

NUMERICAL MATRIX EXPONENTIAL FUNCTION DERIVATIVE VIA LAPLACE TRANSFORM APPROACH

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Abstract. The paper deals with a method how to determine a derivative of a matrix exponential function with respect to a parameter inside a matrix of the exponent. The considered technique is based on a Laplace transform approach when, in the transform domain, the derivative is easily stated. To get a result in the original domain, however, it is necessary to use some numerical technique of an inverse Laplace transform (NILT). In the paper, two such methods are presented. To ensure numerical stability of the computation the NILT method is always preceded by scaling to decrease a Euclidean norm of the matrix below a predefined value, and followed by squaring to return it to the original value. The method finds its practical application in various fields of the electrical engineering simulation, e.g. for a sensitivity analysis in systems with multiconductor transmission lines as their distributed parts.

1 Introduction

A necessity to compute a derivative of the matrix exponential function can appear in various branches of the electrical engineering simulation. For example, in the field of the circuit theory, it can arise in the process of a sensitivity determination at the electrical circuits containing multiconductor transmission lines (MTL) [1]. The properties of an $(n+1)$ -conductor transmission line in the s -domain can be expressed using a chain matrix

$$\Phi(x, s) = e^{\mathbf{M}(s)x}, \quad (1)$$

where x denotes a geometric coordinate along the MTL and $\mathbf{M}(s)$ is the $2n$ -order square matrix

$$\mathbf{M}(s) = \begin{bmatrix} \mathbf{0} & -\mathbf{Z}_0(s) \\ -\mathbf{Y}_0(s) & \mathbf{0} \end{bmatrix}, \quad (2)$$

with $\mathbf{Z}_0(s) = \mathbf{R}_0 + s\mathbf{L}_0$ and $\mathbf{Y}_0(s) = \mathbf{G}_0 + s\mathbf{C}_0$ as the n -order series impedance and shunting admittance matrices, respectively, \mathbf{R}_0 , \mathbf{L}_0 , \mathbf{G}_0 and \mathbf{C}_0 as MTL per-unit-length matrices, and $\mathbf{0}$ as a zero matrix. The sensitivity with respect to a component (a parameter γ) of the MTL per-unit-length matrix can be got from the first derivative of (1), because $\gamma \in \mathbf{M}(s)$ [1]. Similarly γ can be any physical parameter influencing these per-unit-length matrices.

To generalize this approach for any type of a matrix, however, let us denote the matrix in view as $\mathbf{M} \equiv \mathbf{M}(\gamma)$, when its dependence on γ is evident. The matrix exponential function is then formulated as

$$\Phi(\gamma, x) = e^{\mathbf{M}(\gamma)x}, \quad (3)$$

with x standing for any real parameter, and $\partial\Phi(\gamma, x)/\partial\gamma$ is what we want to determine. It is known that a chain rule of a differentiation is applicable only in a special case, $\mathbf{M}(\gamma) = \mathbf{M}\gamma$, when $\partial\Phi(\gamma, x)/\partial\gamma = \mathbf{M}x\Phi(\gamma, x) = \Phi(\gamma, x)\mathbf{M}x$ is simple to find. In the above motivation example, it would correspond to a case we want to determine a sensitivity w.r. to a length l of the MTL, when $\gamma \equiv l$ and then $\partial\Phi(\gamma, l)/\partial l = \mathbf{M}(s)\Phi(\gamma, l) = \Phi(\gamma, l)\mathbf{M}(s)$. Generally, however, this rule cannot be used as can be proven by expanding (3) into a Taylor series and differentiating it term by term. To solve this problem a few approaches can be considered, e.g. based just on the Taylor series expansion, an augmented matrix specially created, a similarity transformation, a convolution integral evaluation or a Padé approximation method [2], [3]. Besides, a method based on a Laplace transformation, in conjunction with a proper technique for a numerical inversion of Laplace transforms (NILT), e.g. [4], [5], can be used. So that the method is applicable just for the example above, it has to be generally able to process complex matrices $\mathbf{M}(\gamma)$.

2 Principle of the method

The method is based on the fact that the derivative $\partial\Phi(\gamma, x)/\partial\gamma$ is easier to find after a Laplace transform of the matrix exponential function with respect to x is performed. It means its q -domain image is first found, then differentiated and finally inverted into the x domain again. A commutativity property of the integration and derivative operations leads to an equality

$$\frac{\partial\Phi(\gamma, x)}{\partial\gamma} = \mathbb{L}_x^{-1} \left\{ \frac{\partial}{\partial\gamma} \mathbb{L}_x \{ \Phi(\gamma, x) \} \right\}, \quad (4)$$

where the Laplace transform can analytically be stated as

$$\mathbb{L}_x \{ \Phi(\gamma, x) \} = \int_0^\infty e^{M(\gamma)x} e^{-qx} dx = \int_0^\infty e^{-(q\mathbf{I}-M(\gamma))x} dx = (q\mathbf{I}-M(\gamma))^{-1}, \tag{5}$$

with \mathbf{I} as an identity matrix. Based on (4) and (5), and doing some arrangements, the result is

$$\frac{\partial \Phi(\gamma, x)}{\partial \gamma} = \mathbb{L}_x^{-1} \left\{ (q\mathbf{I}-M(\gamma))^{-1} \frac{\partial M(\gamma)}{\partial \gamma} (q\mathbf{I}-M(\gamma))^{-1} \right\}. \tag{6}$$

The derivative $\partial M(\gamma)/\partial \gamma$ can already be stated easily. The inverse Laplace transform has to be done numerically in general. To guarantee the sufficiently fast convergence the scaling and squaring are applied [3]. The derivative is found for a transformed matrix exponential function $\Phi(\gamma, x/r)$, for such the r in order to meet the Euclidean norm $\|M(\gamma)x/r\| < 0.5$. This temporal result is transformed back to get the original derivative. These procedures are explained here as follows. First, considering a formula based on (3) as

$$\Phi(\gamma, x) = \Phi^r(\gamma, x/r), \tag{7}$$

then if $r = 2^M$, M integer, a restoring $\Phi(\gamma, x)$ from $\Phi(\gamma, x/r)$ can fast be performed by a squaring process,

$$\Phi(\gamma, x/2^{m-1}) = \Phi^2(\gamma, x/2^m), \tag{8}$$

successively for $m = M, M-1, \dots, 1$. Then, a recursive formula to restore $\partial \Phi(\gamma, x)/\partial \gamma$ from $\partial \Phi(\gamma, x/r)/\partial \gamma$ has a form

$$\frac{\partial \Phi(\gamma, x/2^{m-1})}{\partial \gamma} = \frac{\partial \Phi(\gamma, x/2^m)}{\partial \gamma} \Phi(\gamma, x/2^m) + \Phi(\gamma, x/2^m) \frac{\partial \Phi(\gamma, x/2^m)}{\partial \gamma}, \tag{9}$$

resulting directly from (8). In such a case the inverse Laplace transform is solved for $x = 1$.

3 NILT techniques considered

It is possible to use various types of the NILT techniques, however, they must keep the complexity of the original. That is why the Bromwich integral is considered as

$$\hat{f}(x) = \mathbb{L}_x^{-1} \{ \hat{F}(q) \} = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \hat{F}(q) e^{qx} dq, \tag{10}$$

where $\hat{F}(q)$, q -domain Laplace transform, can be a function of more complex variables, which is designated by a hood above F . Similarly, an x -domain original $\hat{f}(x)$ designates a complex function in general. In the paper, two proper NILT methods will be compared as for their accuracy and numerical stability. First, the NILT [4] will newly be modified to enable such a computation, second, the NILT [5] will be summarized.

3.1 Transform core approximation

A first method results from the theory in [4] where approximate formulae for the NILT based on approximations of the core e^{qx} in a Bromwich integral are developed. They are, however, intended to invert standard transforms leading to real functions in the original domain. Here, these formulae will be modified into their more general forms so that complex originals (10) are enabled. Namely, if the approximation is used as

$$e^{qx} \doteq \frac{e^a}{2 \sinh(a-qx)}, \quad \text{with} \quad \frac{1}{\sinh z} = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 \pi^2 + z^2}, \tag{11}$$

then, by using residue theorem, an approximate formula can be derived as

$$\hat{f}_s(x) = \frac{e^a}{2x} \left\{ \hat{F}\left(\frac{a}{x}\right) + \sum_{n=1}^{\infty} (-1)^n \left[\hat{F}\left(\frac{a}{x} + jn\frac{\pi}{x}\right) + \hat{F}\left(\frac{a}{x} - jn\frac{\pi}{x}\right) \right] \right\}. \tag{12}$$

Similarly, the approximation

$$e^{qx} \doteq \frac{e^a}{2 \cosh(a-qx)}, \quad \text{with} \quad \frac{1}{\cosh z} = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n (n+1/2)}{(n+1/2)^2 \pi^2 + z^2}, \tag{13}$$

leads to the formula

$$\hat{f}_c(x) = \frac{e^a}{j2x} \sum_{n=1}^{\infty} (-1)^n \left[\hat{F}\left(\frac{a}{x} + j(n-1/2)\frac{\pi}{x}\right) - \hat{F}\left(\frac{a}{x} - j(n-1/2)\frac{\pi}{x}\right) \right]. \tag{14}$$

Note that for $\hat{F}(q) \equiv F(q)$ being a real function of the only complex variable q , when $F(q^*) = F^*(q)$, where $*$ denotes a complex conjugate, the formulae (12) and (14) lead to clean real original functions, due to equalities $F(q) + F^*(q) = 2 \operatorname{Re}\{F(q)\}$ and $F(q) - F^*(q) = j2 \operatorname{Im}\{F(q)\}$, respectively. The absolute errors of the formulae (12) and (14) can be derived expanding left parts in (11) and (13) into McLaurine series, respectively [4]. When further considering a basic assumption of the validity of (10), $|\hat{f}(x)| < Me^{\alpha x}$, with M real positive, α as an exponential order, a relationship between a coefficient a and a relative error δ_M required can be written as

$$a = \alpha x + \frac{1}{2} \ln \left(1 + \frac{1}{\delta_M} \right). \tag{15}$$

It can be shown that when an arithmetic mean

$$\hat{f}_m(x) = \left(\hat{f}_s(x) + \hat{f}_c(x) \right) / 2 \tag{16}$$

is used instead of (12) or (14) own, the error is further suppressed, while (15) is valid replacing $\frac{1}{2}$ for $\frac{1}{4}$ only. As is obvious the errors can be controlled by choosing a proper value a . In practice, bounded functions are most often considered, with $\alpha = 0$, and consequently a taken as a constant. However, due to the necessity to sum up a finite number of terms only in (12) and (14) the errors differ from these theoretical. To get nearer to them an Euler transform is convenient to be used to accelerate a convergence of alternating infinite series [4].

3.2 Bromwich integral numerical integration

A second NILT method follows the work [5]. Substituting $q = c + j\omega$ into (10) and using a rectangular rule of the integration, with the generalized frequency step $\Omega = 2\pi/\tau$, an approximate formula can be derived as

$$\hat{f}_l(x) = \frac{e^{cx}}{\tau} \left[\hat{F}(c) + \sum_{n=1}^{\infty} \left(\hat{F}(c + jn\Omega) e^{jn\Omega x} + \hat{F}(c - jn\Omega) e^{-jn\Omega x} \right) \right]. \tag{17}$$

Detailed error analysis in [5] shown that a relative error δ_M satisfies to the formula

$$c = \alpha + \frac{1}{\tau} \ln \left(1 + \frac{1}{\delta_M} \right), \tag{18}$$

with α as an exponential order, valid for $x \in O_{err} = \langle 0, \tau \rangle$. It allows to set up an integration path by c accordingly. Note an interesting thing, namely if $\tau = 2x$ is chosen, (17) and (18) come to (12) and (15), respectively, when denoting $a = cx$ and considering an identity $e^{\pm jn\pi} = (-1)^n$. Here, however, (17) will be evaluated on an interval in which a constant relative error can be kept, on a set of discrete points $\{x_k\}$. It can be advantageous e.g. just for distributed circuits simulation, as determining above mentioned sensitivity if needed to get it along MTL wires. Expressing (17) in $x_k = kX$, for $k = 0, 1, \dots, N-1$, i.e. on the interval $O_{max} = \langle 0, (N-1)X \rangle$, where X acts as the sampling period in the original domain, one can get a discrete formula

$$\hat{f}_l(kX) = C_k \left(\sum_{n=0}^{N-1} \hat{F}_n z_k^n + \sum_{n=0}^{\infty} \hat{G}_n z_k^n + \sum_{n=0}^{N-1} \hat{F}_{-n} z_k^{-n} + \sum_{n=0}^{\infty} \hat{G}_{-n} z_k^{-n} - \hat{F}_0 \right). \tag{19}$$

To fulfil a necessary condition $O_{max} \subset O_{err}$, then $\tau = NX$ can suitable be set up, leading to $C_k = e^{ckX}/NX$. Further, $z_k = e^{jkX\Omega}$, $\hat{F}_n = \hat{F}(c + jn\Omega)$, with $\Omega = 2\pi/NX$, and $\hat{G}_n = \hat{F}_{N+n}$, $z_k^N = e^{j2k\pi} = 1, \forall k$, were considered. In practice, $X = x_m/(N/2 - 1)$ is used, with x_m as the upper limit of an interval of computation, $O_{comp} = \langle 0, x_m \rangle$.

It is evident that when choosing $N = 2^m$, m integer, the first and third finite sum in (19) can very effectively be evaluated by the IFFT and FFT algorithms, respectively. Besides a few terms of the infinite sums in (19) enter either an epsilon (ϵ) or a quotient-difference ($q-d$) algorithm to accelerate a convergence of the summations, coming near to the theoretical error (18) by this [5].

4 Examples and conclusion

First the above stated method will be applied to evaluate a derivative of the matrix exponential function (3), provided that $\mathbf{M}(\gamma) = [m_{ij}(\gamma)]$, $i, j = 1, 2$, is a second-order matrix. In this case an analytical solution can be formulated helping us to determine relative errors of the method. Thus, considering eigenvalues of $\mathbf{M}(\gamma)$ as distinct, $\lambda_1(\gamma) \neq \lambda_2(\gamma)$, a Baker's formula can be utilized to derive

$$\Phi'(\gamma, x) = \frac{1}{D(\gamma)} \left[\left(D_1'(\gamma, x)x - D_1(\gamma, x) \frac{D'(\gamma)}{D(\gamma)} \right) \mathbf{M}(\gamma) + D_1(\gamma, x) \mathbf{M}'(\gamma) + \left(D_0'(\gamma, x) - D_0(\gamma, x) \frac{D'(\gamma)}{D(\gamma)} \right) \mathbf{I} \right], \tag{20}$$

$D(\gamma) = \lambda_2(\gamma) - \lambda_1(\gamma)$, $D_0(\gamma, x) = \lambda_2(\gamma)e^{\lambda_1(\gamma)x} - \lambda_1(\gamma)e^{\lambda_2(\gamma)x}$, $D_1(\gamma, x) = e^{\lambda_2(\gamma)x} - e^{\lambda_1(\gamma)x}$, and $D'(\gamma) = \lambda_2'(\gamma) - \lambda_1'(\gamma)$, $D_0'(\gamma, x) = (\lambda_2(\gamma)xe^{\lambda_1(\gamma)x} - e^{\lambda_2(\gamma)x})\lambda_1'(\gamma) - (\lambda_1(\gamma)xe^{\lambda_2(\gamma)x} - e^{\lambda_1(\gamma)x})\lambda_2'(\gamma)$, $D_1'(\gamma, x) = e^{\lambda_2(\gamma)x}\lambda_2'(\gamma) - e^{\lambda_1(\gamma)x}\lambda_1'(\gamma)$ denote their derivatives w.r. to γ . The eigenvalues are $\lambda_{1,2}(\gamma) = \text{tr}(\mathbf{M}(\gamma))/2 \pm \sqrt{[\text{tr}(\mathbf{M}(\gamma))/2]^2 - \det(\mathbf{M}(\gamma))}$, and their derivatives $\lambda_{1,2}'(\gamma) = [\text{tr}(\mathbf{M}'(\gamma))\lambda_{1,2}(\gamma) - \det'(\mathbf{M}(\gamma))]/[2\lambda_{1,2}(\gamma) - \text{tr}(\mathbf{M}(\gamma))]$, with $\det'(\cdot)$ as a derivative of the determinant. As the examples, let us successively consider two matrices as in (21) and (22), namely

$$\mathbf{M}(\gamma) = \begin{bmatrix} 2\gamma & 1-\gamma^2 \\ -3\gamma & \gamma \end{bmatrix}, \quad (21) \quad \text{and} \quad \mathbf{M}(s) = \begin{bmatrix} 0 & -(R_0 + sL_0) \\ -(G_0 + sC_0) & 0 \end{bmatrix}. \quad (22)$$

The matrix (21) is chosen randomly, being a function of a parameter γ , but satisfying the condition $\lambda_1(\gamma) \neq \lambda_2(\gamma)$. Relative errors defined via Euclidean norms, $\delta_E = \|\Phi'_{num} - \Phi'_{the}\|/\|\Phi'_{the}\|$, were computed based on (6) and (20), while the NILT (16) was applied for a numerical evaluation, see Figure 1 (at left). The matrix (22) corresponds to (2), being used for a single transmission line simulation. Its components are $R_0 = 75 \Omega/m$, $L_0 = 494.6 nH/m$, $G_0 = 0.1 S/m$ and $C_0 = 62.8 pF/m$, a length of the line is $l = 0.03m$ (is substituted for x), and $\gamma \equiv R_0$ is chosen. A relative error is computed by the same method as in the previous case, but now as a function of the imaginary part of the complex frequency $s = \sigma + j\omega$, see Figure 1 (at right). The same range of s as described should roughly be used if applying a subsequent NILT method (12) or (14) to get results in the time domain.

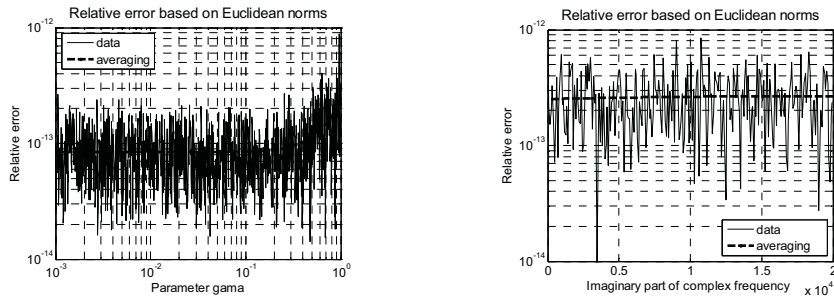


Figure 1. Relative error for $\partial\Phi(\gamma, x)/\partial\gamma$, considering (21), as a function of γ (at left), and relative error for $\partial\Phi(s, x)/\partial R_0$, considering (22), as a function of $\omega = \text{Im}\{s\}$ (at right)

Finally, Figure 2 shows the NILT (19) is applied to get sensitivities in (x, t) domain on a given MTL wire. In this case the matrices (2) of the 4th order are processed. The method is accurate and stable enough for a practical use.

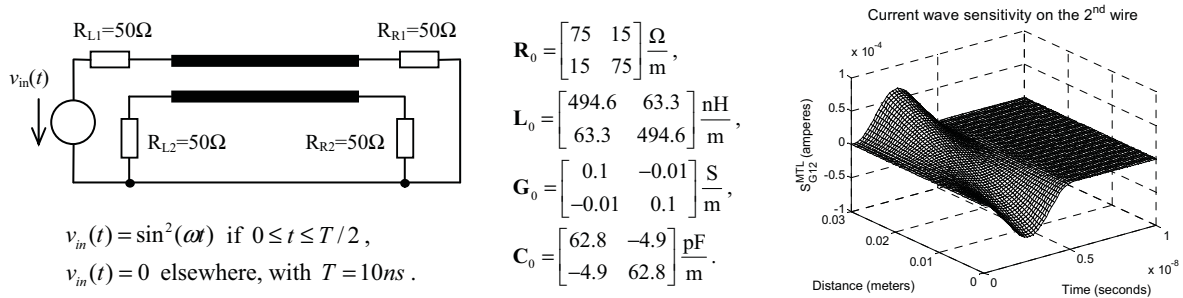


Figure 2. The (2+1)-conductor TL (at left), MTL p.-u.-l. matrices (at middle), and (x, t) -domain sensitivity (at right)

5 References

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