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A NEW METHOD TO IMPROVE PERFORMANCE OF CONTROLLERS DESIGNED BY LQR METHOD

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Abstract. Linear Quadratic Regulator, LQR, is a powerful method for designing control which normally yields smooth closed loop response. However, this method suffers from lack of any means to enhance performance of controlled system. In this paper an algorithm is proposed to enhance the performance of controllers designed by LQR method. The objective is to find proper value of the state variable weighting matrix, **Q**, to obtain acceptable settling time. To this end, a diagonal increment for the chosen **Q** matrix is introduced which can push the dominant poles towards left. The method imposes no restriction on the form of initial **Q** matrix, as it works equivalently well for both diagonal and general form of **Q**.

1 Introduction

LQR is a powerful method for control of multi inputs multi outputs (MIMO), linear systems. Using LQR theory, it is known that for a controllable linear time invariant system, a set of optimal feedback gains may be found which minimizes a quadratic index and makes the closed loop system stable. Johnson and Grimble reported many applications of LQR in the past years [1]. The main difficulty in working with LQR method is to choose proper values for **Q** and **R**, the weighting matrices of state vector and inputs, to achieve acceptable performance for closed loop system. Commonly a trial-and-error has been used to construct the matrices **Q** and **R**. This method is very simple to use, but very difficult to produce good control performance. Also it takes a long time to choose the best value for **Q** and **R**. Bryson [2] developed a method to solve this problem using a simple iteration algorithm to optimize the elements of matrices **Q** and **R**. The basic concept behind this technique is to normalize the contribution of the output and the control term to the quadratic cost function. Robandi et al. [3] proposed a method based on genetic algorithm to produce optimal values for matrices **Q** and **R** to improve the settling time of the system. The computational cost of these methods is high and they do not guarantee a solution to the problem. Here a method is proposed to improve matrix **Q** such that the dominant poles are pushed beyond a specific value, which can guarantee desired performance.

In this paper we present a formulation for finding appropriate matrix Q which may be used in a LQR design process to put the closed loop pole of a first order linear time invariant system, LTI, at a desired location. Then a similar formulation is given for second order system with two complex conjugate poles. It is also shown that for two similar LTI systems related trough a similarity transformation, T, two weighting functions Q and Q' which are transformed with the same transformation matrix would result in similar closed loop systems. Taking advantage of these developments, a method is presented which can improve the choice of Q in a step by step procedure to push the poles beyond a specified value.

After this introduction a review on LQR method and some important properties of that is given in second section. The next section is devoted to some essential development to the proposed method which is how to place poles of first order and second order systems using LQR technique. The algorithm for improving **Q** matrix is explained in fourth section, followed by some numerical example which shows validity of the proposed method and compares the results of this method with some other available methods.

2 LQR Method

Let us consider a linear time invariant system S defined as:

$$\text{S:}\ \ \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{1}$$

In which **x** and **u** denote vectors of state variables and inputs of the system, where **A** and **B** are some constant matrices.

It is well known that for a controllable pair of (**A**, **B**), one may always find a feedback gain **K** which makes the system asymptotically stable and minimizes an index function of the form:

$$J = \int_0^\infty (\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u})dt \tag{2}$$

in which **Q** and **R** are respectively, some positive semi definite and positive definite weighting matrices. It is proven [2, 4, 5] that such feedback gain matrix can be calculated as

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P} \tag{3}$$

Where **P** is a positive definite matrix which satisfies the algebraic Riccati equation known as:

$$\mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P} + \mathbf{A}^{\mathbf{T}}\mathbf{P} + \mathbf{P}\mathbf{A} = \mathbf{0} \tag{4}$$

This method of computation of feedback gain is known as LQR which stands for Linear Quadratic Regulator.

In this section we investigate some properties of LQR method which are essential to our purpose of finding a suitable matrix **Q** which guaranties desired performance.

2.1 Successive design

Theorem 1: Let us consider the system S, defined in (1), is controlled by

$$\mathbf{u} = -\mathbf{K}_1\mathbf{x} \tag{5}$$

where $\mathbf{K_1}$ is the feedback gain computed for systems based on LQR method with the pair of $\mathbf{Q_1}$ and **R** matrices. Defining system S' to be:

$$\text{S}':\ \dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K}_1)\mathbf{x} + \mathbf{B}\mathbf{u}_2 \tag{6}$$

We may state that control of S' with feedback gain $\mathbf{K_2}$ obtained from LQR method with the pair of $\mathbf{Q_2}$ and **R** would result in a similar closed loop system as the one obtained by control of S with **K** calculated based on LQR with the pair of $\mathbf{Q=Q_1+Q_2}$ and **R**.

Proof: Recalling definition of feedback gains obtained based on LQR method as defined in equations (3) and (4), one might write:

$$\mathbf{K}_1 = \mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}_1 \tag{7}$$

where $\mathbf{P_1}$ satisfies the Riccatti equation:

$$\mathbf{Q}_1 - \mathbf{P}_1\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}_1 + \mathbf{A}^{\mathbf{T}}\mathbf{P}_1 + \mathbf{P}_1\mathbf{A} = \mathbf{0} \tag{8}$$

and:

$$\mathbf{K}_2 = \mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}_2 \tag{9}$$

where $\mathbf{P_2}$ satisfies the Ricatti equation:

$$\mathbf{Q}_2 - \mathbf{P}_2\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}_2 + (\mathbf{A} - \mathbf{B}\mathbf{K}_1)^{\mathbf{T}}\mathbf{P}_2 + \mathbf{P}_2(\mathbf{A} - \mathbf{B}\mathbf{K}_1) = \mathbf{0} \tag{10}$$

Control of S' with feedback gain $\mathbf{K_2}$ would results in closed loop system of the form:

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K}_1)\mathbf{x} + \mathbf{B}(-\mathbf{K}_2\mathbf{x}) = (\mathbf{A} - \mathbf{B}\mathbf{K}_1 - \mathbf{B}\mathbf{K}_2)\mathbf{x} \tag{11}$$

Similarly one might obtain feedback gain **K** as follow:

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P} \tag{12}$$

where **P** should satisfy the Riccati equation:

$$\mathbf{Q}_1 + \mathbf{Q}_2 - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P} + \mathbf{A}^{\mathbf{T}}\mathbf{P} + \mathbf{P}\mathbf{A} = \mathbf{0} \tag{13}$$

Summation of equations (8) and (10) results in:

$$\begin{aligned}\mathbf{Q}_1 + \mathbf{Q}_2 - [(\mathbf{P}_1 + \mathbf{P}_2)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}(\mathbf{P}_1 + \mathbf{P}_2) - \mathbf{P}_1\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}_2 - \mathbf{P}_2\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}_1] + \\ + \mathbf{A}^{\mathbf{T}}(\mathbf{P}_1 + \mathbf{P}_2) + (\mathbf{P}_1 + \mathbf{P}_2)\mathbf{A} - (\mathbf{B}\mathbf{K}_1)^{\mathbf{T}}\mathbf{P}_2 - \mathbf{P}_2.\mathbf{B}\mathbf{K}_1 = \mathbf{0}\end{aligned} \tag{14}$$

Substituting for $\mathbf{K_1}$ from equation (7), into equation (14) gives:

$$\mathbf{Q}_1 + \mathbf{Q}_2 - (\mathbf{P}_1 + \mathbf{P}_2)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}(\mathbf{P}_1 + \mathbf{P}_2) + \mathbf{A}^{\mathbf{T}}(\mathbf{P}_1 + \mathbf{P}_2) + (\mathbf{P}_1 + \mathbf{P}_2)\mathbf{A} = \mathbf{0} \tag{15}$$

which means that the matrix $\mathbf{P=P_1+P_2}$ satisfies the Riccati equation (13). So the feedback gain would be:

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}(\mathbf{P}_1 + \mathbf{P}_2) = \mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}_1 + \mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}_2 = \mathbf{K}_1 + \mathbf{K}_2 \tag{16}$$

and the closed loop system will be:

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K}_1 - \mathbf{B}\mathbf{K}_2)\mathbf{x} \tag{17}$$

This is the same as the closed loop system obtained by successive controls defined in equation (11). This theory suggests that we may find the suitable **Q** through a step by step correction algorithm from an initial choice of **Q**. To find a correction matrix which pushes the poles of closed loop system behind a proper value to guarantee desired settling time. It is more convenient to work with canonical form of the system. In the next section, relation between **Q** matrices which results in similar closed loop system for two systems related through similarity transformation is discussed.

2.2 Similarity transformation

Let us consider a system S' which is obtained through a similarity transformation **T** from system S as follow:

$$\text{S}': \ \dot{\mathbf{z}} = \mathbf{A}'\mathbf{z} + \mathbf{B}'\mathbf{u} \tag{18}$$

in which **z** is the transformed state variable vector satisfying **x=Tz**. The system S' is similar with S, i.e. it has similar dynamic properties as S:

$$\begin{aligned} \mathbf{A}' &= \mathbf{T}^{-1}\mathbf{A}\mathbf{T} \\ \mathbf{B}' &= \mathbf{T}^{-1}\mathbf{B} \end{aligned} \tag{19}$$

Theorem 2: Control of system S with the feedback gain **K** obtained based on LQR with the pair of **Q** and **R** would result in similar closed loop system as the one obtained from controlling S' with **K'** obtained based on LQR with the pair of **Q'** and **R'** where:

$$\begin{aligned} \mathbf{Q}' &= \mathbf{T}^{\mathbf{T}}\mathbf{Q}\mathbf{T} \\ \mathbf{R}' &= \mathbf{R} \end{aligned} \tag{20}$$

Proof: The feedback gain **K** obtained from LQR solution of S with pairs (**Q**, **R**) should satisfy the following relations:

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P} \tag{21}$$

where **P** is a positive definite solution of Riccati equation:

$$\mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P} + \mathbf{A}^{\mathbf{T}}\mathbf{P} + \mathbf{P}\mathbf{A} = \mathbf{0}\,. \tag{22}$$

In a similar way **K'** should satisfy:

$$\mathbf{K}' = \mathbf{R}'^{-1}\,\mathbf{B}'^{\mathbf{T}}\,\mathbf{P}' \tag{23}$$

when **P'** should satisfy the Riccati equation:

$$\mathbf{Q}' - \mathbf{P}'\mathbf{B}'\mathbf{R}'^{-1}\,\mathbf{B}'^{\mathbf{T}}\,\mathbf{P}' + \mathbf{A}'^{\mathbf{T}}\,\mathbf{P}' + \mathbf{P}'\mathbf{A}' = \mathbf{0}\,. \tag{24}$$

Substituting for **A'** and **B'** from equation (19) and **Q'** and **R'** from (20) and $\mathbf{P}'=\mathbf{T}^{\mathbf{T}}\mathbf{P}\mathbf{T}$ in equation (24) results in:

$$\mathbf{T}^{\mathbf{T}}\left(\mathbf{A}^{\mathbf{T}}\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P} + \mathbf{Q}\right)\mathbf{T} = \mathbf{0}\,, \tag{25}$$

which means that if P satisfies equation (22) the $\mathbf{P}'=\mathbf{T}^{\mathbf{T}}\mathbf{P}\mathbf{T}$ is a solution to equation (24). As a result, **K'** would be:

$$\mathbf{K}' = \mathbf{R}'^{-1}\mathbf{B}'^{\mathbf{T}}\mathbf{P}' = \mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}(\mathbf{T}^{\mathbf{T}})^{-1}\mathbf{T}^{\mathbf{T}}\mathbf{P}\mathbf{T} = \mathbf{R}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}\mathbf{T}\,. \tag{26}$$

Substituting for **K** from equation (21), **K'** can be found as:

$$\mathbf{K}' = \mathbf{K}\mathbf{T}\,. \tag{27}$$

The closed loop system for systems S', controlled with **K'**, is:

$$\dot{\mathbf{z}} = (\mathbf{A}' - \mathbf{B}'\mathbf{K}')\mathbf{z}\,. \tag{28}$$

Substituting for **A'** and **B'** from (19) and **K'** from (27) in (28), one obtains:

$$\dot{\mathbf{z}} = (\mathbf{T}^{-1}\mathbf{A}\mathbf{T} - \mathbf{T}^{-1}\mathbf{B}\mathbf{K}\mathbf{T})\mathbf{z}\,. \tag{29}$$

Pre multiplying equation (29) with matrix **T** and considering definition of **z** in terms of **x**, we get:

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x} \tag{30}$$

Equation (30) is the same as the closed loop system obtained by controlling S with feedback gain **K**, which completes the proof.

2.3 Diagonal form

Any real time invariant diagonalizable system S can be divided into some independent real first and second order subsystems defined as:

$$\mathrm{S_i}\text{:}\ \dot{\mathbf{x}}_\mathbf{i} = \mathbf{p_i x_i} + \mathbf{B_i u} \tag{31}$$

where the second order subsystems have complex conjugate poles.

Now let us consider that LQR designed gain matrices $\mathbf{K_i}$ which are obtained using pairs of weighting matrices $\mathbf{Q_i}$ and **R** results in closed loop poles of σ_i. It is observed that control of S with a matrix gain **K** obtained from a LQR design of S based on weighting matrices **Q** and **R** whose **Q** is a diagonal matrix of $\mathbf{Q_i}$'s will push the closed loop poles of S toward σ, where σ is the array of closed loop poles of subsystems $\mathrm{S_i}$.

This observation in conjunction with the results of previous section suggests that if we can devise some formulation which can give proper weighting matrices to put the closed loop poles of first and second order subsystems (controlled by LQR method) at desired location we might be able to push the closed loop poles of our system toward desired location.

3 Pole placement through LQR method

As explained in previous section, we are interested in finding weighting matrix **Q** which if used in a LQR control design process, poles of the closed loop system is placed at a desired location. Such formulation must be developed for first order system as well as second order real systems with complex conjugate eigenvalues.

3.1 First order system

A first order system S is defined as:

$$\mathrm{S}\text{:}\ \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \tag{32}$$

Defining **D** as:

$$\mathbf{D} \equiv \mathbf{BR}^{-1}\mathbf{B}^\mathbf{T} \tag{33}$$

where **R** is an arbitrary positive definite matrix. Assuming that S is controlled by a gain **K** obtained from LQR design with a pair of weighting matrices **Q** and **R**, the closed loop poles satisfy the following equation:

$$\mathbf{A}\text{ - }\mathbf{DP} = \lambda, \tag{34}$$

which means that for closed loop poles to be λ, solution of the Riccati equation must satisfy:

$$\mathbf{P} = \mathbf{D}^{-1}(\mathbf{A} - \lambda), \tag{35}$$

and using Riccati equation **Q** can be calculated as:

$$\mathbf{Q} = \mathbf{PDP}\text{ - }\mathbf{A}^\mathbf{T}\mathbf{P}\text{ - }\mathbf{PA}. \tag{36}$$

Substituting for P from (35) and recalling that for a first order system, dimension of matrices A and D are one, we may rewrite equation (36) as:

$$Q = \frac{\lambda^2 - A^2}{D} \tag{37}$$

where:

$$\mathbf{Q} = [Q] \tag{38}$$

3.2 Second order system

Let us consider a second order system S defined as:

$$\mathrm{S}\text{:}\ \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \tag{39}$$

We wish to find a matrix Q which if used to design a LQR controller for S, puts the closed loop poles at $\lambda \pm \mu.i$. To this end, let us use the canonical form of S defined as:

$$\text{S': } \dot{\mathbf{z}} = \mathbf{A}'\mathbf{z} + \mathbf{B}'\mathbf{u} = \begin{bmatrix} a_1 + a_2 i & 0 \\ 0 & a_1 - a_2 i \end{bmatrix} \mathbf{z} + \begin{bmatrix} \vec{b}^T \\ \bar{\vec{b}}^T \end{bmatrix} \mathbf{u} \tag{40}$$

in which **A'** is a diagonal matrix and **B'** is a matrix whose rows are complex conjugate and also:

$$\mathbf{x} = \mathbf{T}\mathbf{z} \tag{41}$$

where **T** is the transformation matrix whose columns are complex conjugate.

Now we try to find matrix **Q'** which if used to design a LQR control for S', puts the closed loop poles at $\lambda \pm \mu.i$. Recalling equation (3), one can see that poles of closed loop system are eigenvalues of the matrix **A'-D'P'** in which **P'** is the solution of the Riccati equation and **D'** is defined as:

$$\mathbf{D}' \equiv \mathbf{B}'\mathbf{R}^{-1}\mathbf{B}'^{\mathbf{T}} \tag{42}$$

Considering the fact that for a 2×2 matrix sum of eigenvalues equals its trace and products of eigenvalues equals its determinant, one can write:

$$tr(\mathbf{D}'\mathbf{P}') = tr(\mathbf{A}) - 2\lambda \tag{43}$$

$$\det(\mathbf{A}' - \mathbf{D}'\mathbf{P}') = \lambda^2 + \mu^2 \tag{44}$$

On the other hand, considering the form of the transformation matrix **T**, it can be shown that for **Q** to be real, it is necessary for **Q'** and **P'** to have the following forms:

$$\mathbf{Q}' = \begin{bmatrix} Q'_{11} & Q'_{12} \\ \overline{Q'_{12}} & Q'_{11} \end{bmatrix} \tag{45}$$

and:

$$\mathbf{P}' = \begin{bmatrix} P'_{11} & P'_{12} \\ \overline{P'_{12}} & P'_{11} \end{bmatrix} \tag{46}$$

as the matrix **P'** must be symmetric, one immediately conclude that P'_{11} is real, which means that:

$$\mathbf{P}' = \begin{bmatrix} P'_{11r} & P'_{12r} + P'_{12i} i \\ P'_{12r} - P'_{12i} i & P'_{11r} \end{bmatrix} \tag{47}$$

Considering the form of **P'**, one can see that equations (43) and (44) are a set of two nonlinear functions of three unknowns, i.e. P'_{11r}, P'_{12r}, P'_{12i}. One might choose P'_{11r} and calculate P'_{12r} and P'_{12i} from equations (43) and (44).

As the matrix **P'** must be positive definite, P'_{11r} must be chosen sufficiently large such that it satisfies the following relation:

$$\det(P') = P'^2_{11r} - (P'^2_{12r} + P'^2_{12i}) > 0 \tag{48}$$

Once **P'** is determined **Q'** can be calculated using algebraic Riccati equation as:

$$\mathbf{Q}' = \mathbf{P}'\mathbf{D}'\mathbf{P}' - \mathbf{A}'^{\mathbf{T}}\mathbf{P}' - \mathbf{P}'\mathbf{A}' \tag{49}$$

and **Q** can be calculated as:

$$\mathbf{Q} = \mathbf{T}^{-1^{\mathbf{T}}}\mathbf{Q}'\mathbf{T}^{-1} \tag{50}$$

4 Q improvement algorithm

We are now at the position to introduce an algorithm which can improve the weighting matrix **Q** push the closed loop poles of system controlled with the LQR designed gain matrix towards design location. The algorithm would be as follow:

Step 1: Compute the LQR control with any arbitrary pair of **Q** and **R.**

Step 2: Transform closed loop system into its canonical form through transformation matrix **T** to obtain system S' defined as:

$$\text{S}':\ \dot{\mathbf{z}} = \mathbf{Az} + \mathbf{Bu} \tag{51}$$

Step 3: Find the dominant poles and their related subsystems. Compute $\mathbf{Q_i}$ which can place the dominant poles at desired location.

Step 4: Calculate the improved **Q** as:

$$\mathbf{Q} = \mathbf{Q} + \mathbf{T}^{-1^{\mathbf{T}}}\mathbf{Q}'\mathbf{T}^{-1} \tag{52}$$

where **Q'** is diagonal matrix of $\mathbf{Q_i}$'s.

Step 5: Find the closed loop poles of the controlled system with the gain obtained from LQR design with the pair of weighting matrices **Q** and **R**.

End this procedure if the closed loop poles are satisfactory. Otherwise either, choose the desired poles further and go to step 3, or choose some closer location for desired poles and go to step 2, i.e. try step by step approach.

5 Illustrative example

Problem: A system S defined as:

$$\text{S:}\ \begin{cases} \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{Y} = \mathbf{Cx} \end{cases} \tag{53}$$

where:

$$\mathbf{A} = \begin{bmatrix} -11 & 7 & 5 & 19 & -18 & -4 \\ 4 & 3 & 2 & -5 & 6 & 0 \\ 9 & -2 & 5 & 17 & 5 & 2 \\ 0 & 3 & 4 & -3 & 2 & 7 \\ 7 & 0 & 4 & 9 & 5 & 15 \\ 5 & 4 & 0 & 1 & 3 & 0 \end{bmatrix} \ \&\mathbf{B} = \begin{bmatrix} 2 & 5 \\ 7 & 1 \\ 3 & 0 \\ 9 & 4 \\ 6 & 1 \\ 7 & 3 \end{bmatrix} \ \&\mathbf{C} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{54}$$

is assumed. We wish to control the system such that it is stabilized and its settling time is less than one second.

Solution: As the pair of (**A**, **B**) is controllable, in first attempt we design a LQR control for that with a Q as shown in table 1 and $\mathbf{R}=\mathbf{I_2}$.

The subsystems of diagonalized system corresponding to dominant poles of closed loop system are:

$$\begin{aligned} \dot{\mathbf{x}}_3 &= \begin{bmatrix} -1.0594 + 3.9011i & 0 \\ 0 & -1.0594 - 3.9011i \end{bmatrix}\mathbf{x}_3 + \begin{bmatrix} 0.9799 + 7.7607i & 5.8748 - 1.8775i \\ 0.9799 - 7.7607i & 5.8748 + 1.8775i \end{bmatrix}\mathbf{u} \\ \dot{x}_4 &= \begin{bmatrix} -0.6230 \end{bmatrix} x_4 + \begin{bmatrix} -5.4664 & 9.6164 \end{bmatrix}\mathbf{u} \end{aligned} \tag{55}$$

To have settling time less than one second the dominant poles must be pushed behind line $\sigma = -5$. So we may choose:

$$p_{d3} = -5 \pm 4i \qquad \textit{and} \qquad p_{d4} = -5 \tag{56}$$

Using formulation introduced for computation of $\mathbf{Q_3}$ and $\mathbf{Q_4}$ results in:

$$\begin{aligned} \mathbf{Q_3} &= \begin{bmatrix} 0.2650 & 0.0930 + 0.0825i \\ 0.0930 - 0.0825i & 0.2650 \end{bmatrix} \\ Q_4 &= \begin{bmatrix} 0.2011 \end{bmatrix} \end{aligned} \tag{57}$$

and **Q'** is subsequently equal to:

$$\mathbf{Q}' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2650 & 0.0930 + 0.0825i & 0 \\ 0 & 0 & 0 & 0.0930 - 0.0825i & 0.2650 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2011 \end{bmatrix} \tag{58}$$

The results of this attempt are also shown in table 1 which is not satisfactory. We may now choose to put the poles at a further distance as:

$$p_{d3} = -7 \pm 4i \qquad \textit{and} \qquad p_{d4} = -11 \tag{59}$$

and start over from step 3 of algorithm. Results of this attempt, which are satisfactory, are given in table 1 too.

The same problem is solved with the method proposed by Bryson and Ho [2]. The results are given in table 2. One should note that continuing with this method provides no further improvement. Figure 1 shows time history of the response of closed loop system for the LQR designed control system as well as systems controlled by a LQR controlled improved by proposed method and also the method proposed by Bryson and Ho [2]. In all simulations initial condition is considered to be:

$$\mathbf{x_0} = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]^{\mathrm{T}} \tag{60}$$

Time history of inputs for these two improvement methods are given in Figure 2. As one can see the proposed method fulfilled the requirement yet the inputs are still reasonable. Comparison of the results obtained from these two methods proves versatility of proposed method.

Attempt #	Q	σ
Open loop	---	16.7 $-10.01 \pm 7.67i$ $0.83 \pm 4.62i$ 0.6796
First	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	-39.43 $-11.52 \pm 7.72i$ -1.06 ± 3.9 -0.62
Second	$\begin{bmatrix} 2.04 & 6.14 & -0.18 & -2.45 & 2.43 & -0.46 \\ 6.14 & 26.39 & -5.09 & -16.27 & 7.91 & -5.95 \\ -0.18 & -5.09 & 3.53 & 6.05 & -0.51 & 1.58 \\ -2.45 & -16.27 & 6.05 & 13.51 & -3.58 & 4.81 \\ 2.43 & 7.91 & -0.51 & -3.58 & 3.19 & -1.09 \\ -0.46 & -5.95 & 1.58 & 4.81 & -1.09 & 4.01 \end{bmatrix}$	-39.43 $-11.52 \pm 7.72i$ $-5.17 \pm 3.21i$ -3.18
Third	$\begin{bmatrix} 4.85 & 19.59 & -3.37 & -11.58 & 6.94 & -4.34 \\ 19.59 & 90.94 & -20.67 & -60.28 & 29.34 & -24.27 \\ -3.37 & -20.67 & 8.55 & 17.74 & -5.65 & 4.75 \\ -11.58 & -60.28 & 17.74 & 44.43 & -18.17 & 16.25 \\ 6.94 & 29.34 & -5.65 & -18.17 & 10.74 & -7.28 \\ -4.34 & -24.27 & 4.75 & 16.25 & -7.28 & 10.45 \end{bmatrix}$	-39.43 $-11.52 \pm 7.72i$ -11.70 $-5.13 \pm 1.17i$

Table 1. Numerical results for modifying matrix **Q** based on proposed method for reducing settling time to 1 second

Attempt #	Q	σ
First	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	-39.43 $-11.52 \pm 7.72i$ -1.06 ± 3.9 -0.62
Second	$\begin{bmatrix} 0.9756 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.9721 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8643 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9553 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9756 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.9654 \end{bmatrix}$	-22.5566 -12.4933 ±11.8413i -5.5845 ±3.3033i -3.0578

Table 2. Numerical results for modifying matrix Q based on Bryson Method

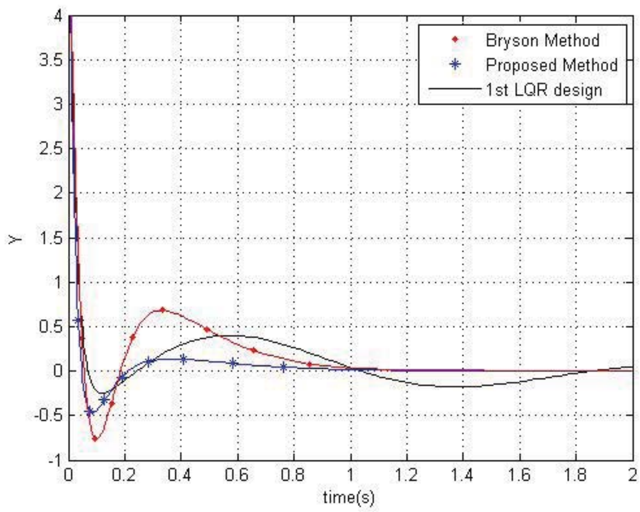

Figure 1. Time response of the closed loop systems designed by LQR method

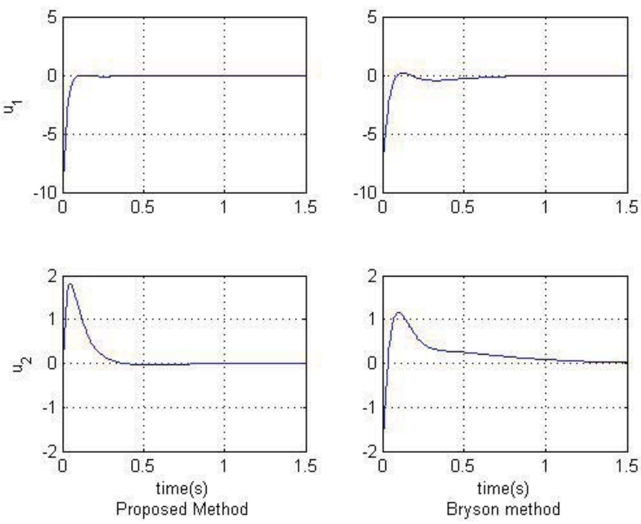

Figure 2. Time history of inputs to the system
Proposed improvement method (at left)
Improvement method proposed by Bryson and Ho [2] (at right)

6 Conclusion

This paper investigates the problem of performance enhancement of linear time invariant systems controlled by LQR method. The system in addition to controllability property needed for LQR technique has to be diagonalizable. A formulation is given base on which the weighting matrix **Q** can be calculated in order to place the closed loop poles of first and second order system at desired location. Based on this formulation an algorithm is then devised which can be used to improve the weighting matrix Q to push the closed loop poles towards desired values. The algorithm is used in an illustrative example whose results show versatility of algorithm.

7 References

[1] Johnson, M.A, and Grimble, M.J.: *Recent trends in linear optimal linear quadratic multivariable control system design*. IEEE proceeding, part D, 1987, pp 53-71.

[2] Bryson, A. E. and Ho, Yu-Chi.: *Applied Optimal Control*. John Wiley & Sons, 1975.

[3] Robandi, I. and Nishimori, K. and Nishimura, R. and Ishihara, N.: *Optimal Feedback Control Design Using Genetic Algorithm in Multi machine Power System*. Electrical Power & Energy Systems, Elsevier, 2001, pp 263-271.

[4] Ogata, K.: *Modern Control Engineering*. 2nd Edition, Prentice-Hall, 1990.

[5] Lewis, F.L. and Syrmos, V.L.: *Optimal Control*. 2nd Edition, Willey, 1995.

[6] Wylie, C.R.: *Advanced Engineering Mathematics*. 3rd Edition, McGraw-Hill, 1996.