COUPLED MODELS OF COMBINED DRY FRICTION BASED ON PADE EXPANSIONS

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Abstract. It is presented a new approach for dry friction modeling under conditions of combined kinematics. The main distinguish feature of this approach is building of friction models which are suitable for using in differential equations of motion. Coupled models of sliding and whirling friction for axially symmetric areas of contact and coupled models of sliding, whirling and rolling friction for circle areas of contact are represented in this article. Under the proposed models of friction are understudied the interrelations between friction force components, torques and velocities. The models involves the replacement of exact integral expressions for the net vector and torque of the dry friction forces, formed with the assumption that Coulomb's friction law is valid at each point of the contact area, by appropriate Pade approximations. This approach substantially simplifies the combined dry friction modeling, making the calculation of double integrals over the contact area unnecessary. Unlike available models, the model based on the Pade approximations enables one to account adequately for the relationship between force and kinematical characteristics over the entire range of angular and linear velocities. The approximate model preserves all properties of the model based on the exact integral expressions and correctly describes the behaviour of the net vector and torque of the friction forces and their first derivatives at zero and infinity. Moreover, one does not have even to calculate the integrals to determine the coefficients of the Pade approximation. The corresponded coefficients can be identified from experiments. Consequently, the models based on Pade approximations may be considered as reological models of combined dry friction. One of the main advantages of proposed models is obviating a necessity to solve the problem of the theory of elasticity and exactly define the boundaries of area of contact.

1 Introduction

There are many works in the scientific literature devoted to the dry friction, classification of that at the dependence on the aims of investigations can be found at [1]. At the most of these publications authors are using the Coulomb model of dry friction supposed that the friction force at the point of contact is direct opposite the relative velocities of sliding and it is not depend on the module of velocity. However, there are many experimental facts about the violation of this law at case when the rubbed bodies are participated simultaneously in the translational, whirling and rolling motions. Following the experimental results from the tyre manufactory in the work [2] was established the empiric dependence of the distribution of normal contact stresses at area of contact from the velocity of rolling. At the corresponded this dependence the influence of whirling is shifting of the symmetric form of contact stresses distribution in the direction of rolling. This shift is good approximated by the liner function with one coefficient depended on the direction and value of the rolling velocity. Asymmetry at distribution of the normal contact stresses at the case of circle areas of contact cause the appearance of the component of the friction force directed on normal to the trajectory of motion that leads to drift of the trajectory of the heavy boil rolling on the rubbed plane from the straight line [3].

2 Coupled models of the sliding, whirling and rolling friction for circle areas of contact.

Construction of combined model of friction of rolling and sliding is performed at he supposition the validities of the Coulomb law at the differential form for the small element of area dS inside of spot of contact, in correspondence with the differentials of the net vector $d\mathbf{F}$ and torque dM_C of the friction forces relatively the center of contact circle are defined by the formulas

$$d\mathbf{F} = -f\tilde{\sigma}\frac{\mathbf{V}}{|\mathbf{V}|}dS, \quad dM_C = -f\tilde{\sigma}\frac{|\mathbf{r}\times\mathbf{V}|}{|\mathbf{V}|}dS, \qquad (1)$$

$$\mathbf{V} = (v - \omega y, \omega x)$$

where f - coefficient of friction, $\mathbf{r} = (x, y)$ - radius vector of the elementary square inside of spot of contact (фиг.1), $\tilde{\sigma}$ - distribution of normal contact stresses, v - linear velocity of sliding and ω - angle velocity of whirling of contact spot center.

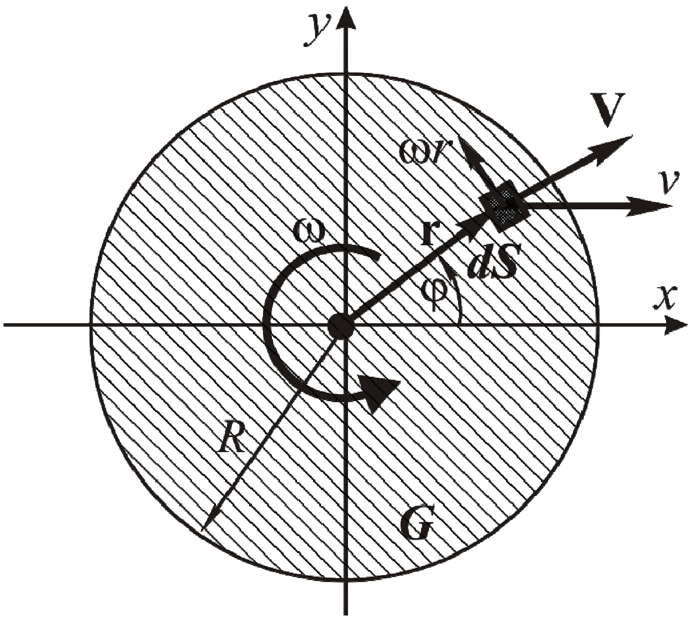


Figure 1.

Asymmetry at the symmetric distribution of normal contact stresses $\sigma(x,y)$, arisen at the non zero velocity of rolling Ω_r, in the rectangular coordinate system $\{xOy\}$, axis x of which is directed alone the velocity of sliding (fig.1) is described by the following dependence:

$$\tilde{\sigma}(x,y)=\sigma(x,y)\left(1+k_r\frac{\xi(x,y)}{R}\right),\ |k_r|\le 1,\ k_r\equiv 0 \text{ при } \omega_r=0 \tag{2}$$

where R - radius of contact circle, ξ - axis of rectangular coordinate system directed perpendicularly to the instantaneous velocity of rolling Ω_r (fig. 2), and k_r - dimensionless coefficient the sign of that is dependent on the direction of motion

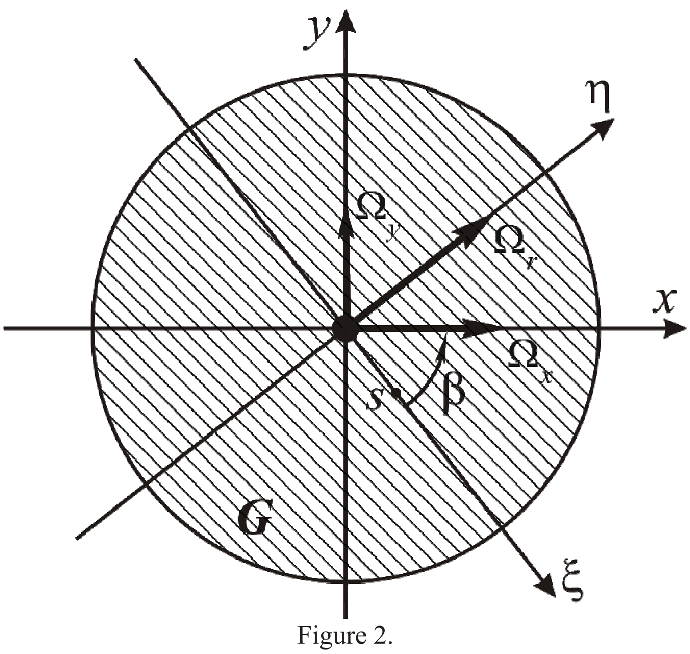


Figure 2.

Connection of the coordinate systems $\{xOy\}$ and $\{\xi O\eta\}$ are given by rotate transform on the angle $\beta\in[0,\pi/2]$ that is defined from the values of projections Ω_x,Ω_y of the instantaneous velocity of rolling Ω_r on the axis x and y (fig. 2):

$$\begin{aligned}&\xi=x\cos\beta-y\sin\beta,\eta=x\sin\beta+y\cos\beta,\\&\cos\beta=\Omega_y/\Omega_r,\ \sin\beta=\Omega_x/\Omega_r,\ \Omega_r=\sqrt{\Omega_x^2+\Omega_y^2}\end{aligned} \tag{3}$$

Substitution of expressions (3) to the formula (2) gives dependence of distribution of the normal contact stresses on the value and direction of rolling velocity:

$$\tilde{\sigma}(x,y) = \sigma(x,y)\left(1 + \frac{k_r}{R\Omega_r}(x\Omega_y - y\Omega_x)\right) \tag{4}$$

The typical behavior of function (4) for different values of the rolling coefficient k_r at the supposition that at the absence of rolling distribution of the normal contact stresses (solid line) is described by Hertz low

$$\sigma(x,y) = \frac{3N}{2\pi R^2}\sqrt{1 - \frac{x^2}{R^2} - \frac{y^2}{R^2}} \tag{5}$$

is presented at fig.3 by the dash lines.

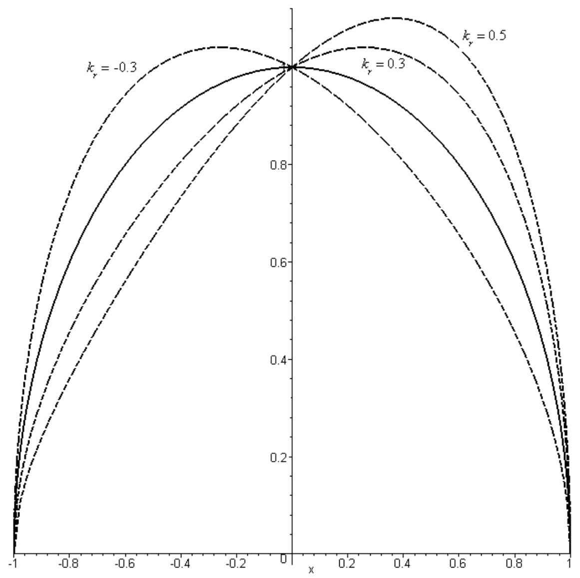


Figure 3.

Integration of the expressions (1) on the spot contact taking in account the formula (4) gives the exact integral model of combined friction of sliding and rolling, that in dimensionless variables: $x = \hat{x}R$, $y = \hat{y}R$, $\sigma(\hat{x},\hat{y}) = \hat{\sigma}(\hat{x},\hat{y})\,N/R^2$ in supposition that distribution of contact stress at the absence of rolling has central symmetry $\sigma(x,y) = \sigma(r)$, has in polar coordinate system with origin at the center of contact circle $x = r\cos\varphi,\ y = r\sin\varphi,\ r \in [0,1],\ \varphi \in [0,2\pi]$ following form

$$\begin{aligned}
F_{\parallel} &= fN\int_0^{2\pi}\int_0^1 \frac{(v - ur\sin\varphi)r\sigma(r)\left(1 - k_r\Omega_x r\sin\varphi/\Omega_r\right)}{\sqrt{u^2r^2 + v^2 - 2uvr\sin\varphi}}\,dr\,d\varphi \\
F_{\perp} &= \frac{fNk_r\Omega_y}{\Omega_r}\int_0^{2\pi}\int_0^1 \frac{ur^3\sigma(r)\cos^2\varphi}{\sqrt{u^2r^2 + v^2 - 2uvr\sin\varphi}}\,dr\,d\varphi \\
M_C &= fRN\int_0^{2\pi}\int_0^1 \frac{(ur^2 - vr\sin\varphi)r\sigma(r)\left(1 - k_r\Omega_x r\sin\varphi/\Omega_r\right)dr\,d\varphi}{\sqrt{u^2r^2 + v^2 - 2uvr\sin\varphi}}
\end{aligned} \tag{6}$$

where $F_{\parallel}$ and $F_{\perp}$ are the components of the friction force directed correspondently on the tangent and normal to the trajectory of motion, and M_C is the torque of whirling respectively the center of circle area directed perpendicularly to plane of whirling.

Transition at the model (6) from the consideration of the connection of the friction of rolling and sliding in term of projection Ω_x and Ω_y of the velocity of rolling to the its absolute value Ω_r and to the angle β between direction of rolling and sliding gives the equivalent form of this model

$$F_{\|} = fN\int_0^{2\pi}\int_0^1 \frac{(v-ur\sin\varphi)r\sigma(r)\left(1-k_r r\sin\varphi\sin\beta\right)}{\sqrt{u^2r^2+v^2-2uvr\sin\varphi}}drd\varphi$$

$$F_{\perp} = fNk_r\int_0^{2\pi}\int_0^1 \frac{ur^3\sigma(r)\cos^2\varphi\cos\beta}{\sqrt{u^2r^2+v^2-2uvr\sin\varphi}}drd\varphi \tag{7}$$

$$M_C = fRN\int_0^{2\pi}\int_0^1 \frac{(ur^2-vr\sin\varphi)r\sigma(r)\left(1-k_r r\sin\varphi\sin\beta\right)}{\sqrt{u^2r^2+v^2-2uvr\sin\varphi}}drd\varphi$$

One of the distinguish feature of model (6)-(7) is appearance of none zero component of friction force normally directed to the trajectory of motion. At the presence of combined motion of rolling and sliding the net vector of friction forces is not opposite directed to the vector of sliding velocity.

At supposition that the distribution of the contact stresses $\tilde{\sigma}(x,y)$ is play role of density the violation at its central symmetry defined by the formula (4) leads to shift of the gravity center of contact circle respectively the geometric centre in the direction of whirling (along axe ξ (fig.2)) on value s, the projections of which to axes x and y are defined by the formulas:

$$s_x = sk_r\frac{\Omega_y}{\Omega_r} \equiv sk_r\cos\beta,\ \ s_y = -sk_r\frac{\Omega_x}{\Omega_r} \equiv -sk_r\sin\beta,$$
$$s \equiv \pi R\int_0^1 \sigma(r)r^3dr \tag{8}$$

The shift of the center of gravity of contact spot, defined by formulas (8) leads to appearance of torque of rolling $\mathbf{M_r}$ parallelly directed to the plane of sliding the projections of that on the directions of the tangent $M_{\|}$ and normal $M_{\perp}$ to the trajectory of motion are defined by expression:

$$M_{\|} = -M_r\frac{\Omega_y}{\Omega_r} \equiv -M_r\cos\beta,\ \ M_{\perp} = -M_r\frac{\Omega_x}{\Omega_r} \equiv -M_r\sin\beta,$$
$$M_r \equiv sk_rN \tag{9}$$

Thus the net torque of friction forces at rectangular coordinate system one axis of that is directed on the tangent of trajectory of motion is

$$\mathbf{M} = \left(M_{\|}, M_{\perp}, -M_C\right) \tag{10}$$

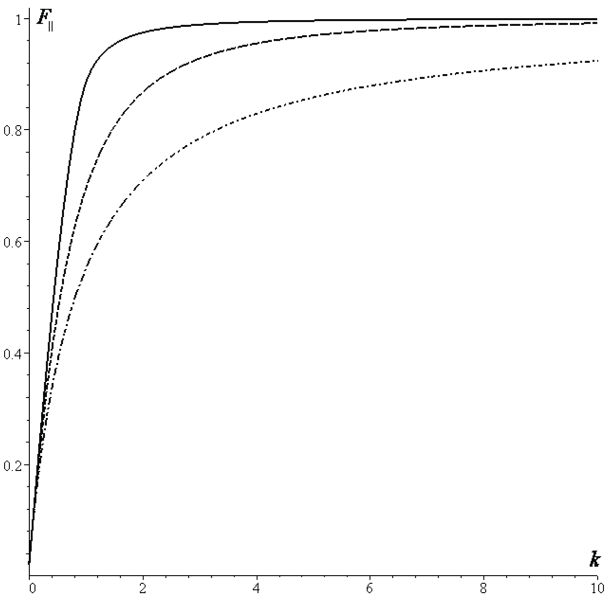
Expressions (6)-(7) for torque M_C and force components $F_{\|}, F_{\perp}$ as function of u, v have several significant properties detailed investigated in [3]. These properties allow simplifying the friction modeling with the aid of replacing of the exact integral models (6)-(7) by the approximate models based on the Pade approximations of corresponded order. This approach permits to escape the integration over the spot of contact. In corresponded with results of the work [3] the combined model friction rolling and sliding of the first order based on the partial-linear Pade approximation has form:

$$M_C = \frac{M_0u + k_rM_{ur}v\,\Omega_x/\Omega_r}{u+mv} \equiv \frac{M_0u + k_rM_{ur}v\sin\beta}{u+mv},$$
$$F_{\|} = \frac{F_0v + k_rF_ru\,\Omega_x/\Omega_r}{v+au} \equiv \frac{F_0v + k_rF_ru\sin\beta}{v+au},$$
$$F_{\perp} = k_rF_r\frac{u}{u+bu}\frac{\Omega_y}{\Omega_r} \equiv \frac{k_rF_ru\cos\beta}{u+bv}, \tag{11}$$
$$F_0 = F_{\|}\big|_{u=0},\ F_r = F_{\|}\big|_{v=0},\ M_0 = M_C\big|_{v=0},\ M_{ur} = M_C\big|_{u=0}$$
$$\frac{1}{m} = \frac{v}{M_0}\frac{\partial M_C}{\partial u}\bigg|_{u=0},\ \ \frac{1}{a} = \frac{u}{F_0}\frac{\partial F_{\|}}{\partial v}\bigg|_{v=0},\ \ \frac{1}{b} = \frac{v}{k_rF_r}\frac{\partial F_{\perp}}{\partial u}\bigg|_{u=0}$$

The model of the first order (11) is sufficient for the dynamics investigation, but for more precise qualitative analysis the model of the second order is required. This model not only good approximates the exact integral models (6)-(7) but conserves all their properties such as behavior of these functions and their first derivatives at zero and infinity.

$$
\begin{aligned}
M_C &= \frac{M_0(u^2+muv)}{v^2+muv+u^2} + \frac{k_r M_{ur} v^2}{u^2+v^2}\frac{\Omega_x}{\Omega_r} \equiv \frac{M_0(u^2+muv)}{u^2+muv+v^2} + \frac{k_r M_{ur} v^2 \sin\beta}{u^2+v^2} \\
F_{\|} &= \frac{F_0(v^2+auv)}{v^2+auv+u^2} + \frac{k_r F_r u^2}{u^2+v^2}\frac{\Omega_x}{\Omega_r} \equiv \frac{F_0(v^2+auv)}{v^2+auv+u^2} + \frac{k_r F_r u^2 \sin\beta}{u^2+v^2} \\
F_{\perp} &= \frac{k_r F_r (u^2+buv)}{v^2+buv+u^2}\frac{\Omega_y}{\Omega_r} \equiv \frac{k_r F_r (u^2+buv)\cos\beta}{v^2+buv+u^2} \\
a &= \frac{u}{F_0}\frac{\partial F_{\|}}{\partial v}\bigg|_{v=0}, \quad m = \frac{v}{M_0}\frac{\partial M_C}{\partial u}\bigg|_{u=0}, \quad b = \frac{v}{k_r F_r}\frac{\partial F_{\perp}}{\partial u}\bigg|_{u=0}
\end{aligned}
\tag{12}
$$

The comparison of the integral model (solid line) and models of the first (11) (dash-dot line) and the second (12) (dash line) for the Hertz distribution of contact stresses (5) as function of parameter $k = v/u$ is presented on the fig.4 for friction force components and on the fig.5 for torque:



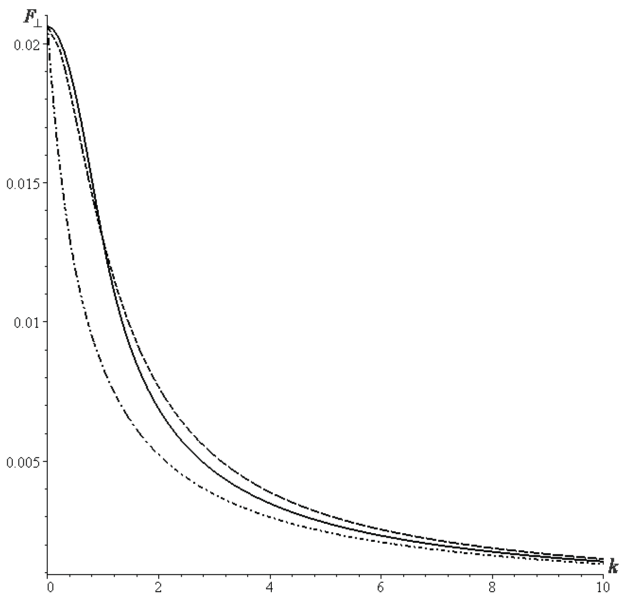


Figure 4.

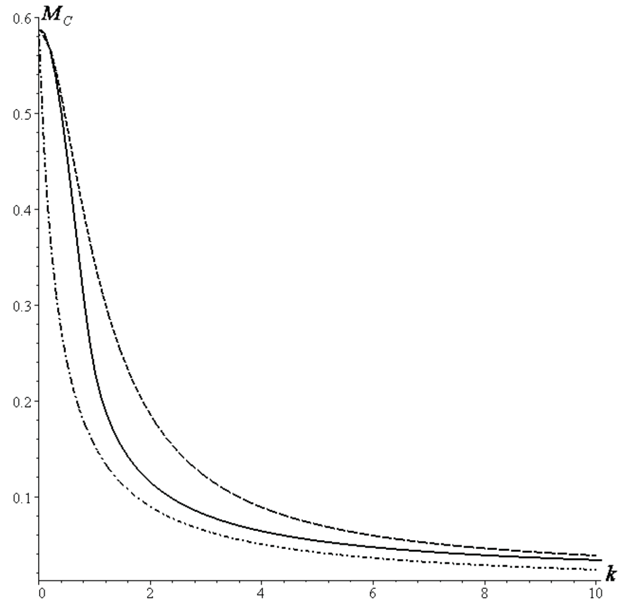


Figure 5.

Models (11)-(12) of combined friction of rolling and sliding based on the Pade approximations can be considered as reological models, because there are no required in solving of real problems to calculate the double integrals, defined the coefficients of Pade approximations. These coefficients can be defined from the experiments.

3 Dynamics of heavy ball on the rubbed plane

Using combined models of friction of rolling and sliding enable correctly describe the deviation of the trajectory of the ball mass center from the straightline. Practically in all previous publication of the different authors this effect had been described under additional suppositions on motion character hardly realized on the practice.

The equations of motion of the heavy ball of the radius R_b and mass m_b in the projections on the axes of the fixed coordinate system $\{OXYZ\}$ (fig.6) has form

$$J\dot{\boldsymbol{\omega}} = \mathbf{M},\ m_b\ddot{\mathbf{r}} = \mathbf{F},\ (J = 2/5 m_b R_b^2) \tag{13}$$

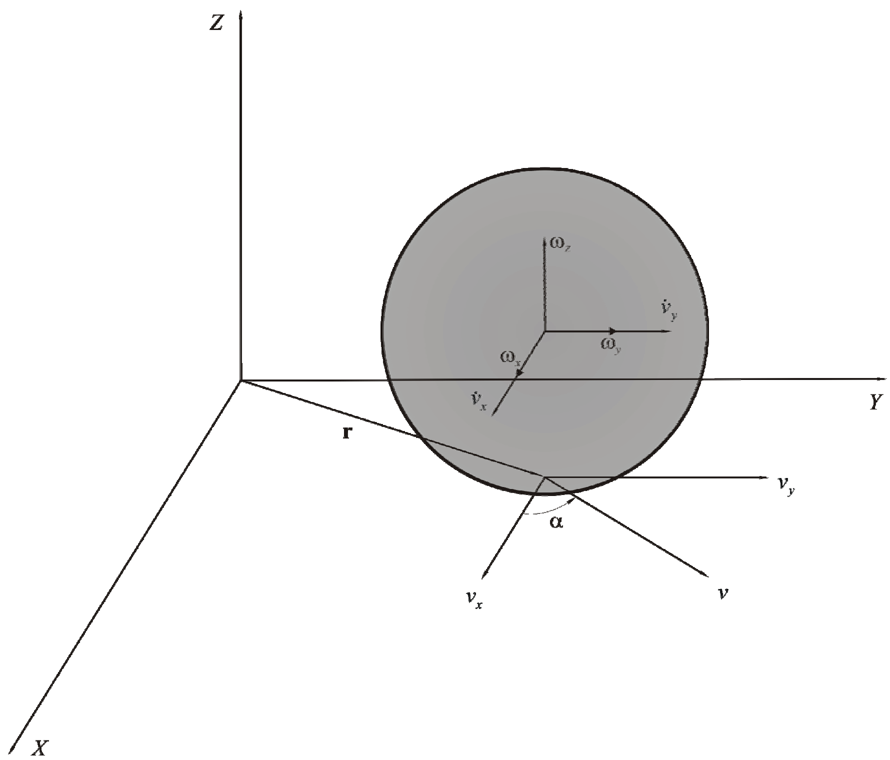


Figure 6.

Connection of components of the net vector and torque at this coordinate systems with their components defined at the section 2 are given by the formulas

$$
\begin{aligned}
F_x &= -F_{\|}\cos\alpha + F_{\perp}\sin(\alpha),\\
F_y &= -F_{\|}\sin\alpha - F_{\perp}\cos(\alpha),\\
M_z &= -M_C\\
M_x &= R_b F_y - M_{\|}\cos\alpha + M_{\perp}\sin(\alpha),\\
M_y &= -R_b F_x - M_{\|}\sin\alpha - M_{\perp}\cos(\alpha),
\end{aligned}
\tag{14}
$$

where components $F_{\|}, F_{\perp}$ of friction force and torque M_C are defined on the base of model of combined friction of rolling and sliding of the first order (11), coefficients of which at the supposition that distribution of contact stresses is describe by the Hertz law (5) are

$$
\begin{aligned}
&F_0 = fN, F_r = 3\pi fN/32, M_0 = 3\pi NfR/16, s = R/5,\\
&a = 8/3\pi, b = 15\pi/32, m = 15\pi/16, M_{ur} = fNR/5
\end{aligned}
$$

Velocity of point of ball coincident with the center of spot of contact

$$
v_x = \dot{x} - R_b\omega_y,\ v_y = \dot{y} + R_b\omega_x \tag{15}
$$

is convenient to express in polar coordinates [3]

$$
v = \sqrt{v_x^2 + v_y^2} \ge 0,\ \cos\alpha = v_x / v,\ \sin\alpha = v_y / v \tag{16}
$$

Thus, the full equations system of dynamics of the heavy ball rolling with friction on rubbed plane is

$$
J\dot{\omega}_x = M_x,\ J\dot{\omega}_y = M_y,\ J\dot{\omega}_z = M_z,\ m\ddot{x} = F_x,\ m\ddot{y} = F_y \tag{17}
$$

Transition at the equations (17) from the variables $\omega_x, \omega_y, \omega_z, \dot{x}, \dot{y}$ to $\Omega_x, \Omega_y, u, v, \alpha$ gives

$$
\begin{aligned}
&\dot{x} = (v + R_b\Omega_y)\cos\alpha + R_b\Omega_x\sin(\alpha)\\
&\dot{y} = (v - R_b\Omega_y)\sin\alpha + R_b\Omega_x\cos(\alpha)\\
&J\dot{\Omega}_x = -F_{\perp}R_b - M_{\|} + \Omega_y\dot{\alpha}\\
&J\dot{\Omega}_y = F_{\|}R_b - M_{\perp} - \Omega_x\dot{\alpha}\\
&J\dot{u} = -RM_C\\
&m_b\dot{v} = -7F_{\|}/2 + 5M_{\perp}/2R_b\\
&m_b v\dot{\alpha} = -7F_{\perp}/2 - 5M_{\|}/2R_b
\end{aligned}
\tag{18}
$$

From the last equations of this system immediately follows conclusion about the deviation of the trajectory of the ball mass center from the straightline. The numeric solutions of the initial problem

$$
v(0) = 1.0,\ u(0) = 3.0,\ x(0) = y(0) = \alpha(0) = \Omega_x(0) = \Omega_y(0) = 0
$$

for the full system got with the aid of Runge-Kutta method of the fourth order is presented on the fig.7 -8. Figure 7 demonstrates the real trajectory of the ball on the plane at coordinates $\{x, y\}$.

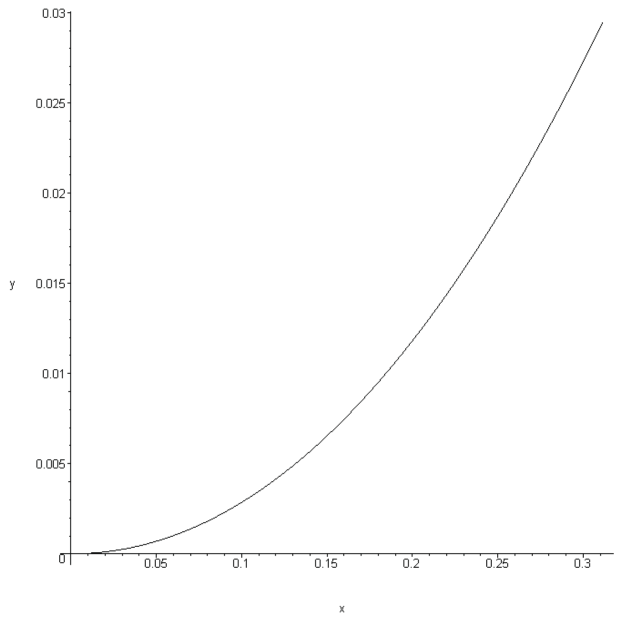


Figure 7.

Figure 8 illustrate numerically got result that the velocities of sliding v and whirling u convert to zero simultaneously.

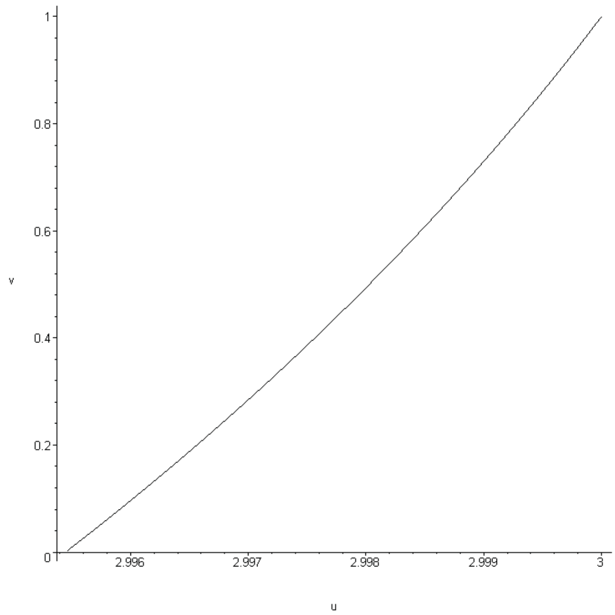


Figure 8.

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