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NOVEL DELAY-DEPENDENT STABILITY CRITERION FOR UNCERTAIN DYNAMIC SYSTEMS WITH INTERVAL TIME-VARYING DELAYS

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Abstract. In this paper, the problem of stability analysis for uncertain dynamic systems with interval time-varying delays is considered. The considered delays are assumed to vary in a given interval. Based on the Lyapunov stability theory, a new delay-dependent stability criterion for the system is established in terms of LMIs (linear matrix inequalities). Numerical example is given to show the effectiveness of proposed method.

1 Introduction

Since many industrial systems such as neural networks, chemical processes, network control systems, laser models, large-scale systems, and etc, can be modelled as time-delay systems, a great deal of effort has been done to the stability analysis or stabilization of time-delay system [1]. It is well known that the occurrence of time-delay deteriorate the system performance, cause oscillation.

Recently, delay-dependent stability analysis, which provides an maximum upper bound of delay for guaranteeing asymptotic stability, has been extensively investigated. In this field, an important index for checking the conservatism of stability criterion is the maximum allowable value of time-delay. Therefore, how to choose Lyapunov-Krasovskii functional and derive the time derivative of this with appropriate free-weighting matrices play key roles to increase the delay bounds for guaranteeing stability. To do this, various Lyapunov-Krasovskii's functionals and techniques in obtaining an upper bound of time-derivative value of Lyapunov-Krasovskii's functionals are proposed recently [2]-[8]. In this regard, Park [2] proposed a new bounding technique of cross terms and showed that the proposed stability criteria of time-delay systems have larger maximum allowable delay bound. Yue *et al.* [5] introduced neutral model transformation to get new stability criteria. Parameter neutral model transformation was used to reduce the conservatism of stability criteria [6]-[7]. Park and Ko [8] increase delay bounds by including all possible information of states when constructing some appropriate integral inequalities proposed in [2]. However, there are still room for reducing the conservatism of stability criterion yet.

In this paper, a new delay-dependent stability criterion for uncertain dynamic systems with interval time-varying delays is presented. In order to derive less conservative results, a new Lyapunov functional which divides delay interval is proposed and different free-weighting matrices in divided delay intervals are included in taking upper bounds of integral terms of time-derivative Lyapunov functionals. Then, a novel condition for delay-dependent stability criterion is established in terms of LMIs (Linear Matrix Inequalities) which can be solved efficiently by various convex optimization algorithms [9]. A numerical example will be included to show the effectiveness of the proposed method.

In the sequel, the following notation will be used. $\mathscr{R}^n$ is the n-dimensional Euclidean space. $\mathscr{R}^{m\times n}$ denotes the set of $m\times n$ real matrix. $\star$ denotes the symmetric part. $X>0$ $(X\geq 0)$ means that X is a real symmetric positive definitive matrix (positive semi-definite). I denotes the identity matrix with appropriate dimensions. $\|\cdot\|$ refers to the induced matrix 2-norm. $diag\{\cdots\}$ denotes the block diagonal matrix. $\mathscr{C}_{n,h}\equiv\mathscr{C}([-h,0],\mathscr{R}^n)$ denotes the Banach space of continuous functions mapping the interval $[-h,0]$ into $\mathscr{R}^n$, with the topology of uniform convergence.

2 Problem statements

Consider the following uncertain dynamic systems with time-varying delays:

$$\begin{aligned}\dot{x}(t) &= (A+\Delta A(t))x(t)+(A_d+\Delta A_d(t))x(t-h(t)),\\ x(s) &= \phi(s), s\in[-h_U,\ 0].\end{aligned}\tag{1}$$

Here, $x(t)\in\mathscr{R}^n$ is the state vector, $A\in\mathscr{R}^n$ and $A_d\in\mathscr{R}^n$ are known constant matrices, $\phi(s)\in\mathscr{C}_{n,h_U}$ are vector-valued initial functions, $h(t)$ means time-varying delays which satisfy $0\leq h_L\leq h(t)\leq h_U$ and $\dot{h}(t)\leq h_D$, $\Delta A(t)$, and $\Delta A_d(t)$ are the uncertainties of system matrices of the form

$$\left[\begin{array}{cc}\Delta A(t) & \Delta A_d(t)\end{array}\right]=DF(t)\left[\begin{array}{cc}E & E_d\end{array}\right],\tag{2}$$

in which the time-varying nonlinear function $F(t)$ satisfies

$$F^T(t)F(t) \leq I, \ \forall t. \tag{3}$$

System (1) can be rewritten as:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-h(t)) + Dp(t), \\ p(t) &= F(t)q(t), \\ q(t) &= Ex(t) + E_d x(t-h(t)). \end{aligned} \tag{4}$$

The purpose of this paper is to present a delay-dependent stability criterion for system (4).
Before deriving our main results, we need the following facts and lemma.
Fact 1. (Schur complement) Given constant symmetric matrices $\Sigma_1, \Sigma_2, \Sigma_3$ where $\Sigma_1 = \Sigma_1^T$ and $0 < \Sigma_2 = \Sigma_2^T$, then $\Sigma_1 + \Sigma_3^T \Sigma_2^{-1} \Sigma_3 < 0$ if and only if

$$\begin{bmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & -\Sigma_2 \end{bmatrix} < 0, \ \text{or} \ \begin{bmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^T & \Sigma_1 \end{bmatrix} < 0. \tag{5}$$

Fact 2. For any real vectors $a, \ b$ and any matrix $Q > 0$ with appropriate dimensions, it follows that:

$$\pm 2a^T b \leq a^T Q a + b^T Q^{-1} b.$$

To derive a less conservative stability criterion, we will use the following lemma which will be utilized in deriving an upper bound of integral terms.

Lemma 1. For any scalar $h(t) \geq 0$, and any constant matrix $Q \in \mathscr{R}^{n \times n}$, $Q = Q^T > 0$, the following inequality holds:

$$-\int_{t-h(t)}^{t} \dot{x}^T(s) Q \dot{x}(s) ds \leq h(t) \zeta^T(t) \mathscr{X} Q^{-1} \mathscr{X}^T \zeta(t) + 2\zeta^T(t) \left[x(t) - x(t-h(t))\right], \tag{6}$$

where

$$\zeta^T(t) = \left[x^T(t) \ x^T(t-h(t)) \ x^T(t-h_L) \ x^T\left(t - \left(\frac{h_U + h_L}{2}\right)\right) \ x^T(t-h_U) \ \dot{x}^T(t) \ p(t) \right] \tag{7}$$

and $\mathscr{X}$ is free weighting matrix with appropriate dimensions.
Proof. From Fact 2, the following inequality holds:

$$-2\int_{t-h(t)}^{t} (\mathscr{X}^T \zeta(t))^T \dot{x}(s) ds \leq \int_{t-h(t)}^{t} \left[\zeta^T(t) \mathscr{X} Q^{-1} \mathscr{X}^T \zeta(t) + \dot{x}^T(s) Q \dot{x}(s)\right] ds. \tag{8}$$

From Eq. (8), we obtain

$$\begin{aligned} -\int_{t-h(t)}^{t} \dot{x}^T(s) Q \dot{x}(s) ds &\leq \int_{t-h(t)}^{t} \zeta^T(t) \mathscr{X} Q^{-1} \mathscr{X}^T \zeta(t) ds + 2\int_{t-h(t)}^{t} (\mathscr{X}^T \zeta(t))^T \dot{x}(s) ds \\ &= h(t) \zeta^T(t) \mathscr{X} Q^{-1} \mathscr{X}^T \zeta(t) ds + 2\zeta^T(t) \mathscr{X} \left[x(t) - x(t-h(t))\right]. \end{aligned} \tag{9}$$

This completes the proof. ■

3 Main results

In this section, a new delay-dependent stability criterion for uncertain dynamic systems with interval time-varying delays (1). Before introducing the main results, the notations of several matrices are defined for simplicity.

$$\begin{aligned} \Sigma &= \Sigma_{(i,j)}, \ (i, \ j = 1, ..., 7), \\ \Sigma_{(1,1)} &= R_2 + R_3 - Q_3 + P_1 A + A^T P_1^T, \ \Sigma_{(1,2)} = P_1 A_d, \ \Sigma_{(1,3)} = Q_3, \ \Sigma_{(1,4)} = 0, \ \Sigma_{(1,5)} = 0, \\ \Sigma_{(1,6)} &= R_1 - P_1 + A^T P_2^T, \ \Sigma_{(1,7)} = P_1 D, \ \Sigma_{(2,2)} = -(1-h_D) R_2, \ \Sigma_{(2,3)} = 0, \ \Sigma_{(2,4)} = 0, \ \Sigma_{(2,5)} = 0, \\ \Sigma_{(2,6)} &= A_d^T P_2^T, \ \Sigma_{(2,7)} = 0, \ \Sigma_{(3,3)} = -R_3 + N_{11} - Q_3, \ \Sigma_{(3,4)} = N_{12}, \ \Sigma_{(3,5)} = 0, \ \Sigma_{(3,6)} = 0, \ \Sigma_{(3,7)} = 0, \\ \Sigma_{(4,4)} &= N_{22} - N_{11}, \ \Sigma_{(4,5)} = -N_{12}, \ \Sigma_{(4,6)} = 0, \ \Sigma_{(4,7)} = 0, \ \Sigma_{(5,5)} = -N_{22}, \ \Sigma_{(5,6)} = 0, \ \Sigma_{(5,7)} = 0, \\ \Sigma_{(6,6)} &= \left(\frac{h_U - h_L}{2}\right)(Q_1 + Q_2) + h_L^2 Q_3 - P_2 - P_2^T, \ \Sigma_{(6,7)} = P_2^T D, \ \Sigma_{(7,7)} = 0, \\ \mathscr{X} &= \begin{bmatrix} 0 & X_1^T & X_2^T & 0 & 0 & 0 & 0 \end{bmatrix}^T, \ \mathscr{Y} = \begin{bmatrix} 0 & Y_1^T & 0 & Y_2^T & 0 & 0 & 0 \end{bmatrix}^T, \\ \overline{\mathscr{X}} &= \begin{bmatrix} 0 & \overline{X}_1^T & 0 & \overline{X}_2^T & 0 & 0 & 0 \end{bmatrix}^T, \ \overline{\mathscr{Y}} = \begin{bmatrix} 0 & \overline{Y}_1^T & 0 & 0 & Y_2^T & 0 & 0 \end{bmatrix}^T, \end{aligned}$$

$$
\begin{aligned}
\Upsilon &= \begin{bmatrix} 0 & -\mathscr{X}+\mathscr{Y} & \mathscr{X} & -\mathscr{Y} & 0 & 0 & 0 \end{bmatrix}, \overline{\Upsilon} = \begin{bmatrix} 0 & -\overline{\mathscr{X}}+\overline{\mathscr{Y}} & 0 & \overline{\mathscr{X}} & -\overline{\mathscr{Y}} & 0 & 0 \end{bmatrix}, \\
\Pi_1 &= \begin{bmatrix} 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \end{bmatrix}, \Pi_2 = \begin{bmatrix} 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}, \\
\Xi_1 &= \Sigma + \left(\frac{h_U-h_L}{2}\right)^{-1} \Pi_1^T \begin{bmatrix} -Q_2 & Q_2 \\ \star & -Q_2 \end{bmatrix} \Pi_1,\ \Xi_2 = \Sigma + \left(\frac{h_U-h_L}{2}\right)^{-1} \Pi_2^T \begin{bmatrix} -Q_1 & Q_1 \\ \star & -Q_1 \end{bmatrix} \Pi_2, \\
\Psi &= \begin{bmatrix} E & E_d & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{10}
$$

Now, we have the following theorem.

Theorem 1. For given scalars $h_U > 0$, $h_L > 0$ and h_D, system (4) is asymptotically stable for $0 \le h_L \le h(t) \le h_U$ and $\dot{h}(t) \le h_D$ if there exist positive definite matrices $R_i(i=1,2,3) > 0$, $Q_i(i=1,2,3) > 0$, $H > 0$, $\begin{bmatrix} N_{11} & N_{12} \\ \star & N_{22} \end{bmatrix} > 0$ and any matrices P_1, P_2, X_i, Y_i, $\overline{X}_i$, $\overline{Y}_i(i=1,2)$ satisfying the following four LMIs:

$$
\begin{bmatrix} \Xi_1+\Upsilon+\Upsilon^T & \Psi^T H & \left(\frac{h_U-h_L}{2}\right)\mathscr{Y} \\ \star & -H & 0 \\ \star & \star & -\left(\frac{h_U-h_L}{2}\right)Q_1 \end{bmatrix} < 0, \quad \begin{bmatrix} \Xi_1+\Upsilon+\Upsilon^T & \Psi^T H & \left(\frac{h_U-h_L}{2}\right)\mathscr{X} \\ \star & -H & 0 \\ \star & \star & -\left(\frac{h_U-h_L}{2}\right)Q_1 \end{bmatrix} < 0, \tag{11}
$$

$$
\begin{bmatrix} \Xi_2+\overline{\Upsilon}+\overline{\Upsilon}^T & \Psi^T H & \left(\frac{h_U-h_L}{2}\right)\overline{\mathscr{Y}} \\ \star & -H & 0 \\ \star & \star & -\left(\frac{h_U-h_L}{2}\right)Q_2 \end{bmatrix} < 0, \quad \begin{bmatrix} \Xi_2+\overline{\Upsilon}+\overline{\Upsilon}^T & \Psi^T H & \left(\frac{h_U-h_L}{2}\right)\overline{\mathscr{Y}} \\ \star & -H & 0 \\ \star & \star & -\left(\frac{h_U-h_L}{2}\right)Q_2 \end{bmatrix} < 0. \tag{12}
$$

Proof. For positive definite matrices $R_i(i=1,2,3)$, $Q_i(i=1,2,3)$, $\begin{bmatrix} N_{11} & N_{12} \\ \star & N_{22} \end{bmatrix}$, let us consider the Lyapunov-krasovskii functional candidate $V = \sum_{i=1}^{3} V_i$ where

$$
\begin{aligned}
V_1(t) &= x^T(t)R_1x(t) + \int_{t-h(t)}^{t} x^T(s)R_2x(s)ds, \\
V_2(t) &= \int_{t-h_L}^{t} x^T(s)R_3x(s)ds + \int_{t-\left(\frac{h_U+h_L}{2}\right)}^{t-h_L} \begin{bmatrix} x(s) \\ x\left(s-\left(\frac{h_U-h_L}{2}\right)\right) \end{bmatrix}^T \begin{bmatrix} N_{11} & N_{12} \\ \star & N_{22} \end{bmatrix} \begin{bmatrix} x(s) \\ x\left(s-\left(\frac{h_U-h_L}{2}\right)\right) \end{bmatrix} ds, \\
V_3(t) &= \int_{t-\left(\frac{h_U+h_L}{2}\right)}^{t-h_L} \int_{s}^{t} \dot{x}^T(u)Q_1\dot{x}(u)duds + \int_{t-h_U}^{t-\left(\frac{h_U+h_L}{2}\right)} \int_{s}^{t} \dot{x}^T(u)Q_2\dot{x}(u)duds \\
&\quad + h_L \int_{t-h_L}^{t} \int_{s}^{t} \dot{x}^T(u)Q_3\dot{x}(u)duds.
\end{aligned} \tag{13}
$$

To derive less conservative results, we add the following zero equation with free variables P_1 and P_2:

$$
0 = 2\left[x^T(t)P_1 + \dot{x}^T(t)P_2\right]\left[-\dot{x}(t) + Ax(t) + A_dx(t-h(t)) + Dp(t)\right]. \tag{14}
$$

(i) When $h_L \le h(t) \le \frac{h_U+h_L}{2}$, we have from (13) - (14) and by applying Lemma 1 and S-procedure [9], the $\dot{V} = \sum_{i=1}^{3} \dot{V}_i$ has a new upper bound as

$$
\dot{V} \le \zeta^T(t)\Omega_1\zeta(t) \tag{15}
$$

where

$$
\Omega_1 = \Xi_1 + \Upsilon + \Upsilon^T + \Psi^T H\Psi + (h(t)-h_L)\mathscr{X}Q_1^{-1}\mathscr{X}^T + \left(\frac{h_U+h_L}{2} - h(t)\right)\mathscr{Y}Q_1^{-1}\mathscr{Y}^T, \tag{16}
$$

and Ξ_1, Υ, Ψ are defined in (10).

Since

$$
(h(t)-h_L)\mathscr{X}Q_1^{-1}\mathscr{X}^T + \left(\frac{h_U+h_L}{2} - h(t)\right)\mathscr{Y}Q_1^{-1}\mathscr{Y}^T \tag{17}
$$

is a convex combination of the matrices $\mathscr{X}Q_1^{-1}\mathscr{X}^T$, and $\mathscr{Y}Q_1^{-1}\mathscr{Y}^T$ on $h(t)$, $\Omega_1 < 0$ for $h_L \le h(t) \le \frac{h_U+h_L}{2}$ can be handled by two corresponding boundary LMIs:

$$
\Xi_1 + \Upsilon + \Upsilon^T + \Psi^T H\Psi + \left(\frac{h_U-h_L}{2}\right)\mathscr{Y}Q_1^{-1}\mathscr{Y}^T < 0, \tag{18}
$$

$$
\Xi_1 + \Upsilon + \Upsilon^T + \Psi^T H\Psi + \left(\frac{h_U-h_L}{2}\right)\mathscr{X}Q_1^{-1}\mathscr{X}^T < 0. \tag{19}
$$

Using Fact 1, the inequalities (18) and (19) are equivalent to the LMIs (11), respectively.
(ii) When $\frac{h_U+h_L}{2} < h(t) \leq h_U$, one can easily prove that the stability condition can be equivalent to the LMIs (12) by using similar method for the case $h_L \leq h(t) \leq \frac{h_U+h_L}{2}$. This completes our proof. ■

Remark 1. In deriving upper bounds of integral terms in $\dot{V}_3$, different free-weighting matrices($\mathscr{X}, \mathscr{Y}, \bar{\mathscr{X}}, \bar{\mathscr{Y}}$) are introduced in two intervals $h_L \leq h(t) \leq \frac{h_L+h_U}{2}$ and $\frac{h_L+h_U}{2} \leq h(t) \leq h_U$, which are not considered in other literature. This may lead to obtain an improved feasible region for delay-dependent stability criteria.

Example 1: Consider the following nominal systems with time-varying delays :

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t-h(t)) \tag{20}$$

For unknown h_D and different values of h_L, the comparison of delay bounds with the recent results in [3] and [4] are shown in Table 1. From the table, our result for this example provides larger delay bounds than the ones in [3] and [4].

Method	h_L	1	2	3	4
He et al.	h_U	1.7424	2.4328	3.2234	4.0644
Shao	h_U	1.8737	2.5049	3.2591	4.0744
Theorem 1	h_U	1.9959	2.6890	3.4250	4.1422

Table 1: Maximum delay bounds h_U for various h_L.

4 Conclusions

In this paper, a new delay-dependent stability criterion for uncertain dynamic systems with interval time-varying delays has been proposed. To reduce a conservatism of stability criterion, a new Lyapounv-Krasovskii functional is proposed and different free weighting matrices in two divided delay intervals has been introduced by utilizing a bounding technique of integral terms (Lemma 1) with the LMI framework for obtaining the stability criterion of the system. The effectiveness of the proposed stability criterion is shown by one numerical example.

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5 References

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