I. Troch, F. Breitenecker, eds. ISBN 978-3-901608-35-3

THE OPTIMAL PORTFOLIO SELECTION: THE CASE OF INTEGRAL BUDGET

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Abstract. We consider a nonlinear optimal control problem with weakly singular integral equation as the control object, subject to control constraints. This integral equation corresponds to a fractional moment of stochastic process involving short-range dependencies. We derive the first order necessary optimality conditions in the form of Euler-Lagrange equation, and then apply them for numerical modelling of the problem of optimal portfolio selection.

1 Introduction

Let us formulate the optimal portfolio selection problem in general form. Let the time interval $[t_0, t_1]$ be fixed, $x \in \mathbb{R}$ denote the state variable, and $u = (u_1, \ldots, u_p, v) \in \mathbb{R}^{p+1}$ denote the vector of control variables. Set $F(t, x(t), u(t)) = e^{-\rho t} v^{\gamma}(t) x(t)$. Then the cost function has the Bolza form

$$J(x(\cdot), u(\cdot)) = \int_{t_0}^{t_1} F(t, x(t), u(t)) dt + e^{-\rho t_1} h(x(t_1)) \to \max_u, \tag{1}$$

where F and h are smooth (C^1) functions. Further, we present control constraints in the abstract form of inequality constraint

$$\varphi(u(t)) \leq 0, \tag{2}$$

where $\varphi(u)$ is a smooth (C^1) vector function of the dimension m. We assume that the gradients $\varphi_i'(u)$, $i \in I(u)$ of the active constraints are linearly independent at each point u such that $\varphi(u) \leq 0$. Here,

$$I(u) = \{i \in \{1, \ldots, m\} \mid \varphi_i(u) = 0\}$$

is the set of active indices at the point u.
Now consider the stochastic object equation by an integral equation

$$x(t) - x(t_0) = \int_{t_0}^{t} \gamma \left[\left(1 - \sum_{k=1}^{p} u_k(\tau) \right) r(t) + \sum_{k=1}^{p} u_k(\tau) \mu_k(\tau) - v(\tau) \right] x(\tau) d\tau \tag{3}$$

$$+ \int_{t_0}^{t} \frac{\gamma(\gamma+1)}{2} x(\tau) \sum_{k=1}^{p} u_k^2(\tau) \varsigma_k^2(\tau) (d\tau)^{2H_k},$$

(where $\mu_k(t) > 0$ is the appreciation rate, $\varsigma_k(t) > 0$ is the volatility of the stocks, $H_k \in]0,1[$ is the Hurst parameter) with the initial condition $x(t_0) = a$, where $a \in \mathbb{R}$ is fixed. Following reasoning is strongly dependent on Hurst parameter value H as far as it changes the role of integration with respect to $(d\tau)^H$, namely [5]

$$\int_{t_0}^{t} \kappa(\tau) (d\tau)^{2H} = 2H \int_{t_0}^{t_1} (t_1 - \tau)^{2H-1} \kappa(\tau) d\tau, \quad 0 < H < 1/2, \tag{4}$$

Portfolio can contain stocks with different statistical properties. Hence for the pink noise, denoting $\beta_k = 2H_k$, $0 < \beta_k < 1$, we rewrite (3) as

$$x(t) - x(t_0) = \int_{t_0}^{t} \left[\Phi_0(u(\tau)) + \sum_{k=1}^{p} \frac{\beta_k}{(t-\tau)^{1-\beta_k}} \Phi_k(u(\tau)) \right] x(\tau) d\tau, \tag{5}$$

where

$$\Phi_0(u(t)) = \gamma\left[\left(1 - \sum_{k=1}^{p} u_k(t)\right) r(t) + \sum_{k=1}^{p} u_k(t)\mu_k(t) - \nu(t)\right],$$

$$\Phi_k(u(t)) = \frac{\gamma(\gamma-1)}{2} u_k^2(t)\varsigma_k^2(t).$$

The goal of this paper is to obtain the solution of problem (1), (2), (5) with the initial condition $x(t_0) = a$.

2 Necessary optimality conditions

For simplicity, we study problem (1), (2) and (5) in the case $p = 1$ (so that we omit index k), using the Dubovitski-Milyutin method, presented in [1], [3], and [8]. Let $(x(t), u(t))$ be an optimal process, where $x(t_0) = a$, $x \in C([t_0, t_1], \mathbb{R})$, $u \in L^\infty([t_0, t_1], \mathbb{R}^2)$, $u = (u_1, \nu)$.

Using results [8], we introduce the variation equation [7]

$$\begin{aligned}\ell(\bar{x}, \bar{u}) = c_0\bar{x}(t_0) + \int_{t_0}^{t_1} \bar{x}(t)d\sigma(t) - \int_{t_0}^{t_1}\int_{t_0}^{t_1}\Big[& \bar{x}(\tau)\Phi_0(u(\tau)) \\ & + x(\tau)\Phi_0'(u(\tau))\bar{u}(\tau) \\ & + \frac{\beta\bar{x}(\tau)}{(t-\tau)^{1-\beta}}\Phi(u(\tau)) + \frac{\beta x(\tau)}{(t-\tau)^{1-\beta}}\Phi'(u(\tau))\bar{u}(\tau)\Big] d\tau\, d\sigma(t)\end{aligned}$$

where $d\sigma(t)$ is the measure of Lebesgue-Stieltjes on $[t_0, t_1]$, respectively $\sigma(t)$ is the function of bounded variation on $[t_0, t_1]$, $c_0 \in \mathbb{R}$.

We denote by $[\sigma](t) = \sigma(t+0) - \sigma(t-0)$ the jump of the function σ at the point t. If the measure $d\sigma(t)$ is given, then the jumps $[\sigma](t)$ are known for all $t \in [t_0, t_1]$, particularly, $[\sigma](t_0)$ and $[\sigma](t_1)$ are known. This means, that the left hand limit $\sigma(t_0 - 0)$ and the right hand limit $\sigma(t_1 + 0)$ are also defined. Using this reasoning it is easy to get the Euler-Lagrange equation for the optimal process $(x(t), u(t))$

$$\begin{aligned}&-\alpha_0 \int_{t_0}^{t_1} \left(F_x(t, x(t), u(t)) \cdot \bar{x}(t) + F_u(t, x(t), u(t)) \cdot \bar{u}(t)\right) dt \\ &-\alpha_0 e^{-\rho t_1} h'(x(t_1))\bar{x}(t_1) + \ell(\bar{x}, \bar{u}) + \lambda(\varphi'(u(t))\bar{u}) = 0, \ \forall \bar{u} \in L^\infty, \ \bar{x} \in C,\end{aligned} \tag{6}$$

where $\lambda \in (L^\infty)^*$, $\lambda \ge 0$, $\lambda = (\lambda_1, ..., \lambda_m)$, λ_i is concentrated on the set $\{t \mid \varphi_i(u(t)) = 0\}$, $i = 1, \dots, m$; moreover, $\alpha_0 \ge 0$ and $\alpha_0 + \|\lambda\| + |c_0| + \|\sigma\| > 0$. Due to the hypothesis of linear independence of the gradients $\varphi_i'(u)$, we deduce from equation (6) that the functionals λ_i are absolutely continuous, i.e., $\lambda_i = \lambda_i^a \in L^1$, $i = 1, \dots, m$. Then setting $\bar{x} = 0$ in this equation and using the fact that $\bar{u} \in L^\infty$ is an arbitrary element, we deduce the local maximum principle:

$$\begin{aligned}&-\alpha_0 F_u(t, x(t), u(t)) - x(t)\Phi_0'(u(t))\int_t^{t_1} d\sigma(\tau) - \\ &-x(t)\Phi'(u(t))\int_t^{t_1} \tfrac{\beta}{(\tau-t)^{1-\beta}} d\sigma(\tau) + \lambda^a(t)\phi'(u(t)) = 0,\end{aligned} \tag{7}$$

where $\lambda^a(t) \ge 0$ and $\lambda^a(t)\varphi(u(t)) = 0$.

If $\bar{u} = 0$ and $\bar{x}$ is arbitrary, then we deduce from equation (6) that

$$\begin{aligned}&-\alpha_0 F_x(t, x(t), u(t))dt - \alpha_0 e^{-\rho t_1} h'(x(t_1))\bar{x}(t_1) + c_0\delta(t - t_0)dt \\ &+ d\sigma(t) - \int_t^{t_1}\left\{\Phi_0(u(t)) + \tfrac{\beta}{(\tau-t)^{1-\beta}}\Phi(u(t))\right\} d\sigma(\tau)\, dt = 0,\end{aligned} \tag{8}$$

where $\delta(t)$ is the Dirac delta-function. It follows that

$$d\sigma(t) = \sigma^a(t)dt - c_0\delta(t - t_0)dt + \alpha_0 e^{-\rho t_1} h'(x(t_1))\delta(t - t_1)dt,$$

where $\sigma^a \in L^1$. Set $\psi(t) = \int_t^{t_1} d\sigma(\tau) = \int_t^{t_1} \sigma^a(\tau)d\tau + \alpha_0 e^{-\rho t_1} h'(x(t_1)), t > t_0$, then the following theorem holds.

Theorem. Let $(x(t),u(t))$ be an optimal process on the interval $[t_0,t_1]$, where $x(\cdot) \in C([t_0,t_1],\mathbb{R})$, $u(\cdot) \in L^\infty([t_0,t_1],\mathbb{R}^2)$. Then there exists a tuple of Lagrange multipliers $(\alpha_0, \psi(\cdot), \lambda(\cdot))$ such that α_0 is a number, $\psi : [t_0,t_1] \to \mathbb{R}$ is an absolutely continuous function, $\lambda : [t_0,t_1] \to \mathbb{R}^{m^*}$ is an integrable function, and the following conditions are fulfilled:

- nonnegativity: $\alpha_0 \geq 0$ and $\lambda(t) \geq 0$ a.e. on $[t_0,t_1]$;
- nontriviality: $\alpha_0 + \|\psi\| > 0$;
- complementarity: $\lambda(t)\varphi(u(t)) = 0$ a.e. on $[t_0,t_1]$;
- adjoint equation

$$-\psi'(t) = \alpha_0 F_x(t, x(t), u(t)) + \Phi_0(u(t))\psi(t) - \Phi(u(t)) \int_t^{t_1} \frac{\beta}{(\tau-t)^{1-\beta}} \psi'(\tau)\, d\tau; \tag{9}$$

- local maximum principle:

$$x(t)\Phi_0'(u(t))\,\psi(t) - x(t)\Phi'(u(t)) \int_t^{t_1} \frac{\beta}{(\tau-t)^{1-\beta}} \psi'(\tau)d\tau + \alpha_0 F_u(t, x(t), u(t)) - \lambda(t)\phi'(u(t)) = 0; \tag{10}$$

- transversality condition:

$$\psi(t_1) = \alpha_0 e^{-\rho t_1} h'(x(t_1)). \tag{11}$$

3 Algorithm

To illustrate the proposed methodology, we will consider an example of the portfolio, which contains one bond and two stocks. In this case the wealth connected with the portfolio at time t can be written as

$$\begin{aligned} dZ(t) = {} & Z(t)[(1-u_1(t)-u_2(t))r(t)]dt - v(t)Z(t)dt \\ & + u_1(t)Z(t)\left(\mu_1(t)dt + \varsigma_1(t)w(t)(dt)^{H_1}\right) \\ & + u_2(t)Z(t)\left(\mu_2(t)dt + \varsigma_2(t)w(t)(dt)^{H_2}\right), \end{aligned} \tag{12}$$

where constrains for the invested capital can be presented in the next manner:

- first stock $u_1(t) \geq 0$ and second stock $u_2(t) \geq 0$

$$\varphi_1(u_1(t)) = -u_1(t), \varphi_2(u_2(t)) = -u_2(t), \tag{13}$$

- both stocks $u_1(t) + u_2(t) \leq 1$

$$\varphi_3(u_1(t), u_2(t)) = u_1(t) + u_2(t) - 1; \tag{14}$$

- the consumption value $v(t) \geq 0$

$$\varphi_4(v(t)) = -v(t). \tag{15}$$

This gives possibility to write a vector of Lagrange multipliers $\lambda(t) = (\lambda_1(t), ..., \lambda_4(t))$. Denote optimal control as $u^{opt}(t) = u^{opt}(x(t), \sigma^a(t))$.

3.1 The case of white noise

For $H_1 = H_2 = 0.5$, we can get classical formulation of the optimal control task for maximum principle and easily get optimal values $u^{opt}(t)$, namely

$$u_{1w}^{opt} = \frac{(\mu_1 - r)}{2(1-\gamma)\varsigma_1^2},$$

$$u_{2w}^{opt}=\frac{(\mu_2-r)}{2(1-\gamma)\varsigma_2^2},$$

$$v_w^{opt}(t)=\frac{\Upsilon_2\exp\left[\frac{\rho t}{\gamma-1}\right]}{\exp[-\Upsilon_1 t+\Upsilon_2]-\exp[(-\Upsilon_1+\Upsilon_2)t]},$$

where $\Upsilon_1=\frac{\gamma}{4}\left[\frac{(\mu_1-r)^2}{\varsigma_1^2}+\frac{(\mu_2-r)^2}{\varsigma_2^2}+4(1-\gamma)r\right]$, $\Upsilon_2=\frac{\Upsilon_1(1-\gamma)-\rho}{1-\gamma}$.

Solution (17) - (19) is the same solution as in [6] and does not present any interest, however it will be useful in further reasoning.

3.2 The case with short-range dependence

Let the object equation has a form

$$\begin{aligned}x(t)-x(t_0)=&\int_0^t\gamma[(1-u_1(\tau)-u_2(\tau))r(\tau)-v(\tau)]x(\tau)d\tau\\&+\int_0^t\gamma[u_1(\tau)\mu_1(\tau)+u_2(\tau)\mu_2(\tau)]x(\tau)d\tau\\&+\int_0^t\frac{\gamma(\gamma-1)}{2}x(\tau)u_1^2(\tau)\varsigma_1^2(\tau)(d\tau)^{2H_1}\\&+\int_0^t\frac{\gamma(\gamma-1)}{2}x(\tau)u_2^2(\tau)\varsigma_2^2(\tau)(d\tau)^{2H_2}.\end{aligned}\tag{20}$$

Let $\beta_1=2H_1$, $\beta_2=2H_2$, $c_1=\gamma(\mu_1(\tau)-r(\tau))$, $c_2=\gamma(\mu_2(\tau)-r(\tau))$, $c_3=\gamma(\gamma-1)\varsigma_1^2(\tau)\beta_1$ and $c_4=\gamma(\gamma-1)\varsigma_2^2(\tau)\beta_2$, then we can introduce the following relations

$$\begin{aligned}&\Theta_0(u(t))=c_1u_1(t)+c_2u_2(t)-v(t)+r(t),\\&\Theta_1(u(t))=c_3u_1^2(t),\Theta_2(u(t))=c_4u_2^2(t).\end{aligned}$$

We also suppose that $0<H_1<1/2$ and $0<H_2<1/2$. All this gives us the possibility to introduce the adjoint equation (9)

$$\begin{aligned}&\sigma^a(t)=\alpha_0F_x(t,x(t),\underline{u}(t))\\&+\int_t^{t_1}\left(\Theta_0(u(t))+\frac{\Theta_1(u(t))}{(\tau-t)^{1-\beta_1}}+\frac{\Theta_2(u(t))}{(\tau-t)^{1-\beta_2}}\right)\sigma^a(\tau)d\tau\end{aligned}\tag{21}$$

and the local maximum principle (10)

$$\begin{aligned}&x(t)\int_t^{t_1}\left(\Theta_{0u}'(u(t))+\frac{1}{(\tau-t)^{1-\beta_1}}\Theta_{1u}'(u(t))+\frac{1}{(\tau-t)^{1-\beta_2}}\Theta_{2u}'(u(t))\right)\sigma^a(\tau)d\tau\\&+\alpha_0F_u(t,x(t),u(t))-\lambda(t)\varphi_u(t,x(t),u(t))=0,\end{aligned}\tag{22}$$

which gives the system of three equations

$$\begin{aligned}\frac{\partial H_1}{\partial u}=&\quad x(t)\gamma\int_t^{t_1}\left(c_1+\frac{2c_3u_1(t)}{(\tau-t)^{1-\beta_1}}\right)\sigma^a(\tau)d\tau+\lambda_1(t)+\lambda_3(t)=0,\\\frac{\partial H_2}{\partial u}=&\quad x(t)\gamma\int_t^{t_1}\left(c_2+\frac{2c_4u_2(t)}{(\tau-t)^{1-\beta_2}}\right)\sigma^a(\tau)d\tau+\lambda_2(t)+\lambda_3(t)=0,\\\frac{\partial H_3}{\partial u}=&\quad -x(t)\gamma\int_t^{t_1}\sigma^a(\tau)d\tau+\alpha_0\gamma e^{-\rho t}v^{\gamma-1}(t)x(t)+\lambda_4(t)=0\end{aligned}\tag{23}$$

The solution to system (23) gives an optimal control (16), which depends on the solutions of the equation (21) and equation (22). The problem, which arises here, is connected with the fact that the system (23) is undetermined. Thus, in order to find the optimal control, we propose to use the iterative algorithm.

Step 0. Find the optimal control (17) - (19) as if $H_1=H_2=0.5$ and put $i=0$ and $u_1^{(i)}(t)=u_{1w}^{opt}$, $u_2^{(i)}(t)=u_{2w}^{opt}$ and $v^{(i)}(t)=v_w^{opt}(t)$. If there is no any active constraint, then go to **step 1**, otherwise go to **step 6**.

Step 1. Put $\lambda_1(t)=\lambda_2(t)=\lambda_3(t)=\lambda_4(t)=0$ and rewrite (23) as

$$\begin{aligned}\int_t^{t_1}\left(c_1+\frac{2c_3u_1(t)}{(\tau-t)^{1-\beta_1}}\right)\sigma^a(\tau)d\tau &= 0,\\ \int_t^{t_1}\left(c_2+\frac{2c_4u_2(t)}{(\tau-t)^{1-\beta_2}}\right)\sigma^a(\tau)d\tau &= 0,\\ \int_t^{t_1}\sigma^a(\tau)d\tau-\alpha_0e^{-\rho t}v^{\gamma-1}(t) &= 0.\end{aligned} \tag{24}$$

Step 2. Using local maximum principle (24) rewrite the adjoint equation as

$$\begin{aligned}\sigma^a(t)= {} & \alpha_0e^{-\rho t}\left(v^{(i)}(t)\right)^\gamma\\ & +\gamma\left(\frac{c_1u_1^{(i)}(t)}{2}+\frac{c_2u_2^{(i)}(t)}{2}-v^{(i)}(t)+r\right)\int_t^{t_1}\sigma^a(\tau)d\tau,\end{aligned} \tag{25}$$

and replace it by an ordinary differential equation $Q'(t)=-q(t)Q(t)+f(t)$, where $Q(t)=\int_t^{t_1}\sigma^a(\tau)d\tau$, $q(t)=\gamma\left(\frac{c_1u_1^{(i)}(t)}{2}+\frac{c_2u_2^{(i)}(t)}{2}-v^{(i)}(t)+r\right)$ and $f(t)=e^{-\rho t}\left(v^{(i)}(t)\right)^\gamma$.

Step 3. Find $\sigma^a(\tau)$ and estimate $\int_t^{t_1}\sigma^a(\tau)d\tau$.

Step 4. Define the new values of optimal control (16)

$$\underline{u}_1^{opt}(t)=-\frac{c_1}{2c_3}\frac{\int_t^{t_1}\sigma^a(\tau)d\tau}{\int_t^{t_1}\frac{\sigma^a(\tau)d\tau}{(t-\tau)^{1-\beta_1}}}, \tag{26}$$

$$\underline{u}_2^{opt}(t)=-\frac{c_2}{2c_4}\frac{\int_t^{t_1}\sigma^a(\tau)d\tau}{\int_t^{t_1}\frac{\sigma^a(\tau)d\tau}{(t-\tau)^{1-\beta_2}}}, \tag{27}$$

$$v^{opt}(t)=\left(\frac{e^{-\rho t}}{\int_t^{t_1}\sigma^a(\tau)d\tau}\right)^{\frac{1}{1-\gamma}}. \tag{28}$$

Step 5. Check constraints (13) - (15). If any of the constraints is active, then go to **step 6**, else check the convergence of the solution. If desired convergence is reached, then go to **step 7**, else put $i=i+1$, $u_1^{(i)}(t)=u_1^{opt}(t)$, $u_2^{(i)}(t)=u_2^{opt}(t)$ and $v^{(i)}(t)=v^{opt}(t)$ and return to **step 3**.

Step 6. Write the adjoint equation taking into account active constraints, solve it as a weak singular Volterra equation of second kind, recalculate (28) and $u_1^{(i)}(t)=u_1^{(i-1)}(t)$, $u_2^{(i)}(t)=u_2^{(i-1)}(t)$ and $v^{(i)}(t)=v^{opt}(t)$. Put $i=i+1$ and return to **step 3**.

Step 7. Solve object equation and define the value of goal function

$$J^{opt}=\int_{t_0}^{t_1}x(\tau)e^{-\rho\tau}\left(v^{opt}(\tau)\right)^\gamma d\tau.$$

4 Numerical example

In order to illustrate proposed methodology of portfolio optimal control we determined parameters of Polish financial market during the period since 27/12/2006 until 07/03/2007, namely the riskless interest rate $r(t)\equiv 0.06$, the discount rate $\rho\equiv 0.05$ and the constant relative risk aversion $\gamma\equiv 0.23$. We also selected two stocks: first one is Arka BZ WBK ASiWE FIO (with $Y_{10}=50.03PLN$) and second one is Pioneer E FIO (with

$Y_{20} = 43.47PLN$). Daily prices are presented on Figure 1. Using the semiparametric estimation method [Filatova] we found that for first stock Hurst parameter is $H_1 = 0.2013$ and for second one it takes values $H_2 = 0.26071$.

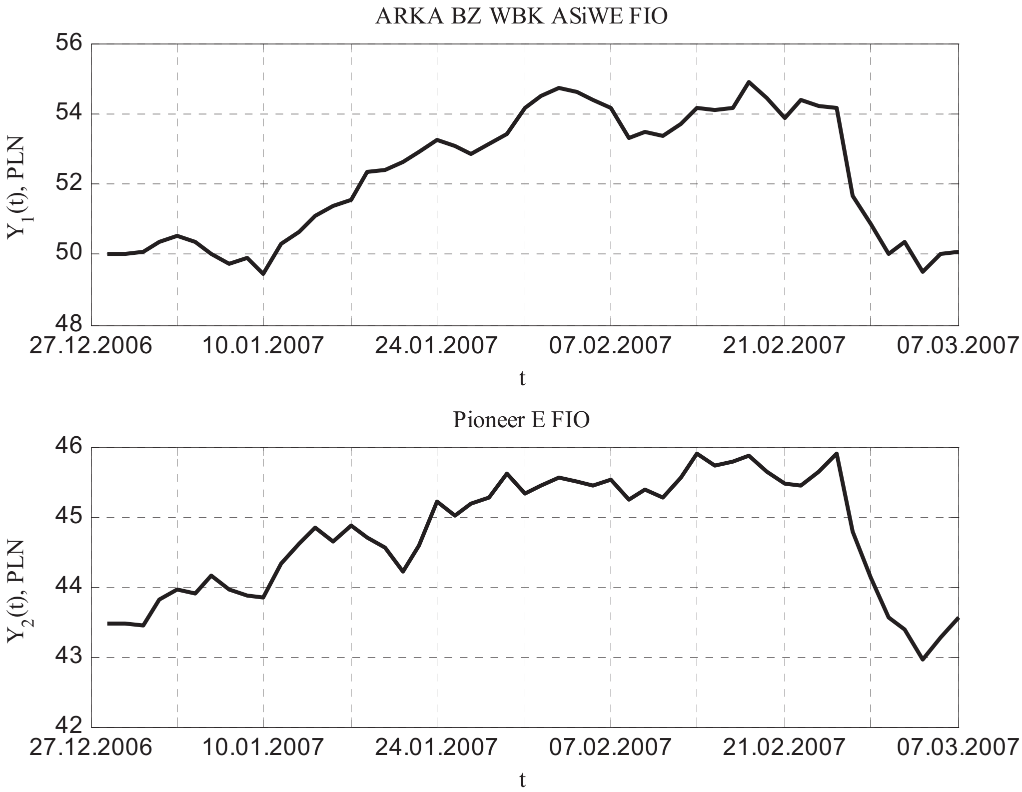

Figure 1. Stocks prices

Let stochastic differential equation (it corresponds to our object equation and to the situation above) is given by

$$dY_t = f(Y_t, \theta)dt + g(Y_t, \theta)dB_t^H, \tag{29}$$

where $f(Y_t, \theta)$ and $g(Y_t, \theta)$ are some functions, θ is a vector of unknown parameters, dB_t^H is the increment of the fractional Brownian motion (FBM) with Hurst parameter H, $0 < H < 1$, in the sense that

$$Y_t = Y_{t_0} + \int_{t_0}^{t} f(Y_s, \theta)ds + \int_{t_0}^{t} g(Y_s, \theta)dB_s^H, \tag{30}$$

where second integral is the stochastic Skorohod integral with respect to the FBM [2].

To find the estimates $\hat{\theta}$ on the basis of a sample of $(N+1)$ observations $Y_0, Y_1, ..., Y_N$ of the stochastic process (29) at known times $t_0, t_1, ..., t_N$ it is possible to use ideas of [4]. So, the estimates of the parameters are

$$\hat{\mu} = \frac{1}{N\Delta} \sum_{k=0}^{N-1} \frac{Y_{k+1} - Y_k}{Y_k} \tag{31}$$

and

$$\hat{\varsigma}^2 = \frac{1}{\Delta^{2H} N} \sum_{k=0}^{N-1} \frac{\left[Y_{k+1} - Y_k - \hat{\mu} Y_k \Delta\right]^2}{Y_k^2}. \tag{32}$$

To find estimates (l5) and (l6) we calibrated time scale taking $\Delta t = 0.01$, that gave us $\hat{\mu}_1 = 0.2701$ and $\hat{\varsigma}_1 = 0.7120$ for first stock and $\hat{\mu}_2 = 0.3802$ and $\hat{\varsigma}_2 = 0.9325$ for second one.

The optimal solution of selected stock portfolio is presented on Figure 2 and Figure 3.

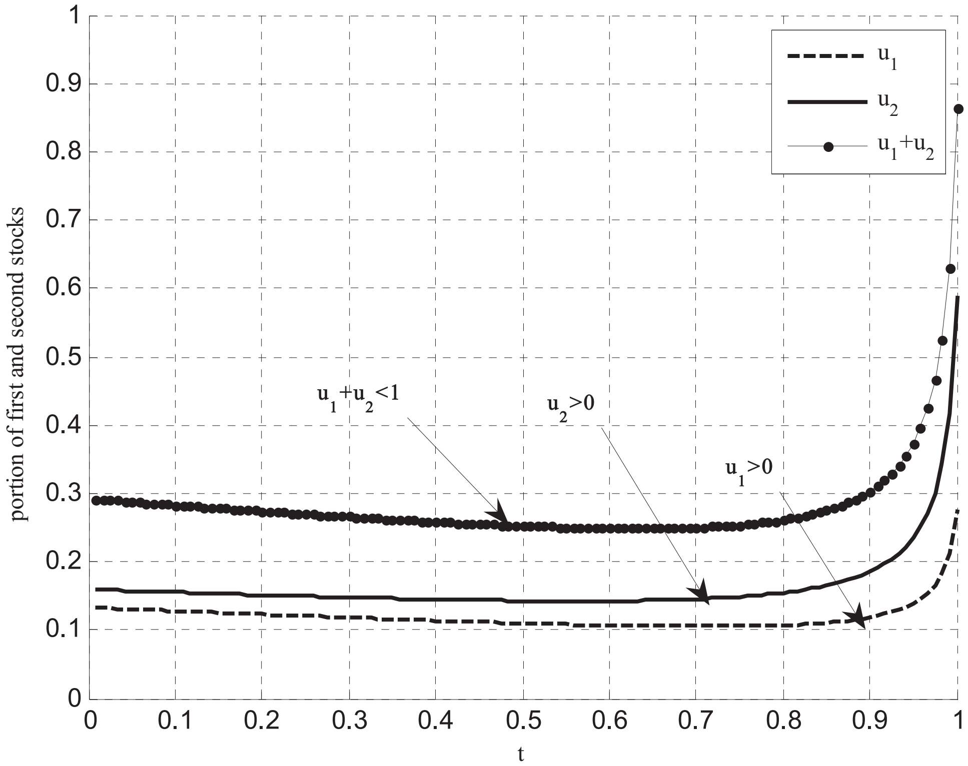

Figure 2. Optimal strategy of investments

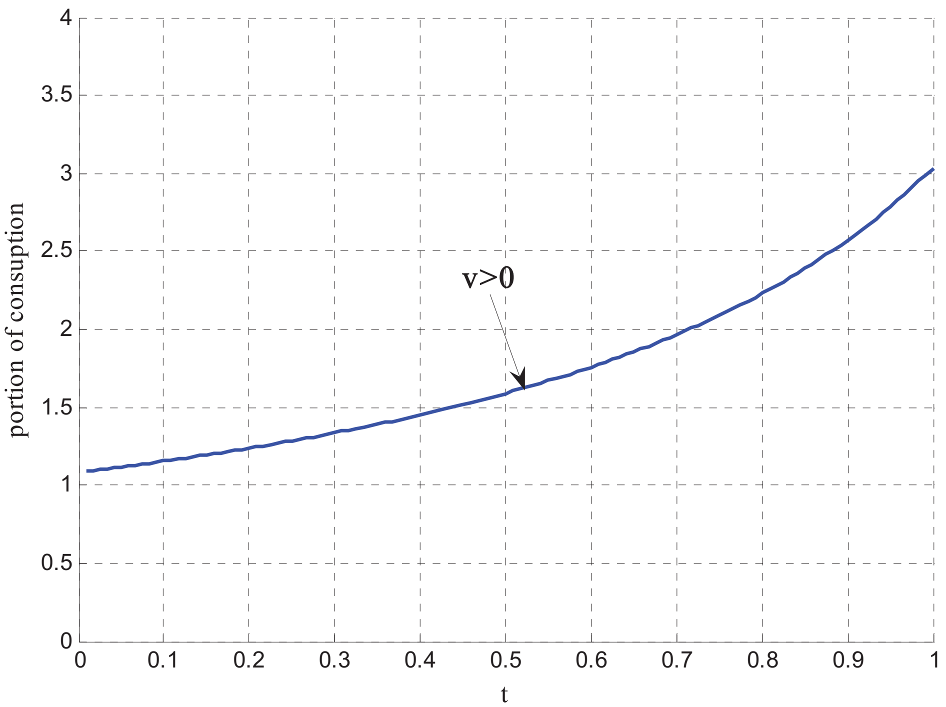

Figure 3. Optimal strategy for consumption

5 Acknowledgement

We grateful to Professor Nikolai Osmolovskii for the fruitful discussion and comments

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