ENERGY-BASED MODEL REDUCTION OF LINEAR SYSTEMS

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Abstract. Modeling metrics and algorithms that assist the development of dynamic system models are essential for efficient use of modeling and simulation in everyday engineering tasks. Various modeling techniques have been proposed in order to make the modeling procedure more systematic and easier to use by inexperienced modelers. The authors, in particular, previously developed an energy-based modeling metric called "element activity" that was implemented in the Model Order Reduction Algorithm (MORA). The motivation underlying the development of the element activity and MORA was to address model reduction of nonlinear systems. However, the purpose of this paper is to apply MORA to linear systems in order to obtain additional clarifying interpretations. Given the linearity and superposition property of the models in hand, the steady state response to sinusoidal inputs is considered. For this condition, the element activity can be calculated analytically for any given excitation frequency and, thus, a series of reduced models that depend on the excitation frequency can be generated. This methodology is applied to a linear quarter car model and the reduced models generated by MORA are compared to the results obtained from a frequency-based model deduction algorithm. The results show that MORA generates a series of frequency dependant proper models that are similar to those explicitly generated by frequency metrics. Thus, MORA, an energy-based model reduction technique appropriate for nonlinear systems, can also generate frequency-based reduced models with more refined reduction as compared to frequency-based metrics.

1 Introduction

Modeling and simulation have yet to achieve wide utilization as commonplace engineering tools. Part of the reason for this is that current modeling and simulation techniques are inadequate. One drawback is that they require sophisticated users who are often not domain experts and thus lack the ability to effectively utilize the model and simulation tools to uncover the important design trade-offs. Another is that models are often large and complicated, with a large number of parameters, making the physical interpretation of the model outputs, even by domain experts, difficult. This is particularly true when "unnecessary" features are included in the model. It is the premise of this work that more effective use of modeling and simulation necessitates the need for proper models, that is, models with physically meaningful states and parameters that are of necessary but sufficient complexity to meet the engineer's objective.

A variety of algorithms have been developed and implemented to help automate the production of proper dynamic system models. Wilson and Stein developed the Model Order Deduction Algorithm (MODA) that deduces the needed system model complexity from subsystem models of variable complexity using a frequency-based metric, [11]. Additional work on deduction algorithms for generating proper models has been reported in order to extend the applicability of the algorithm, [1] [2] [10]. These algorithms have been implemented and demonstrated in a computer automated modeling environment, [9].

In an attempt to overcome the limitations of the frequency-based metrics the authors introduced a new model reduction technique that also generates proper models, [4]. This approach uses an energy-based metric (element activity) that in general can be applied to nonlinear systems and considers the importance of all energetic elements (generalized inductance, capacitance and resistance) in the model, [5] [6] [7]. The contribution of each element in the model is ranked according to the energy metric under specific excitation. Elements with small contribution are eliminated from the model to produce a reduced model. The element activity metric has more flexibility from the existing frequency-based metrics that address the issue of model complexity by only adding compliant elements, leaving the importance of inertial and resistive unresolved. In contrast, the activity metric considers the importance of all energetic elements, and therefore, the significance of all energy elements in the model can be described. It is the purpose of this paper to apply this metric to linear systems in order to compare the complexity of the proper models produced by MORA over a specified frequency versus proper models produced by MODA.

This paper is organized as follows: first background about the energy-based metric is provided in Section 2. Section 3 shows the application of the element activity to linear systems where the steady state response is considered. Section 4 provides an illustrative example of a linear quarter car model, to demonstrate the generation of proper models of linear systems using the activity metric. Finally, a discussion and set of conclusions are given in Sections 5 and 6, respectively.

2 Background

The authors' original work on the energy-based metric for model reduction, [4], is briefly described here for convenience. The main idea behind this model reduction technique is to evaluate the "element activity" of the individual energy elements of a full system model under a stereotypic set of input and initial conditions. The activity of each energy element establishes a hierarchy for all the elements. Those elements below a user-defined threshold of acceptable level of activity are eliminated from the model, a reduced model is generated and a new set of governing differential equations is derived.

2.1 Element Activity

A measure of the power response of a dynamic system, which has physical meaning and a simple definition, is used to develop the modeling metric, element activity (or simply activity). The element activity, A, is defined for any energy element as:

$$A = \int_0^{\tau} \left|\mathcal{P}(t)\right| \cdot dt \tag{1}$$

where $\mathcal{P}(t)$ is the element power and τ is the time over which the model has to predict the system behavior. The activity has units of energy, representing the amount of energy that flows in and out of the element over the given time τ. The energy that flows in and out of an element is a measure of how active this element is, i.e., how much energy passes through it, and consequently the quantity in Eq. (1) is termed activity.

The activity is calculated for each energy element in a system based on the response. In the case that the system is modeled using a bond graph formulation, the state equations are derived using the multiport bond graph representation, [3] [8], and the state equations have the general form:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x},\mathbf{u}), \quad \mathbf{x}(0) = \mathbf{x}_o \tag{2}$$

where:

$\mathbf{x} \in \Re^n$ is the state vector, and n is the number of independent states,

$\mathbf{u} \in \Re^m$ is the input vector, and m is the number of inputs,

$\mathbf{F} : \Re^n \times \Re^m \to \Re^n$ is a nonlinear continuous vector function,

$\boldsymbol{x}_o$ are the initial conditions of the states.

Appropriate outputs are defined in order to calculate the power of each element in the model using the constitutive law of each element. For convenience, the outputs are selected to be flow, effort, and flow for inertial, compliant, and resistive elements, respectively. The output vector for this set of variables has the form:

$$\mathbf{y} = \left\{ \begin{array}{c} \mathbf{f}_I \\ \hline \mathbf{e}_C \\ \hline \mathbf{f}_R \end{array} \right\} \tag{3}$$

where $\mathbf{y} \in \Re^k$ and $\mathbf{f}_I, \mathbf{e}_C$, and $\mathbf{f}_R$ are vectors of size k_I, k_C, and k_R respectively. The variables k_I, k_C, and k_R represent the number of inertial, compliant and resistive elements, respectively, and total number of energy elements in the system is $k = k_I + k_C + k_R$. For the output variables given Eq. (3) and using the multiport bond graph representation, the output equations can be written as:

$$\mathbf{y} = \mathbf{H}(\mathbf{x},\mathbf{u}) \tag{4}$$

where $\mathbf{H} : \Re^n \times \Re^m \to \Re^k$ is a nonlinear vector function. Given these variables, the required efforts or flows for calculating the element power are computed from their constitutive law as shown below:

$$\begin{aligned} &\text{I: } f = \Phi_I(p) \Leftrightarrow p = \Phi_I^{-1}(f) \Leftrightarrow e = \dot{p} = \dot{f} \cdot \frac{\partial}{\partial f}\left(\Phi_I^{-1}(f)\right) \\ &\text{C: } e = \Phi_C(q) \Leftrightarrow q = \Phi_C^{-1}(e) \Leftrightarrow f = \dot{q} = \dot{e} \cdot \frac{\partial}{\partial e}\left(\Phi_c^{-1}(e)\right) \\ &\text{R: } f = \Phi_R(e) \Leftrightarrow e = \Phi_R^{-1}(f) \end{aligned} \tag{5}$$

where Φ_I, Φ_C, Φ_R are known functions.

Finally, the power needed for calculating the activity in Eq. (1) is computed from Eq. (5) as:

$$\begin{aligned} &\text{I: } \mathbb{P}_I = e \cdot f = \dot{f} \cdot \frac{\partial}{\partial f}\left(\Phi_I^{-1}(f)\right) \cdot f \\ &\text{C: } \mathbb{P}_C = e \cdot f = e \cdot \dot{e} \cdot \frac{\partial}{\partial e}\left(\Phi_c^{-1}(e)\right) \\ &\text{R: } \mathbb{P}_R = e \cdot f = e \cdot \Phi_R(e) = f \cdot \Phi_R^{-1}(f) \end{aligned} \tag{6}$$

2.2 Activity Index

The activity as defined in Eq. (1) is a measure of the absolute importance of an element as it represents the amount of energy that flows through the element over a given time period. In order to obtain a relative measure of the importance, the element activity is compared to a quantity that represents the "overall activity" of the system. This "overall activity" is defined as the sum of all the element activities of the system, is termed total activity (A^{Total}) and is given by:

$$A^{Total} = \sum_{i=1}^{k} A_i \tag{7}$$

where A_i is the activity of the i^{th} element given by Eq. (1). Thus a normalized measure of element importance, called the element activity index or just activity index, is defined as:

$$AI_i = \frac{A_i}{A^{Total}} = \frac{\int_0^{\tau} \left|\mathbb{P}_i(t)\right| \cdot dt}{\sum_{i=1}^{k} \left\{ \int_0^{\tau} \left|\mathbb{P}_i(t)\right| \cdot dt \right\}} \qquad i = 1,\ldots,k \tag{8}$$

The activity index, AI_i, is calculated for each element in the model and represents the portion of the total system energy that flows through a specific element. This calculation is typically performed on the results produced by numerical integration of the system state equations. MORA then produces a reduced model by eliminating elements from the model based on their activity.

3 Activity Analysis of Linear Systems

The activity metric and MORA have been previously formulated for the general nonlinear case. By restricting the use of MORA to linear systems, analytical expressions for the activity can be obtained. The analysis is simplified even more if, in addition to the linearity assumption, the system is assumed to have a single sinusoidal input, and only the steady state response is examined. These assumptions are motivated from Fourier analysis where an arbitrary function can by decomposed into a series of harmonics. Using this decomposition, the analysis is first carried out for a single input sinusoidal excitation under steady state conditions. Then, the effects of each harmonic are superposed, using the superposition principle, to get the aggregate response of the system caused by a general input $u(t)$. Furthermore, the superposition principle can also be used in the case that the system has more than one input. The response for each input is separately calculated, and then, the individual responses are superposed. Consequently, any dynamic response of a linear system can be calculated using the superposition principle and the response of a single input single frequency excitation. Therefore, the generality of the analysis is maintained even if the previous assumptions are made.

The same analysis for calculating the activity (see Section 2.1) is performed for the case of a linear system. In this case, the system has linear junction structure and linear constitutive laws, which yield linear time invariant state equations. Therefore, for linear systems Eq. (2) and Eq. (4) can be written in the matrix form:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{u} \\ \mathbf{y} &= \mathbf{C} \cdot \mathbf{x} + \mathbf{D} \cdot \mathbf{u} \end{aligned} \tag{9}$$

where $\mathbf{A} \in \Re^{n \times n}, \mathbf{B} \in \Re^{n \times m}, \mathbf{C} \in \Re^{k \times n}, \mathbf{D} \in \Re^{k \times m}$ are the constant state space matrices.

Furthermore, the constitutive laws of linear energy elements are given by:

$$\begin{aligned} &\text{I: } \Phi_I\left(p\right) = \frac{1}{z_I} p \\ &\text{C: } \Phi_C\left(q\right) = \frac{1}{z_C} q \\ &\text{R: } \Phi_R\left(e\right) = \frac{1}{z_R} e \end{aligned} \tag{10}$$

where z_I, z_C, z_R are known constants. A vector, **z**, with all the constitutive law coefficients is introduced as shown below:

$$\mathbf{z} = \begin{Bmatrix} \mathbf{z}_I \\ \mathbf{z}_C \\ \mathbf{z}_R \end{Bmatrix} \tag{11}$$

where $\mathbf{z} \in \Re^k$, $\mathbf{z}_I \in \Re^{k_I}$, $\mathbf{z}_C \in \Re^{k_C}$, and $\mathbf{z}_R \in \Re^{k_R}$. This vector is used later in the analysis to produce concise expressions.

The linear constitutive laws are used to calculate the efforts and flows needed to compute the power. Substitution of the linear constitutive laws of Eq. (10) into the general nonlinear expressions in Eq. (6) simplifies the calculation of the element power to:

$$\begin{aligned} &\text{I: } \mathbb{P}_I = e \cdot f = z_I \cdot f \cdot \dot{f} \\ &\text{C: } \mathbb{P}_C = e \cdot f = z_C \cdot e \cdot \dot{e} \\ &\text{R: } \mathbb{P}_R = e \cdot f = z_R \cdot f^2 \end{aligned} \tag{12}$$

3.1 Single Input Sinusoidal Excitation of Frequency ω

The time response of the required effort and flow outputs, $\mathbf{y}(t)$, in Eq. (3) is computed in order to complete the calculation of the element power. The system in Eq. (9) has m inputs, however, one input at a time is considered to simplify the analysis. Recall that the superposition principle can be used to reconstruct the system response with all the inputs acting on the system at the same time. In addition, the excitation for the j^{th} input is assumed sinusoidal. Therefore, the single excitation of the system can be described as:

$$u_j(t) = U_j \cdot \sin(\omega \cdot t) \tag{13}$$

where U_j is the amplitude of the excitation and ω is the excitation frequency. The steady state response for the excitation in Eq. (13) is calculated using linear system analysis and the outputs are given by:

$$y_{ij}\left(t,\omega\right) = U_j \cdot Y_{ij}\left(\omega\right) \cdot \sin\left(\omega \cdot t + \varphi_{ij}\left(\omega\right)\right), \; i = 1,\ldots,k, \; j = 1,\ldots,m \tag{14}$$

where:

$Y_{ij}\left(\omega\right) = \left|G_{ij}(j \cdot \omega)\right|$, is the amplitude of the i^{th} output due to the j^{th} input

$\varphi_{ij}\left(\omega\right) = \angle G_{ij}(j \cdot \omega)$, is the phase shift of the i^{th} output due to the j^{th} input

$\mathbf{G}(s) = \mathbf{C} \cdot \left(\mathbf{A} - s \cdot \mathbf{I}\right)^{-1} \cdot \mathbf{B} + \mathbf{D}$

The output, $y_{ij}(t,\omega)$ in Eq. (14), is either an effort or a flow needed to calculate the power of each element in Eq. (12). The element power can then be used to calculate the element activity. However, to compute the activity, the upper bound of the integral in Eq. (1) must be determined first. For this case, the periodicity feature of the response is exploited. A periodic function repeats itself every T seconds, and therefore, a single period of this function contains the required information about the response. Thus, the upper bound of the integral is set to the period, $T = 2\pi/\omega$, of the excitation. The activity of the energy elements is calculated by substituting Eq. (14) into Eq. (12) and then by using the definition of the activity in Eq. (1). These substitutions give the following expressions for the steady state activity for the different types of energy elements. Note that the superscript, ss, denotes the activity due to steady state response only.

$$A_{ij}^{ss} = \int_0^T \left| \frac{z_i \cdot \omega \cdot \left(U_j \cdot Y_{ij}\right)^2 \sin\left(2\left(\omega \cdot t + \varphi_{ij}\right)\right)}{2} \right| \cdot dt,\; i = 1,\ldots,k_I + k_C$$
$$A_{ij}^{ss} = \int_0^T \left| z_i \cdot \left(U_j \cdot Y_{ij}\right)^2 \sin^2(\omega \cdot t + \varphi_{ij}) \right| \cdot dt, \quad i = k_I + k_C + 1,\ldots,k \tag{15}$$

After evaluating the integrals in Eq. (15) for a given input frequency ω, the following results are obtained:

$$A_{ij}^{ss}\left(\omega\right) = 2 \cdot z_i \cdot U_j^{\,2} \cdot Y_{ij}^{\,2}\left(\omega\right), \quad i = 1,\ldots,k_I + k_C$$
$$A_{ij}^{ss}\left(\omega\right) = \frac{\pi \cdot z_i \cdot U_j^{\,2} \cdot Y_{ij}^{\,2}\left(\omega\right)}{\omega}, \quad i = k_I + k_C + 1,\ldots,k \tag{16}$$

Next, the activity indices are calculated as defined in Eq. (8). For the activities in Eq. (16) the activity indices become:

$$AI_{ij}^{ss}\left(\omega\right) = \frac{2 \cdot z_i \cdot Y_{ij}^{\,2}}{2\sum_{i=1}^{k_I + k_C} z_i \cdot Y_{ij}^{\,2} + \frac{\pi}{\omega} \sum_{i=k_I + k_C + 1}^{k} z_i \cdot Y_{ij}^{\,2}}, \quad i = 1,\ldots,k_I + k_C$$
$$AI_{ij}^{ss}\left(\omega\right) = \frac{\frac{\pi}{\omega} \cdot z_i \cdot Y_{ij}^{\,2}}{2\sum_{i=1}^{k_I + k_C} z_i \cdot Y_{ij}^{\,2} + \frac{\pi}{\omega} \sum_{i=k_I + k_C + 1}^{k} z_i \cdot Y_{ij}^{\,2}}, \quad i = k_I + k_C + 1,\ldots,k \tag{17}$$

The input amplitude, U_j, does not appear in any of the element activity indices, as it is shown in Eq. (17), since all element activities are proportional to the square of the amplitude.

The above results for the activity index can be used by MORA to generate reduced models. The generated results are only valid for a single input frequency and for only the j^{th} system input. The activity expressions in Eq. (16) are of little direct practical use since the assumed conditions usually do not exist in practice. However, these expressions can be combined to generate a single model that is valid for a more general input. A procedure for accomplishing this is presented in the next section.

3.2 General Excitation

The linear system, as originally defined in Eq. (9), can be excited by m inputs, which can be a general function of time. The analysis of the activity in this case can be treated as a special case of the analysis, described in Section 2, where the state equations and constitutive laws are linear. However, the results from the single input sinusoidal excitation can also be used to calculate the element activity. To make the use of the previous results possible, the input must be decomposed into a series of harmonics.

For a smooth and bounded input function, the Fourier theorem can be used to expand this function into an infinite series (integral) of sine and cosine terms. In addition, the cosine and sine terms can be combined to obtain a single amplitude and phase shift. More specifically the j^{th} input can be expanded as:

$$u_j\left(t\right) = \int_0^\infty U_j\left(\omega\right) \cdot \sin\left(\omega \cdot t + \varphi_j\left(\omega\right)\right) \cdot d\omega \tag{18}$$

In practice, this decomposition can be produced from measurements and usually the results are presented in the frequency domain rather than in the time domain. For example, in the vehicle dynamics area, road measurements are taken in order to provide realistic excitation to vehicle models. These measurements are then analyzed using Fourier analysis and expressed in a frequency domain form such as the one in Eq. (18).

Given the frequency decomposition, the element activity for the general input, $u_j(t)$, is calculated from the steady state activities as previously derived in Eq. (16). The steady state activity in this case is a function of frequency, since the amplitudes of the sine terms of the Fourier expansion in Eq. (18), are a function of the frequency ω. The input, $u_j(t)$, is equal to the integral of sine functions, therefore, the element activity due to this input should be the integral of the steady state activities for a given frequency, ω. This is expressed as:

$$A_{ij} = \int_0^\infty A_{ij}^{ss}\left(\omega\right) \cdot d\omega, \quad i = 1,\ldots,k \tag{19}$$

Substitution of the steady state activities from Eq. (16) into Eq. (19) gives the activities for the different types of energy elements:

$$\begin{aligned} A_{ij} &= \int_0^\infty 2 \cdot z_i \cdot U_j^2(\omega) \cdot Y_{ij}^2(\omega) \cdot d\omega, \quad i = 1,\dots,k_I + k_C \\ A_{ij} &= \int_0^\infty \frac{\pi \cdot z_i \cdot U_j^2(\omega) \cdot Y_{ij}^2(\omega)}{\omega} \cdot d\omega, \quad i = k_I + k_C + 1,\dots,k \end{aligned} \tag{20}$$

The above results can be extended for the multi-input case. The overall activity due to the system excitation by all the m inputs is the sum of the activities due to each input separately. Therefore, the element activity is:

$$\begin{aligned} A_i &= \sum_{j=1}^{m} \left\{ \int_0^\infty 2 \cdot z_i \cdot U_j^2(\omega) \cdot Y_{ij}^2(\omega) \cdot d\omega \right\}, \quad i = 1,\dots,k_I + k_C \\ A_i &= \sum_{j=1}^{m} \left\{ \int_0^\infty \frac{\pi \cdot z_i \cdot U_j^2 \cdot Y_{ij}^2(\omega)}{\omega} \cdot d\omega \right\}, \quad i = k_I + k_C + 1,\dots,k \end{aligned} \tag{21}$$

The results in Eq. (21) can be used to calculate the total activity and the element activity indices. With the activity indices known, MORA can be applied to rank order the importance of each element in the model and then generate reduced models.

4 Illustrative Example

A quarter car model is selected as the linear system to which we apply MORA and compare the results to those obtained from MODA applied to the same system. This model is chosen because it has been extensively used in the automotive literature and the frequency dependent properties of the system are well understood. This example is used to generate reduced models under a broadband excitation representing the road surface roughness. The model consists of the sprung mass, namely, the major mass supported by the suspension, and the unsprung mass, which includes the wheel and axle masses supported by the tire. The suspension is modeled as a spring and a damper in parallel, which are connected to the unsprung mass. The tire is also modeled as a spring and a damper in parallel, which transfer the road force to the unsprung mass (wheel hub). The input to the system is the road profile that prescribes a velocity, $V_r(t)$, to the contact point of the tire with the road. The system is composed of six ideal energy elements described by an equal number of parameters. The bond graph model of the system is depicted in Figure 1 and the parameters are given in the Appendix.

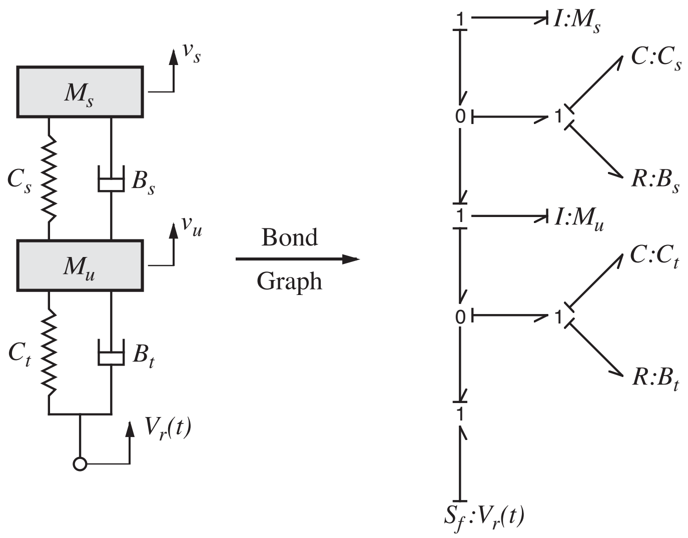

Figure 1: Full quarter car model

4.1 Broadband Frequency Excitation

To generate a reduced model that is valid for an input with broad frequency content, the methodology described in the Section 3.2 is applied. To calculate the element activity for a general input, the amplitude of the input

($U_j(\omega)$ in Eq. (21)) as a function of frequency is also required. In addition, the input is also assumed to have a finite bandwidth, ω_b, such that the reduced models can be compared with the models generated by the frequency-based metrics, e.g., MODA. Therefore, system input that is the road velocity is expanded as:

$$V_r\left(t\right) = \int_0^{\omega_b} U_r\left(\omega\right) \cdot \sin\left(\omega \cdot t\right) \cdot d\omega \tag{22}$$

The amplitude, $U_r\left(\omega\right)$, is usually defined from a specific input (i.e. road profile meter data), however, an input which is "equally weighted" for each frequency is used here. The term "equally weighted" is considered in the context of power flow, since the activity metric is based on energy (power flow). Therefore, the amplitude, $U_r\left(\omega\right)$, is defined such that the power flowing into the system is "equal" for all frequencies. This assumption is made to achieve an equivalent excitation as the one used by MODA. The power flowing into the system, in this example, is calculated as the product of the road velocity and the contact force implied to the road by the system. The road velocity is a prescribed sine function, and thus, the steady state response of contact force can be calculated from the model.

$$F_r\left(\omega,t\right) = U_r\left(\omega\right) \cdot \bar{F}_r\left(\omega\right) \cdot \sin\left(\omega \cdot t + \varphi_r\left(\omega\right)\right) \tag{23}$$

where $\bar{F}_r\left(\omega\right)$ is the steady state amplitude and $\varphi_r\left(\omega\right)$ the phase shift for an excitation of frequency ω. Therefore, the power flowing into the system is:

$$\mathbb{P}_r\left(\omega,t\right) = U_r^{\,2}\left(\omega\right) \cdot \bar{F}_r\left(\omega\right) \cdot \sin\left(\omega \cdot t\right) \cdot \sin\left(\omega \cdot t + \varphi_r\left(\omega\right)\right) \tag{24}$$

The power flowing into the system is "equally weighted" when amplitude of the power flow in Eq. (24) is constant for all frequencies (for convenience it is set to one, see Figure 2), thus:

$$U_r^{\,2}\left(\omega\right) \cdot \bar{F}_r\left(\omega\right) = 1 \tag{25}$$

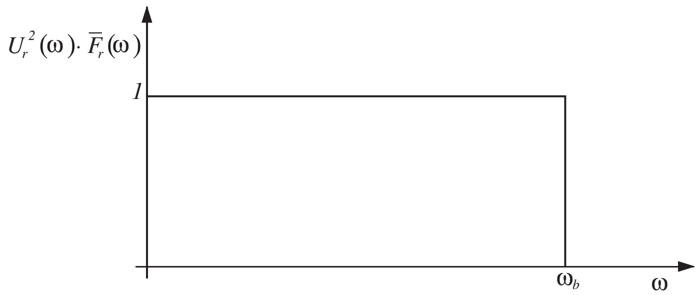

Figure 2: Frequency content of input power flow

Therefore, the amplitude of the road velocity is given by:

$$U_r\left(\omega\right) = \sqrt{\frac{1}{\bar{F}_r\left(\omega\right)}} \tag{26}$$

Finally, the substitution of Eq. (26) into Eq. (21) gives the activities of all the energy elements:

$$\begin{aligned} A_i &= \int_0^{\omega_b} \frac{2 \cdot z_i \cdot Y_i^2\left(\omega\right)}{\bar{F}_r\left(\omega\right)} \cdot d\omega, \quad i = 1,2,3,4 \\ A_i &= \int_0^{\omega_b} \frac{\pi \cdot z_i \cdot Y_i^2\left(\omega\right)}{\omega \cdot \bar{F}_r\left(\omega\right)} \cdot d\omega, \quad i = 5,6 \end{aligned} \tag{27}$$

4.2 Activity Analysis and Model Reduction

Using Eq. (27), the activities are calculated as a function of the road bandwidth. The road bandwidth is varied from *0.1 rad/s* to *10000 rad/s* in order to include frequencies well below the low natural frequency (*8 rad/s*) and above the high natural frequency (*76 rad/s*). The element activities and activity indices are shown in Figure 3. The element activities are monotonically increasing, since the integrands in the expressions in Eq. (27) are always positive. Therefore, as the bandwidth increases, the activity increases too. At low frequencies the most active elements are the two masses (sprung and unsprung). These elements handle the bulk of the energy flow induced into the system by the input. As the bandwidth increases the activity of all the elements increases, and finally at frequencies beyond the second natural frequency all elements have almost the same activity.

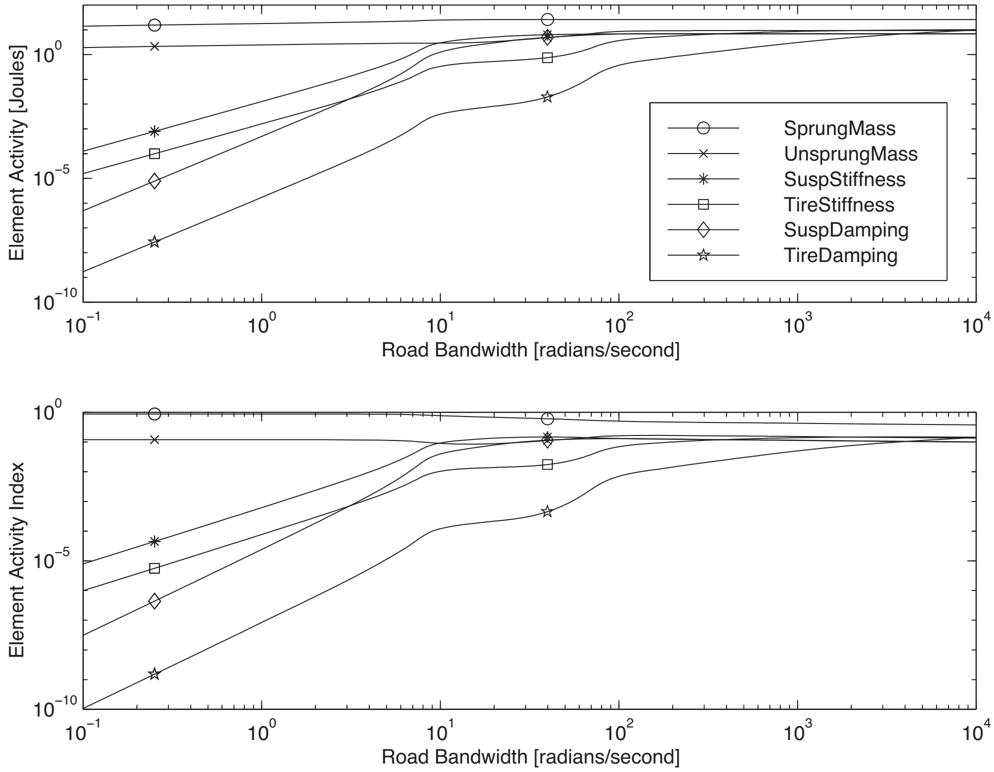

Figure 3: Activity and Activity Indices

MORA is then applied to generate reduced models. Based on a *98%* activity threshold (arbitrary threshold), the model complexity is determined as a function of the road bandwidth. Figure 4 shows the range of frequencies over which each element is included in the reduced model according to MORA. The y-axis represents each element, where the thick line defines the range of frequencies, over which the element should be included, while no line implies the range of frequencies over which the element should be eliminated. The labels of the y-axis represent the names of the energy elements.

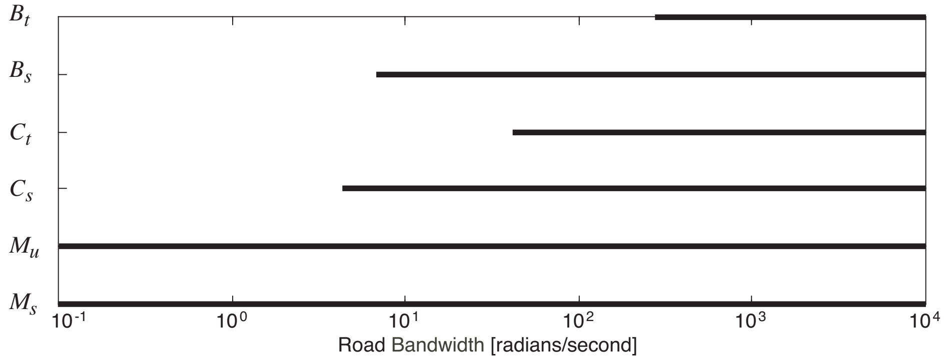

Figure 4: Included elements, Threshold = *98%*

As shown in Figure 4, a low bandwidth excitation requires only the rigid body model with the two masses as the only elements included in the reduced model. As the bandwidth increases up to *4.7 rad/s*, the suspension stiffness must also be included. Just before the first natural frequency at *7.2 rad/s*, the suspension damping is also added to the reduced model as its activity increases above the specified elimination threshold. The tire stiffness is the next element included in the model before the second natural frequency (*43.7 rad/s*). Eventually, at a bandwidth beyond the second natural frequency (*288 rad/s*), MORA does not eliminate any energy elements, and therefore, the reduced model is the same as the full model. This is intuitive, because a rich input excites all system dynamics, and therefore, a model that includes all the dynamics of the system (full model) is needed to accurately predict the system behavior. The reduced model complexity (number of included energy elements) and reduced bond graphs are also shown as a function of the input bandwidth in Figure 5.

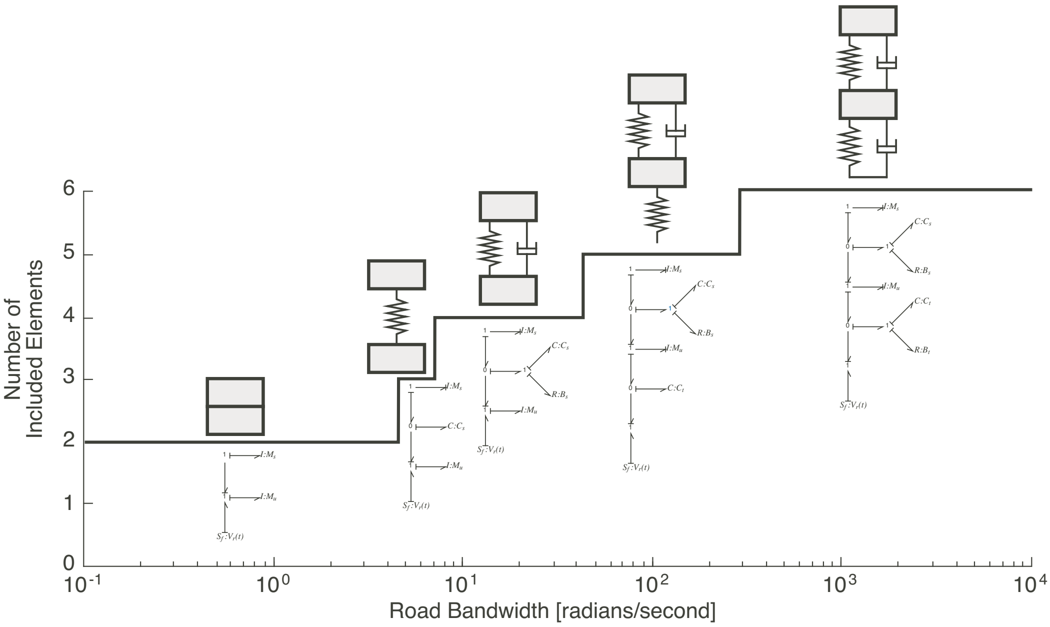

Figure 5: Model complexity, Threshold = *98%*

4.3 Models deduced by MODA

MODA is also used to deduce a series of proper models as a function of the input frequency. A series of proper models are generated for a series of Frequency Range of Interests (FROI), in order to examine the required model complexity needed as a function of frequency. MODA is used to generate a proper model for each FROI within the same frequency range used previously by MORA. The results are shown in Figure 6 where the system complexity along with the proper model bond graphs for the different frequencies are plotted. In addition, Figure 6 shows the reduced models that MORA generates. Note that the model complexity in Figure 6 is plotted against the road bandwidth, and therefore, the FROI is divided by 5 in order to convert the FROI into the road bandwidth. This factor (5) is used in order to account for the proper model accuracy, [11].

Three different proper models are generated depending on the selected FROI. For low frequencies, MODA suggests that the proper model is just the two masses rigidly connected, which is expected, since the system behaves like a rigid body for frequencies well below the fundamental natural frequency. After the FROI is increased beyond the first natural frequency (*8/5 rad/s*), the complexity is increased in order to include the suspension stiffness and damping. The same model is valid up to the second natural frequency (*76/5 rad/s*) where the tire stiffness and damping are also included. The same system model is valid for all FROIs above the second natural frequency.

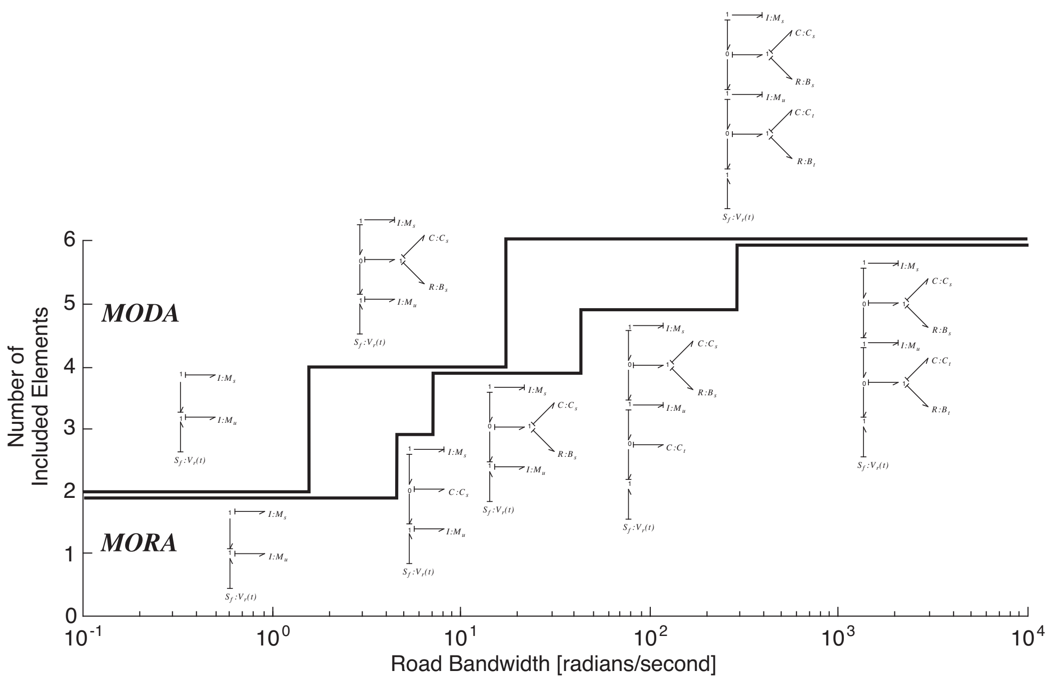

Figure 6: Comparison of MORA with MODA

5 Discussion

5.1 Implications of MORA applied to linear systems

Efficiency: The properties of linear systems simplify the application of activity to determine the importance of energy elements in a model. This makes MORA, which has been previously shown to be useful for nonlinear systems, to be an even more attractive algorithm for automating the creation of proper models of linear systems. The analytical expressions derived in this paper make activity calculations more efficient than computing the activities from numerical simulations of the full model as described in Section 2. The only restriction is that the input can be expressed in a Fourier expansion series.

Complexity vs. Bandwidth: MORA was developed to assess the importance of energy elements in a general nonlinear system model. The assumption is that elements with low activity are of low importance to the accuracy of the model. While the connection between activity and model accuracy has been shown anecdotally, and even shown to be related to the frequency content in the model, [6], the results in this paper show that for linear systems, activity is a function of frequency. More specifically, Eq. (21) shows that as the frequency of excitation increases the activity of the model grows. This result, while not being directly applicable to nonlinear systems, still provides a more solid foundation for the assumption that low activity appears to be related to high bandwidth behavior of a system and, therefore, not very important to the behavior of the high energy, low frequency gross "motion" of a system.

5.2 Comparison to MODA

Algorithm Conservatism: In the illustrative example, MORA is compared with a model deduction algorithm, MODA. For low bandwidths, the two algorithms generate the same model by including only the two masses of the system. As the input bandwidth increases, MODA creates equivalent to more complicated models than MORA. For example, MODA includes the suspension stiffness at a frequency five times smaller than the first natural frequency, whereas MORA includes it just before the first natural frequency. The same phenomenon occurs for the tire stiffness, where MODA includes it at a frequency five times smaller than the second natural frequency, while MORA includes it just before the second natural frequency. Thus for the example shown, MODA appears to be a more conservative algorithm, generating more complicated models than MORA. The conservativeness of MODA is a consequence of the magnification coefficient used to determine the FROI (about 5 times the input frequency). The magnitude of this coefficient is selected to account for the model accuracy. Because no specific definition of accuracy is used by MORA, one could argue that neither MORA nor MODA is inherently more conservative. It depends on the choice of the activity threshold versus the FROI coefficient.

Equality of Assessment: MODA has a modal orientation to model complexity. That is as more modes are used the larger the bandwidth of the model. With finite segment models such as used in the example, more modes means adding two states as another capacitance (with sometimes a common flow resistance) and another inductance to the model. MORA does not have this orientation; it simply treats all energy elements strictly on the basis of their activity. Thus, MORA can identify when resistance elements (independent of other elements) become important and need to be included in the model. For example, in the illustrative example, the suspension damping (resistance) is included just before the first natural frequency. The tire damping does not have a large contribution, and therefore, it is not included except at higher frequencies when it does become important.

6 Summary and Conclusions

A previously developed concept, an energy-based modeling metric called activity, is applied to linear systems to compare model complexity as a function of frequency. It is shown that when considering the sinusoidal steady state response, the derivation of analytical expressions for the activity as a function of the input frequency is possible. Furthermore, the activity due to a general system excitation (e.g., input with a given frequency content or bandwidth) can be computed based on the "steady state" activity and the Fourier expansion of the excitation. It is also shown that the activity varies with the frequency content of the excitation. Thus, a previously published algorithm, MORA, which is based on the activity metric, can generate a series of models. This series of models are shown to be ordered such that as the model complexity increases the model is accurate over a wider frequency range. These results are compared to those obtained using a previously published Model Order Deduction Algorithm (MODA). It is shown that MORA generates models of similar complexity to those generated by MODA for a given frequency range. Finally, in contrast to MODA, MORA is shown to account equally for the contributions of all the energy elements to the total system response. The results of this paper provide more insight into the nature of the reduced ordered models produced by MORA, and therefore, demonstrate that MORA is an even more useful tool than previously realized for the production of proper models of nonlinear systems. Furthermore, it appears to have some advantages over MODA for the creation of proper linear system models.

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9 Appendix

Sprung Mass, M_s	z_1 = 267 Kg
Suspension Stiffness, K_s	z_3 = 18,742 N/m
Suspension Damping, B_s	z_5 = 700 N·s/m
Unsprung Mass, M_u	z_2 = 36.6 Kg
Tire Stiffness, K_t	z_4 = 193,915 N/m
Tire Damping, B_t	z_6 = 200 N·s/m