I. Troch, F. Breitenecker, eds. ISBN 978-3-901608-35-3

SYNCHRONIZATION OF A GENERAL CLASS OF NEURAL NETWORKS WITH STOCHASTIC PERTURBATION

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Abstract. This paper studies the problem of synchronization for a general class of neural networks with time-varying delays and stochastic perturbation. By defining Lyapunov functionals, a new delay-dependent stability criterion for the synchronization is established in terms of linear matrix inequality (LMI). A numerical example is given to illustrate the proposed method.

1 Introduction

During the last decade, neural networks are widely studied, because of their immense potentials of application prospective in a variety of areas such as signal processing, patten recognition, associative memory and combinatorial optimization [1]. Since significant time delays are ubiquitous both in neural processing and in signal transmission, it is necessary to introduce delays into communication channels which lead to delayed neural networks (DCNNs) model. Recently, it has been revealed that if neural network's parameters and time delays are appropriately chosen, the DCNNs can exhibit some complicated dynamics and even chaotic behaviors. Hence, it has attracted many scholars to study the synchronization of chaotic DCNNs [2]. In real nervous systems, synaptic transmission is a noisy process brought on by random fluctuations from the release of neurotransmitters [3]. Therefore, it is of practical importance to study the stochastic effects on the stability property of delayed neural networks [4]. In Li and Cao [5], the synchronization problem for DCNNs with stochastic perturbation has been investigated. On the other hand, due to the complicated dynamic properties of the neural cells in the real world, the existing neural network models in many cases cannot characterize the properties of a neural reaction process precisely. It is natural and important that systems will contain some information about the derivative of the past state to further describe and model the dynamics for such complex neural reactions [6].
This paper considers a general class of neural networks with stochastic perturbation, i.e. neutral-type neutral network. To date, the synchronization problem for neural networks of neutral type with stochastic perturbation has not been investigated fully. By Lyapunov method and LMI framework, a stability criterion for synchronization between two neural networks is obtained in the paper.
Notation: For symmetric matrices X and Y, the notation $X > Y$ (respectively, $X \geq Y$) means that the matrix $X-Y$ is positive definite, (respectively, nonnegative). $diag\{\cdots\}$ denotes the block diagonal matrix. $\star$ represents symmetric part of a given matrix. The notation $\rho(A)$ denotes the spectral radius of A. For $h>0, \mathscr{C}([-h,0],\mathscr{R}^n)$ means the family of continuous functions ϕ from $[-h,0]$ to $\mathscr{R}^n$ with the norm $\|\phi\| = \sup_{-h\leq s\leq 0}|\phi(s)|$. Let $(\Omega, \mathscr{F}, \{F_t\}_{t\geq 0}, \mathscr{P})$ be a complete probability space with a filtration $\{F_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathscr{F}_0$ contains all $\mathscr{P}$-pull sets). $L^p_{\mathscr{F}_0}([-h,0],\mathscr{R}^n)$ the family of all $\mathscr{F}_0$ - measurable $\mathscr{C}([-h,0],\mathscr{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -h \leq \theta \leq 0\}$ such that $\sup_{-h\leq\theta\leq 0}\mathbf{E}|\xi(\theta)|^p < \infty$ where $\mathbf{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure $\mathscr{P}$. Denote by $\mathscr{C}^{2,1}(\mathscr{R}^n \times \mathscr{R}^+, \mathscr{R}^+)$ the family of all nonnegative functions $V(x,t)$ on $\mathscr{R}^n \times \mathscr{R}^+$ which are continuously twice differentiable in x and differentiable in t .

2 Problem Statements

Consider a class of neural networks with time-varying delays

$$d[x(t) - Cx(t-h(t))] \quad = \quad \Big[-Ax(t) + W_0 f(x(t)) + W_1 f(x(t-h(t))) + J\Big]dt, \tag{1}$$

where $x(t) = [x_1(t), x_2(t), ..., x_n(t)]^T \in \mathscr{R}^n$ is the neuron state vector, n denotes the number of neurons in a neural network, $f(x(t)) = [f_1(x_1(t)), ..., f_n(x_n(t))]^T \in \mathscr{R}^n$ denotes the neuron activation function, $A = diag\{a_i\}$ is a positive diagonal matrix, $W_0 = (w^0_{ij})_{n\times n}$, $W_1 = (w^1_{ij})_{n\times n}$, and $C = (c_{ij})_{n\times n}$ are the interconnection matrices representing the weight coefficients of the neurons, $J = [J_1, J_2, ..., J_n]^T$ means a constant input vector, and $h(t)$ is time-varying delay.
In this paper, it is assumed that $0 \leq h(t) \leq \overline{h}$ and $\dot{h}(t) \leq h_d < 1$, and the matrix C satisfies $\rho(C) < 1$.

The activation functions, $f_i(x_i(t)), i = 1,2,...,n$, are assumed to be nondecreasing, bounded and globally Lipschitz; that is,

$$|f_j(\xi_j)| \le |l_j(\xi_j)|, \ \forall\ \xi_j \ne 0,\ j = 1,...,n, \tag{2}$$

where l_i is known real constant.
For synchronization problem, let us take system (1) as a drive network, then, the following response network is constructed:

$$\begin{aligned} d[y(t) - Cy(t-h(t))] &= \Big[-Ay(t) + W_0 f(y(t)) + W_1 f(y(t-h(t))) \\ &+ J + u(t)\Big] dt + [H_0(y(t)-x(t)) + H_1(y(t-h(t)) - x(t-h(t))]\, d\omega(t), \end{aligned} \tag{3}$$

where $y(t) = [y_1(t), y_2(t), ..., y_n(t)]^T \in \mathscr{R}^n$ is the state vector of response network, $u(t)$ is the control input for achieving synchronization, H_0 and H_1 are known constant matrices with appropriate dimensions, $\omega(t)$ is a scalar Wiener Process (Brownian Motion) on $(\Omega, \mathscr{F}, \{F_t\}_{t\ge 0}, \mathscr{P})$ which satisfies $\mathbf{E}\{d\omega(t)\} = 0$ and $\mathbf{E}\{d\omega^2(t)\} = dt$. This type of stochastic perturbation can be regarded as a result from the occurrence of external random fluctuation and other probabilistic causes.
By defining the error signal as $e(t) = y(t) - x(t)$, the error dynamic equation is:

$$\begin{aligned} d[e(t) - Ce(t-h(t))] &= \Big[-Ae(t) + W_0 g(e(t)) + W_1 g(e(t-h(t))) + u(t)\Big] dt \\ &+ [H_0 e(t) + H_1 e(t-h(t))]\, d\omega(t), \end{aligned} \tag{4}$$

where $g(e(t)) = f(x(t)+e(t)) - f(x(t))$.
In this paper, the following feedback controller for synchronization between drive network (1) and response network (3) is proposed:

$$u(t) = Ke(t), \quad K = diag\{k_1, k_2, \cdots, k_n\}. \tag{5}$$

The following definition will be used in deriving main result.
Definition 1. For the stochastic neural networks (4) and every $\phi \in L^2_{\mathscr{F}_0}([-\bar{h}, 0], \mathscr{R}^n)$, the trivial solution is globally asymptotically stable in the mean square if

$$\lim_{t\to\infty} \mathbf{E}|x(t,\phi)|^2 = 0. \tag{6}$$

3 Main results

In this section, we propose a new criterion for synchronization of neural networks with stochastic perturbation described by (1) and (3).

Now, the following theorem gives our main result.
Theorem 1. For given h_d, $\bar{h}$, and $L = diag\{l_1,\ l_2, ...,\ l_n\}$, the equilibrium point of (4) is globally asymptotically stable in the mean square if there exist positive diagonal matrices $P, Y, T_i (i = 1,2)$, and positive definite matrices $Q_i(i = 1,2), R_i(i = 1,2,3)$ satisfying the following LMI:

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & \Psi_{15} & 0 & 0 \\ \star & \Psi_{22} & \Psi_{23} & \Psi_{24} & 0 & 0 & 0 \\ \star & \star & \Psi_{33} & 0 & \Psi_{35} & 0 & 0 \\ \star & \star & \star & \Psi_{44} & \Psi_{45} & 0 & 0 \\ \star & \star & \star & \star & \Psi_{55} & 0 & 0 \\ \star & \star & \star & \star & \star & \Psi_{66} & 0 \\ \star & \star & \star & \star & \star & \star & \Psi_{77} \end{bmatrix} < 0 \tag{7}$$

where

$$\begin{aligned} \Psi_{11} &= -PA - A^T P + Y + Y^T + R_1 + R_2 + H_0^T P H_0 + \bar{h} H_0^T Q_2 H_0 + L^T T_1 L, \\ \Psi_{12} &= A^T PC + H_0^T P H_1 + \bar{h} H_0^T Q_2 H_1 - Y^T C,\ \ \Psi_{13} = PW_0,\ \ \Psi_{14} = PW_1,\ \ \Psi_{15} = -A^T P + Y^T, \\ \Psi_{22} &= H_1^T P H_1 - (1-h_d) R_1 + \bar{h} H_1^T Q_2 H_1 + L^T T_2 L,\ \ \Psi_{23} = -C^T P W_0,\ \ \Psi_{24} = -C^T P W_1, \\ \Psi_{33} &= R_3 - T_1, \Psi_{35} = W_0^T P,\ \ \Psi_{44} = -(1-h_d) R_3 - T_2,\ \ \Psi_{45} = W_1^T P, \\ \Psi_{55} &= \bar{h}^2 Q_1 - 2P,\ \ \Psi_{66} = -Q_1,\ \ \Psi_{77} = -(1-2h_d) R_2. \end{aligned}$$

Then, the control gain K of synchronization controller (5) is $K = P^{-1} Y$.

Proof. System (1) can be represented as

$$d[e(t) - Ce(t-h(t))] = q(t) dt + z(t) d\omega(t). \tag{8}$$

where

$$q(t) \equiv -(A-K)e(t)+W_0g(e(t))+W_1g(e(t-h(t))),\ \ z(t)\equiv H_0e(t)+H_1e(t-h(t)). \tag{9}$$

Let us consider the following Lyapunov-Krasovskii functional candidate:

$$\begin{aligned} V(e_t,t) &= [e(t)-Ce(t-h(t))]^TP[e(t)-Ce(t-h(t))]+\int_{t-h(t)}^{t} e^T(s)R_1e(s)ds \\ &\quad +\int_{t-2h(t)}^{t} e^T(s)R_2e(s)ds+\int_{t-h(t)}^{t} g^T(e(s))R_3g(e(s))ds \\ &\quad +\bar{h}\int_{-\bar{h}}^{0}\int_{t+\theta}^{t} q^T(s)Q_1q(s)dsd\theta+\int_{-\bar{h}}^{0}\int_{t+\theta}^{t} z^T(s)Q_2z(s)dsd\theta \end{aligned} \tag{10}$$

where $e_t = e(t+\theta),\ -2\bar{h}\le\theta\le 0$.
Then, by Ito's formula, the stochastic differential $dV(e_t,t)$ can be obtained as

$$dV(e_t,t)=LV(e_t,t)dt+2[e(t)-Ce(t-h(t))]^TP[H_0e(t)+H_1e(t-h(t))]d\omega(t) \tag{11}$$

where

$$\begin{aligned} LV(e_t,t) &= 2[e(t)-Ce(t-h(t))]^TP[-(A-K)e(t)+W_0g(e(t))+W_1g(e(t-h(t)))] \\ &\quad +[H_0e(t)+H_1e(t-h(t))]^TP[H_0e(t)+H_1e(t-h(t))] \\ &\quad +e(t)^TR_1e(t)-(1-\dot{h}(t))e^T(t-h(t))R_1e(t-h(t)) \\ &\quad +e(t)^TR_2e(t)-(1-2\dot{h}(t))e^T(t-2h(t))R_2e(t-2h(t)) \\ &\quad +g(e(t))^TR_3g(e(t))-(1-\dot{h}(t))g^T(e(t-h(t)))R_3g(e(t-h(t))) \\ &\quad +\bar{h}^2q^T(t)Q_1q(t)-\bar{h}\int_{t-\bar{h}}^{t} q^T(s)Q_1q(s)ds+\bar{h}z^T(t)Q_2z(t)-\int_{t-\bar{h}}^{t} z^T(s)Q_2z(s)ds. \end{aligned} \tag{12}$$

Using Jansen inequality, we have

$$dV(e_t,t)\le \Sigma dt+2[e(t)-Ce(t-h(t))]^TP[H_0e(t)+H_1e(t-h(t))]d\omega(t) \tag{13}$$

where

$$\begin{aligned} \Sigma &= 2[e(t)-Ce(t-h(t))]^TP[-(A-K)e(t)+W_0g(e(t))+W_1g(e(t-h(t)))] \\ &\quad +[H_0e(t)+H_1e(t-h(t))]^TP[H_0e(t)+H_1e(t-h(t))]+e(t)^TR_1e(t) \\ &\quad -(1-h_d)e^T(t-h(t))R_1e(t-h(t))+e(t)^TR_2e(t)-(1-2h_d)e^T(t-2h(t))R_2e(t-2h(t)) \\ &\quad +g(e(t))^TR_3g(e(t))-(1-h_d)g^T(e(t-h(t)))R_3g(e(t-h(t)))+\bar{h}^2q^T(t)Q_1q(t) \\ &\quad -\left(\int_{t-h(t)}^{t} q(s)ds\right)^T Q_1\left(\int_{t-h(t)}^{t} q(s)ds\right)+\bar{h}z^T(t)Q_2z(t)-\int_{t-h(t)}^{t} z^T(s)Q_2z(s)ds. \end{aligned} \tag{14}$$

From Eq. (9), the following equation holds

$$2q^T(t)P[-q(t)-(A-K)e(t)+W_0g(e(t))+W_1g(e(t-h(t)))]=0. \tag{15}$$

Here note that Eq. (2) means that

$$g_j^2(e_j(t))-l_j^2e_j^2(t)\le 0\ \ (j=1,...,n), \tag{16}$$

and

$$g_j^2(e_j(t-h(t)))-l_j^2e_j^2(t-h(t))\le 0\ \ (j=1,...,n). \tag{17}$$

From two inequalities (16) and (17) above, for any diagonal positive matrices $T_1 = diag\{t_{11},...,t_{1n}\}$ and $T_2 = diag\{t_{21},...,t_{2n}\}$, the following inequality holds

$$\begin{aligned} 0 &\le -\sum_{j=1}^{n} t_{1j}\left[g_j^2(e_j(t))-l_j^2e_j^2(t)\right]-\sum_{j=1}^{n} t_{2j}\left[g_j^2(e_j(t-h(t)))-l_j^2e_j^2(t-h(t))\right] \\ &= e^T(t)L^TT_1Le(t)-g^T(e(t))T_1g(e(t)) \\ &\quad +e^T(t-h(t))L^TT_2Le(t-h(t))-g^T(e(t-h(t)))T_2g(e(t-h(t))). \end{aligned} \tag{18}$$

For simplicity, let us define $\zeta(t)$ as

$$\zeta^T(t) = \left[e^T(t)\ e^T(t-h(t))\ g^T(e(t))\ g^T(e(t-h(t)))\ q^T(t)\ \left(\int_{t-h(t)}^{t} q(s)ds\right)^T\ e^T(t-2h(t))\right]. \tag{19}$$

and $Y = PK$.
By utilizing the relationship (14)-(18), we have that

$$dV(e_t,t) \leq \Sigma_1 dt + 2[e(t) - Ce(t-h(t))]^T P[H_0 e(t) + H_1 e(t-h(t))]d\omega(t) \tag{20}$$

where

$$\begin{aligned}
\Sigma_1 \quad = \quad & 2[e(t) - Ce(t-h(t))]^T[-(PA-Y)e(t) + PW_0 g(e(t)) + PW_1 g(e(t-h(t)))] \\
& + [H_0 e(t) + H_1 e(t-h(t))]^T P[H_0 e(t) + H_1 e(t-h(t))] + e(t)^T R_1 e(t) \\
& - (1-h_d)e^T(t-h(t))R_1 e(t-h(t)) + e(t)^T R_2 e(t) - (1-2h_d)e^T(t-2h(t))R_2 e(t-2h(t)) \\
& + g(e(t))^T R_3 g(e(t)) - (1-h_d)g^T(e(t-h(t)))R_3 g(e(t-h(t))) + \bar{h}^2 q^T(t) Q_1 q(t) \\
& - \left(\int_{t-h(t)}^{t} q(s)ds\right)^T Q_1 \left(\int_{t-h(t)}^{t} q(s)ds\right) + \bar{h} z^T(t) Q_2 z(t) - \int_{t-h(t)}^{t} z^T(s) Q_2 z(s) ds \\
& + 2q^T(t)[-Pq(t) - (PA-Y)e(t) + PW_0 g(e(t)) + PW_1 g(e(t-h(t)))] + e^T(t) L^T T_1 L e(t) \\
& - g^T(e(t)) T_1 g(e(t)) + e^T(t-h(t)) L^T T_2 L e(t-h(t)) - g^T(e(t-h(t))) T_2 g(e(t-h(t))) \\
= \quad & \left[\zeta^T(t)\Psi\zeta(t) - \int_{t-h(t)}^{t} z^T(s) Q_2 z(s) ds\right] \leq \zeta^T(t)\Psi\zeta(t).
\end{aligned} \tag{21}$$

Note that if $\Psi < 0$, then there exists a positive scalar γ such that

$$\Psi + diag\{\gamma I, 0, 0, 0, 0, 0, 0\} < 0. \tag{22}$$

By taking the mathematical expectation on both side of (20) and considering (22), we have

$$\frac{\mathbf{E}V(e_t,t)}{dt} \leq \mathbf{E}(\zeta^T(t)\Psi\zeta(t)) \leq -\gamma \mathbf{E}|e(t)|^2, \tag{23}$$

which implies that the error dynamics (4) between master (1) and response network (3) is globally asymptotically stable in the mean square. This completes our proof. ■

Remark 1. The solutions of Theorem 1 can be obtained by solving the eigenvalue problem with respect to solution variables, which is a convex optimization problem. In this Paper, we utilize Matlab's LMI Control Toolbox which implements the interior-point algorithm. This algorithm is faster than classical convex optimization algorithms.

Example 1. In order to show the feasibility of our criterion, we consider the following stochastic neural networks with the following system matrices and parameters:

$$A = diag\{2,2\},\ W_0 = \begin{bmatrix} 1 & 0.2 \\ 0.3 & 0.8 \end{bmatrix},\ W_1 = \begin{bmatrix} 0.3 & -0.2 \\ -0.2 & 0.3 \end{bmatrix},$$
$$C = \begin{bmatrix} 0.2 & -0.1 \\ 0.1 & 0.1 \end{bmatrix},\ H_0 = \begin{bmatrix} 0.2 & 0.2 \\ 0.2 & 0.4 \end{bmatrix},\ H_1 = \begin{bmatrix} 0.1 & 0.05 \\ 0.05 & 0.1 \end{bmatrix},\ L = I,\ h_d = 0.1.$$

By applying Theorem 1 to the system above and using Matlab's LMI Toolbox, one can easily find that the LMI given in Theorem 1 is feasible for any h. For instance, when $\bar{h} = 10$, the gain matrix K for controller (5) is as follows:

$$K = diag\{0.3678, 0.3737\}.$$

Acknowledgments
This work was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD, Basic Research Promotion Fund) (KRF-2008-521-D00134).

References

[1] L.O. Chua and L. Yang, IEEE Trans. Circuit. Syst. I **35** (1988) 1257.
[2] W. Yu and J. Cao, Physica A, Vol. 373 (2007) 252.
[3] S. Haykin, *Neural Networks : A Comprehensive Foundation* (Prentice-Hall, NJ, 1998).
[4] W.H. Chen and X. Lu, Phys. Lett. A **372** (2008) 1061.
[5] X. Li, and J. Cao, Journal of The Franklin Institute, **345** (2008) 779.
[6] J.H. Park, O.M. Kwon, and S.M. Lee, Appl. Math. Comput. **196** (2008) 236.