable (defined in the unit cube), scale convergence reduces to the special case dubbed λ-scale convergence [1]. Equivalently, we can see λ-scale convergence as a special case of "generalized" two-scale convergence with a fixed operator sequence [2]. Utilizing the λ-scale convergence technique it is possible to homogenize e.g. a sequence of stationary heat conduction equations with homogeneous Dirichlet boundary data, i.e.

$$\begin{cases} -\nabla \cdot \left\{ (A \circ \alpha^h) \nabla u^h \right\} = f & \text{in } \Omega, \\ u^h = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

as $h \to \infty$, where A is a Y-periodic heat conduction matrix, f is a heat source function, and α^h defines a possibly non-periodic structure obeying certain restrictions [6]. We assume from now on that Ω is an open bounded set in $\mathbb{R}^N$ and that Y is the unit cube $(0,1)^M$ in $\mathbb{R}^M$, $M = N$. (The assumption $M = N$ is not necessary in general, but is convenient to work with in the present context.) In this contribution, which is a short revision of the detailed preprint [5] adapted for the MATHMOD 2009 Conference Proceedings, we use the results obtained in [6] to homogenize a simple but illuminating two-dimensional "toy model" stationary heat conduction problem of the type (1).

2 λ-scale convergence

Following [1, 6], the definition of λ-scale convergence is given by Definition 1 below, where the sequence $\{\widetilde{\alpha}^h\}$ of operators $\widetilde{\alpha}^h : L^2(\Omega; C_\#(Y)) \to L^2(\Omega)$ is defined according to $(\widetilde{\alpha}^h v)(x) = v(x, \alpha^h(x))$, $x \in \Omega$.

Definition 1 *Assume that $\{\alpha^h\}$ is a sequence of functions $\alpha^h : \Omega \to Y$. A sequence $\{u^h\}$ in $L^2(\Omega)$ is said to λ-scale converge to $u_0 \in L^2(\Omega \times Y)$ if $\langle u^h, \widetilde{\alpha}^h v \rangle_{L^2(\Omega)} \to \langle u_0, v \rangle_{L^2(\Omega\times Y)}$ for any $v \in L^2(\Omega; C_\#(Y))$.*

In the special case $\alpha^h(x) = hx$, $x \in \Omega$, we see that we get the classical, periodic two-scale convergence. What kind of basic restrictions do we need to impose on $\{\alpha^h\}$ to be able to homogenize (1)? This question leads us to the notion of asymptotic uniform distribution defined in Definition 2 below. We first assume that each α^h is a continuous bijection $\mathbb{R}^N \to \mathbb{R}^N$. We note the fact that this α^h is actually an extension of the original α^h. Let $\{\overline{Y}^j\}$ be an enumerable covering of $\mathbb{R}^N$ with unit cubes and $\{\overline{Y}^j_k\}$ a finite covering of each $\overline{Y}^j$ with cubes of side length n^{-1}. Suppose Ω has a Lipschitz boundary, then we define $\Omega_j^h = (\alpha^h)^{-1}(\overline{Y}^j) \cap \Omega$. Furthermore, we assume that there is a sequence $\{q^h\}$ of finite subsets of $\mathbb{Z}_+$ such that $\Omega = \bigcup_{j\in q^h} \Omega_j^h$. We assume that $\Omega_j^h \subset N_{r^h}(x_j^h)$ for some disk $N_{r^h}(x_j^h)$ centred at some $x_j^h \in \Omega_j^h$ and with radius $r^h \to 0$ independent of j,k. Finally, we introduce $\Omega_{jk}^h = (\alpha^h)^{-1}(\overline{Y}_k^j) \cap \Omega$. We are now prepared to formulate the definition.

Definition 2 *Suppose that for all k and any $j \in q^h$ such that $(\alpha^h)^{-1}(Y^j) \cap \Omega$ and $\partial\Omega$ are disjoint, $|\lambda(\Omega_{jk}^h)/\lambda(\Omega_j^h) - n^{-N}| < \varepsilon^h$, where $\varepsilon^h \to 0$. Then $\{\alpha^h\}$ is said to be asymptotically uniformly distributed, abbreviated AUD, on Ω.*

A fundamental property in the context of "generalized" two-scale convergence is the concept of strong two-scale compatibility [2]. The operator sequence $\{\widetilde{\alpha}^h\}$ has this property if it is two-scale compatible, i.e. fulfils certain boundedness properties (with respect to some admissible space), and satisfies some special weak convergence in $L^2(\Omega)$. It turns out that if $\{\alpha^h\}$ is AUD, $\{\widetilde{\alpha}^h\}$ is strongly two-scale compatible (with respect to the admissible space $L^2(\Omega; C_\#(Y))$) [1]. In Proposition 3 below we will see why the strong two-scale compatibility (and thus the AUD) property is so important [6]. Firstly, introduce the Banach space X, $\mathscr{D}(\Omega; C_\#^\infty(Y)) \subset X \subset L^2(\Omega; L_\#^2(Y))$. Secondly, let $Z, Z^\perp \subset L^2(\Omega; L_\#^2(Y)^N)$ be a Banach space, $Z \cap \mathscr{D}(\Omega; C_\#^\infty(Y))$ is dense in Z, and its orthogonal complement, respectively. Thirdly, define the operator sequence $\{\nabla_y \widetilde{\alpha}^h\}$ by $(\nabla_y \widetilde{\alpha}^h v)(x) = \nabla_y v(x, \alpha^h(x))$, $x \in \Omega$.

Proposition 3 *Suppose $\{\widetilde{\alpha}^h\}$ is strongly two-scale compatible with respect to X and that $\nabla_y \widetilde{\alpha}^h v : (\nabla\alpha^h)^{\mathrm{T}} \rightharpoonup 0$ in $L^2(\Omega)$ for every $v \in Z \cap \mathscr{D}(\Omega; C_\#^\infty(Y))$. Furthermore, let $\{u^h\} \subset H^1(\Omega)$ be a bounded sequence. Then, up to a subsequence, there exist $u \in L^2(\Omega)$ and $w_1 \in Z^\perp$ such that $u^h \to u$ in $L^2(\Omega)$, $\langle u^h, \widetilde{\alpha}^h v\rangle_{L^2(\Omega)} \to \langle u, v\rangle_{L^2(\Omega\times Y)}$ for every $v \in X$, and $\langle \nabla u^h, \widetilde{\alpha}^h v\rangle_{L^2(\Omega)^N} \to \langle \nabla u + w_1, v\rangle_{L^2(\Omega\times Y)^N}$ for every $v \in X^N$.*

Hence, the λ-scale limit (up to a subsequence) of $\{u^h\}$ has no dependence in the second-scale variable, and that the λ-scale limit (up to a subsequence) of the gradient sequence $\{\nabla u^h\}$ gets an extra term w_1 containing an explicit dependence on the second-scale variable.

3 H-convergence

What do we exactly mean by "homogenizing" a sequence of partial differential equations, e.g. those of the type (1)? Define $\mathscr{M}(\mathscr{O})$ to be the set of bounded and coercive $N \times N$ matrix functions in $L^\infty(\mathscr{O})^{N\times N}$, where $\mathscr{O} \subset \mathbb{R}^N$ is open. Let (1A) and (1B) refer to (1) but with A^h and B, respectively, instead of $A \circ \alpha^h$. In Definition 4 we define the relevant convergence mode in the homogenization procedure.

Definition 4 *We say that $\{A^h\} \subset \mathscr{M}(\Omega)$ H-converges to $B \in \mathscr{M}(\Omega)$ if for any $f \in H^{-1}(\Omega)$ the sequence $\{u^h\}$ of solutions to (1A) satisfies $u^h \rightharpoonup u$ in $H_0^1(\Omega)$ and $A^h \nabla u^h \rightharpoonup B\nabla u$ in $L^2(\Omega)^N$ where $u \in H_0^1(\Omega)$ uniquely solves (1B).*

Under which assumptions can we homogenize the stationary heat conduction problem (1)? We introduce a convenient class of sequences in Definition 5.

Definition 5 *Suppose that (i) $\{\widetilde{\alpha}^h\}$ is strongly two-scale compatible with respect to a Banach space X as defined above, (ii) there exists a sequence $\{p^h\}$, $p^h \to 0$ in $H^1(\Omega)$, such that $p^h \nabla\alpha^h \to \Pi$ in $L^2(\Omega)^{N\times N}$, where $\Pi \in L^\infty(\Omega)^{N\times N}$, and (iii) there exists a Banach space Z as defined above for which $\nabla_y \widetilde{\alpha}^h v : (\nabla\alpha^h)^{\mathrm{T}} \rightharpoonup 0$ in $L^2(\Omega)$ for every $v \in Z \cap \mathscr{D}(\Omega; C_\#^\infty(Y))$. Then $\{\alpha^h\}$ is said to be of type H_X^Π.*

We can now formulate an important theorem on the homogenization of stationary heat conduction problems with thermal conductivity matrices involving the sequence $\{\alpha^h\}$, see Theorem 6. The main key to prove the theorem is to utilize the λ-scale convergence result of Proposition 3, see [6] for details.

Theorem 6 *Assume that $A \in C_\#(Y)^{N\times N} \cap \mathscr{M}(\mathbb{R}^N)$, and that $\{\alpha^h\}$ is of type H_X^Π, $X = L^2(\Omega; C_\#(Y))$. Furthermore, suppose $w_1 \in Z^\perp$ and the weak limit $u \in H_0^1(\Omega)$ of solutions $\{u^h\}$ to (1) uniquely solve the homogenized problem $\int_\Omega \int_Y A(\nabla u + w_1) \cdot \nabla v = \int_\Omega f v$ for all $v \in H_0^1(\Omega)$ and the local problem $\int_Y A(\nabla u + w_1) \cdot \Pi \nabla_y v = 0$ for all $v \in H_\#^1(Y)/\mathbb{R}$. Then $\{A \circ \alpha^h\}$ H-converges to B satisfying the flow formula $B\nabla u = \int_Y A(\nabla u + w_1)$.*

Note here that the local problem is effectively parameterized over Ω and that in effect, B is expected to vary over Ω in general. In the classical periodic case the homogenized thermal conductivity matrix is given explicitly as the modified arithmetic mean $B = \int_Y A(I + \nabla_y z)$, where $z \in H_\#^1(Y)/\mathbb{R}$ uniquely solves the local problem $-\nabla_y \cdot \{A(I + \nabla_y z)\} = 0$ in Y. In the next section we will study a simple but illuminating example where we can achieve a corresponding result in the non-periodic case.

4 Homogenization of a two-dimensional "toy model"

Detailed proofs of all original propositions and theorems in this section can be found in the preprint [5] available in the arXiv database. Consider a simple two-dimensional "toy model" function sequence $\{\alpha^h\}$ according to $\alpha^h(x) = (hx_1, hx_2|x_2|)$, $x \in \mathbb{R}^2$, and $\Omega = (a_1, b_1) \times (a_2, b_2)$ strictly included in the first quadrant. Note that in the x_1-direction periodicity is preserved, while in the x_2-direction periodicity is violated with an increasing frequency for growing x_2. The first thing we need to do in order to homogenize with respect to $\{\alpha^h\}$ is to verify the AUD property on Ω.

Proposition 7 $\{\alpha^h\}$ *is AUD on* Ω.

Proof. First we label the covering cubes by using natural two-dimensional indices j and k in the obvious manner. The covering cubes are then mapped into Ω to obtain explicit expressions for Ω_j^h and Ω_{jk}^h. We note that the Ω_j^h shrink uniformly so that they can be fit into disks whose radius r^h tend to zero. The proof is concluded by noting that $\lambda\left(\Omega_{jk}^h\right)/\lambda\left(\Omega_j^h\right) \to n^{-2}$. □

We immediately get Proposition 8.

Proposition 8 $\{\widetilde{\alpha}^h\}$ *is strongly-two scale compatible with respect to* $L^2\left(\Omega;C_\#(Y)\right)$.

Proof. Use Proposition 7 together with the observed fact that the AUD property implies the strong two-scale compatibility property. □

Let us define the diagonal and invertible matrix $\Pi(x) = \operatorname{diag}(1, 2x_2)$, $x \in \Omega$, to be employed in Proposition 9 and henceforth.

Proposition 9 $\{\alpha^h\}$ *is of type* H_X^Π, $X = L^2(\Omega;C_\#(Y))$.

Proof. Condition (i) in the definition of type-H_X^Π sequences is exactly Proposition 8. Condition (ii) is easily verified by noting that it suffices to choose $p^h = 1/h$. Condition (iii) readily follows from defining the Banach space $Z = \left\{v \in L^2\left(\Omega;L_\#^2(Y)^2\right) : \nabla_y \cdot (\Pi v) = 0\right\}$. □

The orthogonal complement $Z^\perp$ is derived by utilizing the result on divergence-free functions given below in Lemma 10 (see, e.g., [4]).

Lemma 10 *Let* $f \in L_\#^2(Y)^N$ *be orthogonal to the space of divergence-free functions in* $C_\#^\infty(Y)^N$. *Then there exists a scalar potential* $\phi \in H_\#^1(Y)/\mathbb{R}$ *such that* $f = \nabla_y \phi$.

Proposition 11 *The orthogonal complement of Z in* $L^2\left(\Omega;L_\#^2(Y)^2\right)$ *is* $Z^\perp = \left\{\Pi\nabla_y u_1 : u_1 \in L^2\left(\Omega;H_\#^1(Y)/\mathbb{R}\right)\right\}$.

Proof. First let $v \in Z$ and $w_1 \in Z^\perp$ and use the definition of them being orthogonal noting that $v \cdot w_1 = \Pi v \cdot \Pi^{-1} w_1$. Use Lemma 10 to obtain $w_1 = \Pi\nabla_y u_1$ for some $u_1 \in L^2\left(\Omega;H_\#^1(Y)/\mathbb{R}\right)$, where some extra care has to be taken concerning the required function space since in the lemma $x \in \Omega$ merely appears as a parameter. □

We are now ready to formulate a preliminary homogenization result for our "toy model".

Proposition 12 *Assume that* $A \in C_\#(Y)^{2\times 2} \cap \mathscr{M}(\mathbb{R}^2)$, *and suppose* $u_1 \in L^2\left(\Omega, H_\#^1(Y)\right)$ *and the weak limit* $u \in H_0^1(\Omega)$ *of solutions* $\{u^h\}$ *to* (1) *uniquely solve the homogenized problem* $\int_\Omega \int_Y A(\nabla u + \Pi\nabla_y u_1) \cdot \nabla v = \int_\Omega fv$ *for all* $v \in H_0^1(\Omega)$ *and the local problem* $\int_Y A(\nabla u + \Pi\nabla_y u_1) \cdot \Pi\nabla_y v = 0$ *for all* $v \in H_\#^1(Y)/\mathbb{R}$. *Then* $\{A \circ \alpha^h\}$ *H-converges to B satisfying the flow formula* $B\nabla u = \int_Y A(\nabla u + \Pi\nabla_y u_1)$.

Proof. The claim follows directly from the main homogenization result of Theorem 6 together with Proposition 9 (where we showed that $\{\alpha^h\}$ is of type H_X^Π) and Proposition 11 (where $Z^\perp$ was characterised). □

Note that the local problem is parameterized over Ω. The main original result of this contribution is Theorem 13 which corresponds to the homogenization result mentioned in the end of Section 3. Also in Theorem 13 the local problem in Y is effectively parameterized over Ω, i.e., there is effectively a local problem to solve in Y for each point in Ω.

Theorem 13 *Assume that* $A \in C_\#(Y)^{2\times 2} \cap \mathscr{M}(\mathbb{R}^2)$. *Then* $\{A \circ \alpha^h\}$ *H-converges to B given by* $B = \int_Y A\left(I + \Pi\nabla_y z\right)$, *where* $z \in L^\infty\left(\Omega;H_\#^1(Y)^2\right)$ *uniquely solves the local problem* $-\Pi\nabla_y \cdot \left\{A\left(I + \Pi\nabla_y z\right)\right\} = 0$ *in* Y.

Proof. Let u, u_1 be the assumed solutions in Proposition 12 and introduce the ansatz $u_1 = \nabla u \cdot z$ for some appropriate z. The ansatz gives a sufficient homogenized thermal conductivity matrix $B = \int_Y A\left(I + \Pi\nabla_y z\right)$ according to the flow formula in Proposition 12. Note that we must have $z \in L^\infty(\Omega)^2$ (parametrically over Y). To obtain the local problem, fix some $v \in H_\#^1(Y)/\mathbb{R}$ to be used in the local problem of Proposition 12, which is straightforwardly shown to be is satisfied if $\int_Y \Pi\nabla_y v \cdot A\left(I + \Pi\nabla_y z\right) = 0$. Utilizing partial integration and the divergence theorem, noting that the boundary contribution vanishes, the local problem becomes $-\int_Y v\Pi\nabla_y \cdot \left\{A\left(I + \Pi\nabla_y z\right)\right\} = 0$. This certainly holds true if $-\Pi\nabla_y \cdot \left\{A\left(I + \Pi\nabla_y z\right)\right\} = 0$ in Y, and we have derived the local problem. It remains to prove uniqueness of the solution to the local problem. This is accomplished by defining a transformed second-scale variable $\widehat{y}(y) = \Pi^{-1} y$, $y \in Y$, parameterized over Ω and defining a gradient operator $\nabla_{\widehat{y}} = \Pi\nabla_y$. Then the local problem can be written on the form $-\nabla_{\widehat{y}}\left\{\widehat{A}\left(I + \nabla_{\widehat{y}}\widehat{z}\right)\right\} = 0$ in $\widehat{Y}$ with respect to a new function $\widehat{z}$ to solve for which uniquely determines the desired z. Uniqueness of $\widehat{z}$ follows from the fact that $\widehat{A} \in C_\#\left(\widehat{Y}\right) \cap \mathscr{M}(\mathbb{R}^2)$ so that the uniqueness result in the "classical" case can be employed. □

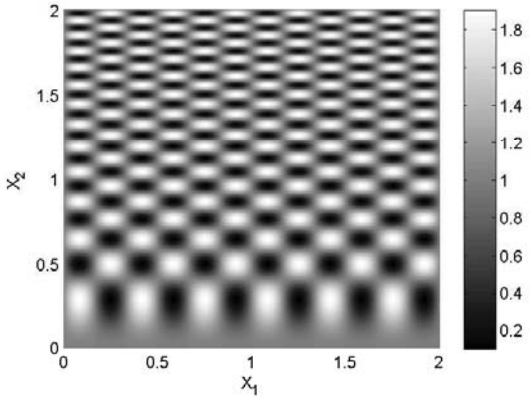

Figure 1: The scalar factor of the non-periodic thermal conductivity matrix $A \circ \alpha^h$ for $h = 3$. Note that the mapped upper sub-cells are square-shaped along $\mathscr{L}$.

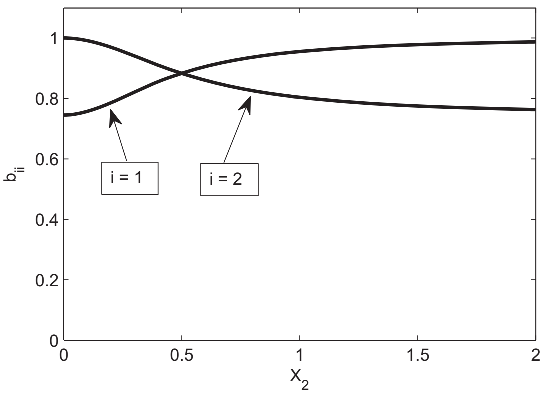

Figure 2: The non-vanishing, i.e. diagonal, entries of the homogenized thermal conductivity matrix B. Note that there is no x_1-dependence, as expected.

5 Numerical illustration of the "toy model"

In order to illustrate the result above, consider $A(y) = \left(1 + \frac{9}{10}\sin 2\pi y_1 \sin 2\pi y_2\right)I$, $y \in \mathbb{R}^2$, and $\Omega = (\delta, 2)^2$, $\delta \gtrsim 0$. Apparently, $A \in C_\#(Y)^{2\times 2} \cap \mathscr{M}(\mathbb{R}^2)$, so Theorem 13 can be used. The non-periodic thermal conductivity matrix is $\left(A \circ \alpha^h\right)(x) = \left(1 + \frac{9}{10}\sin 2\pi h x_1 \sin 2\pi h x_2^2\right)I$, $x \in (\delta, 2)^2$, see Figure 1. The local problem becomes $-\nabla_y \cdot \left(C\nabla_y z\right) = g$ in Y parametrically over Ω, where $C(x,y) = \left(1 + \frac{9}{10}\sin 2\pi y_1 \sin 2\pi y_2\right)\mathrm{diag}\left(1, 4x_2^2\right)$ and $g(x,y) = \frac{9}{5}\pi\left(\cos 2\pi y_1 \sin 2\pi y_2, 2x_2 \sin 2\pi y_1 \cos 2\pi y_2\right)$. Solving the local problem numerically and then plugging the solution (one for each $x \in \Omega$) for z into the formula for the thermal conductivity matrix B in Theorem 13, we get vanishing off-diagonal entries and non-vanishing diagonal entries as given by Figure 2. We see that we have isotropy (i.e., B's diagonal entries b_{11} and b_{22} are equal) for $x_2 = 1/2$, which can be explained heuristically in the following way. When h is large, the periodicity cells are mapped as near-perfect squares along $\mathscr{L} = \left\{x \in \Omega : x_2 = 1/2\right\}$, see Figure 1 where this can be glimpsed for a row of upper sub-cells. This observation together with the fact that A is symmetric in the y_1- and y_2-directions means that we would expect that the homogenized thermal conductivity matrix B should be isotropic along $\mathscr{L} \subset \Omega$, which apparently is the case.

6 Conclusions

In this contribution we have shown how to use the recently developed λ-scale convergence technique to homogenize a simple example of a non-periodic stationary heat conduction problem. The main aim was not to homogenize with respect to the specific $\left\{\alpha^h\right\}$ chosen, but to show that it is possible to homogenize problems with the λ-scale convergence technique to obtain explicit answers concerning the characterisation of the homogenized limit problem, e.g., an explicit homogenized thermal conductivity matrix B. The λ-scale convergence technique developed by Holmbom and Silfver clearly opens up a wide range of more "realistic" problems treatable in a similar manner. The next step would be to find classes of sequences $\left\{\alpha^h\right\}$ for which homogenization results of the type Theorem 13 can be directly applied. Such a class would e.g. be the important "non-entangled" sequences given by $\alpha^h = h\alpha$ for sufficiently well-behaved α.

7 References

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