SENSITIVITY ANALYSIS APPLIED TO A TEST RIG MODEL

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Abstract. This paper presents methods for sensitivity analysis applied to a test rig model. On the one hand, the effect of parameter variations on system variables is analysed by means of parameter sensitivities of first and second order, while on the other hand – for the purpose of comparison – conventional Monte Carlo (MC) simulation is applied. Aspects like modelling, applicability and performance are discussed. The result evaluation leads to the conclusion that sensitivity calculation using parameter sensitivities – especially of second order and with respect to systems with a high number of parameters – is a serious alternative to MC analysis. The contents treated in this paper are based on results, which were developed in the Fraunhofer collaborative project "Computer Aided Robust Design" (CAROD).

1 Introduction

The detection of critical parameter variations on a system's characteristics is one of the major objectives in a robust design process. Tolerances of material parameters, manufacturing processes or assembly operations generally lead to scattering system properties. It must be assured that these properties are within a given range. Numerical modelling and simulation is one method to investigate the consequences of parameter variations on the interesting characteristics. In the following it is assumed that the technical system to be analysed can be modelled by a system of differential-algebraic equations. In this context, sensitivity analysis is an initial task to determine parameters which highly affect the system behaviour. Furthermore, the knowledge of the relation between parameters and system characteristics may help to determine parameter ranges. As a general approach a Taylor series approximation using an appropriate order can be applied. In this context, parameter sensitivities can be determined in an efficient way. Therefore, the determination of first-order sensitivities is widely discussed, while typically higher order sensitivities are not used. Approaches to determine second-order sensitivities will be discussed in the following.

The methods are applied to a model of a multi-axial elastomer test rig. Finally, based on the analysis results, the applicability and performance of the treated methods for sensitivity evaluation are discussed.

2 Mathematical Model

The equations of motion of the considered system are given by a differential-algebraic system of equations (DAEs)

$$F(x, \dot{x}, p, t) = 0 \,. \tag{1}$$

$x(t) \in \mathbb{R}^n$ are implicitly defined waveform values (state variables) and $p \in \mathbb{R}^m$ summarises the real-valued parameters p_i. Applying implicit function rules, the system to determine first-order parameter sensitivities $\frac{\partial x}{\partial p_i}(t)$ for all p_i with $i=1,\dots,m$ can be established [1, 2] as

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial p_i} + \frac{\partial F}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial p_i} = -\frac{\partial F}{\partial p_i} \,. \tag{2}$$

In the same way, the second-order parameter sensitivities $\frac{\partial^2 x}{\partial {p_i}^2}(t)$ can be determined as

$$\begin{aligned} &\frac{\partial^2 F}{\partial x^2}\left(\frac{\partial x}{\partial p_i}\right)^2 + \frac{\partial^2 F}{\partial \dot{x}\partial x}\frac{\partial \dot{x}}{\partial p_i}\frac{\partial x}{\partial p_i} + \frac{\partial^2 F}{\partial p_i \partial x}\frac{\partial x}{\partial p_i} + \frac{\partial^2 F}{\partial x \partial \dot{x}}\frac{\partial x}{\partial p_i}\frac{\partial \dot{x}}{\partial p_i} + \frac{\partial^2 F}{\partial \dot{x}^2}\left(\frac{\partial \dot{x}}{\partial p_i}\right)^2 + \frac{\partial^2 F}{\partial p_i \partial \dot{x}}\frac{\partial \dot{x}}{\partial p_i} + \\ &\frac{\partial F}{\partial x}\frac{\partial^2 x}{\partial {p_i}^2} + \frac{\partial^2 F}{\partial x \partial p_i}\frac{\partial x}{\partial p_i} + \frac{\partial F}{\partial \dot{x}}\frac{\partial^2 \dot{x}}{\partial {p_i}^2} + \frac{\partial^2 F}{\partial \dot{x}\partial p_i}\frac{\partial \dot{x}}{\partial p_i} = -\frac{\partial^2 F}{\partial {p_i}^2} \,, \end{aligned} \tag{3.1}$$

which is equivalent to

$$\frac{\partial F}{\partial x}\frac{\partial^2 x}{\partial {p_i}^2}+\frac{\partial F}{\partial \dot{x}}\frac{\partial^2 \dot{x}}{\partial {p_i}^2}=-\frac{\partial^2 F}{\partial {p_i}^2}-\dots \ . \qquad (3.2)$$

A similar system can be established for cross sensitivities $\frac{\partial^2 x}{\partial p_i \partial p_j}$. It is obvious that higher order sensitivity equations depend on the state variables of the original system (1) and the vectors of lower order sensitivities. Therefore, the adjoint sensitivity approach cannot be applied directly to higher order sensitivity analysis. Thus, we apply algorithms to determine first-order sensitivities on a combination of (1) and (2) using the DAE solver DASPK [3] and the simulator Dymola. An alternative method for a special implementation to determine higher order sensitivities is a staggered solution of (1) and the sensitivity systems. Linear systems of equations with the same coefficient matrix have to be solved in this case.

Dymola and DASPK were used to calculate both the solution and sensitivities. The DAE-system (1) together with the first-order sensitivity system (2) can concurrently be solved by Dymola, if (2) is added explicitly or using the difference quotient method of first order. Without use of first-order difference quotient an explicit determination of system of first-order parameter sensitivities (2) by differentiation of DAE system (1) is necessary. After this, a manual extension of the model has to be determined. The code DASPK can directly solve (1) and (2), where the first-order sensitivity system (2) is generated within the code automatically. Furthermore, parameter sensitivities of second order (3) were computed with Dymola and DASPK. Calculations of second-order sensitivities were carried out using explicit differentiation of (1) regarding specified parameters and manually extension of system (1). Contrary to DASPK, Dymola yields only an approximation of the second-order sensitivities if difference quotients of first respectively second order are used, which nevertheless represents a good approximation yet.

3 Technical Example: Elastomer Test Rig

The mathematical investigations are demonstrated by example of a multi-axial test rig model (figure 1). The physical test rig is mainly used for sign-off tests of automotive elastomer bushings considering service loads. Based on a transmission design with cardan joints, the load directions "axial ($\tilde{x}$)", "lateral (y)" and "torsion (α)" can be realised isolated or in combination respectively [4].

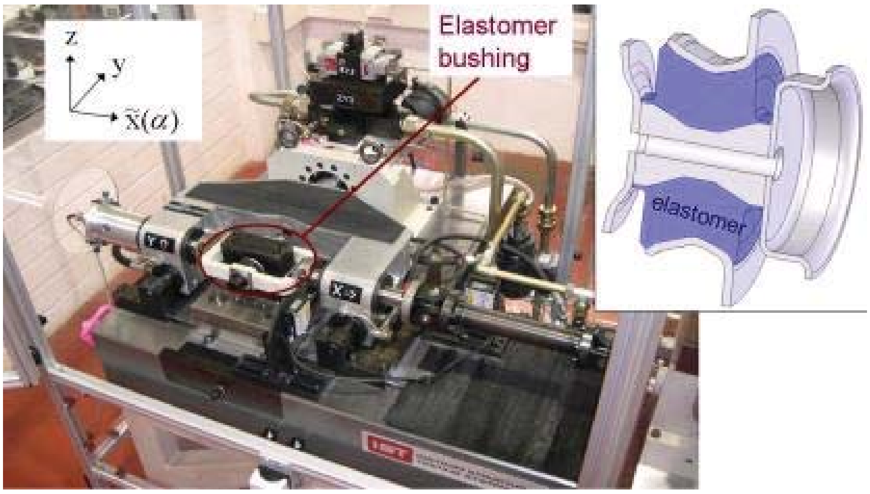

Figure 1. Multi-axial test rig for characterisation and testing of elastomer bushings and sketch of a typical elastomer bushing for automotive application

The investigations target a virtual scenario, which describes the production of a small series of (only) theoretically identical test rigs. Due to manufacturing tolerances the test rig components will differ more or less, leading to scattering of the test rigs behaviour. In the context of this presentation, tolerances of masses, inertias and damping coefficients are considered. The idea is to pre-evaluate the sensitivity of test rig performance due to variations of single parameters. Beside information concerning the performance scatter to be expected, promising "adjusting screws" for system optimization can be derived.

To perform sensitivity calculation, the analytical equations of motion have been set up explicitly using the Lagrange approach. Thus, the system equations are available in symbolic form. The DAE system of the elastomer test rig is described by 3 equations of motion with 3 state variables $\tilde{x}(t)$, $y(t)$ and $\alpha(t)$ as well as 23 system parameters. The analyses cover system excitations by sinusoidal forces and moments (test case 1), noise signals (test case 2) as well as signals of the type of ramp functions (test case 3). In test case 1 and 2 the excitations are realised simultaneously, while test case 3 includes isolated uni-axial loads. The performance and dynamics of the test rig are evaluated by analysing the resulting displacements $\tilde{x}$ and y as well as the torsion angle α at the modelled elastomer bushing.

An interesting aspect presents the sensitivity of scalar evaluation quantities. Sensitivities in general are functions of time t. These functions can be used as a basis for the derivation of scalar evaluation parameters, which are used to describe specific technical attributes, e.g. performance, accuracy or stability of a system. In this case, the temporal mean S_1 and temporal quadratic mean S_2 were used as scalar evaluation parameters. These are defined by

$$S_1 = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} E(t)dt \qquad \text{and} \qquad S_2 = \frac{1}{t_3 - t_1}\int_{t_1}^{t_3=1s} \left(E(t) - \overline{E}\right)^2 dt\,, \tag{4}$$

whereas E is an arbitrary time-dependent function and $\overline{E} = \frac{1}{t_3 - t_1}\int_{t_1}^{t_3} E(t)dt$.

The design objective is to minimise the scalars temporal mean S_1 and temporal quadratic mean S_2.

4 Interpretation of Sensitivity Results

Sensitivity calculation examines the effects of minor parameter deviations from their nominal values on the behaviour of the dynamic systems. The sensitivities were computed to detect critical parameters. Parameters with a significant influence on the interesting variable $\tilde{x}$ regarding test case 1 are single component masses of the test rig model. In terms of test case 3, concerning scalar evaluation quantity S_2, resulting displacement $\tilde{x}$, related to E in (4), the same component masses are dominant. Concerning sensitivities of y regarding test case 1 also component masses have an influence while parameters with a significant influence to α are single component inertias.

The same critical parameters were identified by MC simulation [5]. However, the computational effort is much higher if the MC simulation runs are carried out using the original “full” test rig model.

The Taylor series approach [2] using second-order sensitivities and regarding test case 3 and $\tilde{x}$

$$\tilde{x}\left(t, p_{i,\mathrm{var}}\right) \approx \tilde{x}\left(t, p_i\right) + \frac{\partial \tilde{x}}{\partial \mathrm{p_i}}(t)\left(p_{i,\mathrm{var}} - p_i\right) + \frac{1}{2!}\frac{\partial^2 \tilde{x}}{\partial p_i^2}(t)\left(p_{i,\mathrm{var}} - p_i\right)^2\,, \tag{5}$$

whereas $\partial \tilde{x} / \partial p_i$ is the first-order and $\partial^2 \tilde{x} / \partial p_i^2$ the second-order parameter sensitivity of $\tilde{x}$ concerning parameter p_i, was compared to the result when solving the system’s equation (figure 2) directly. Figure 3 illustrates the comparison of the parameter varied (+10%) nominal solution (blue line) of the special displacement $\tilde{x}$ regarding specified parameter of mass mK and the solution using first-order Taylor approach (linear part of formula (5), red line) as well as second-order Taylor approach ((5), green line). Figure 3 shows obviously, that the Taylor approach of second order is a good approximation for the solution of the DAE system of the elastomer test rig and that the second-order Taylor approach represents a better approximation as the first-order Taylor approach.

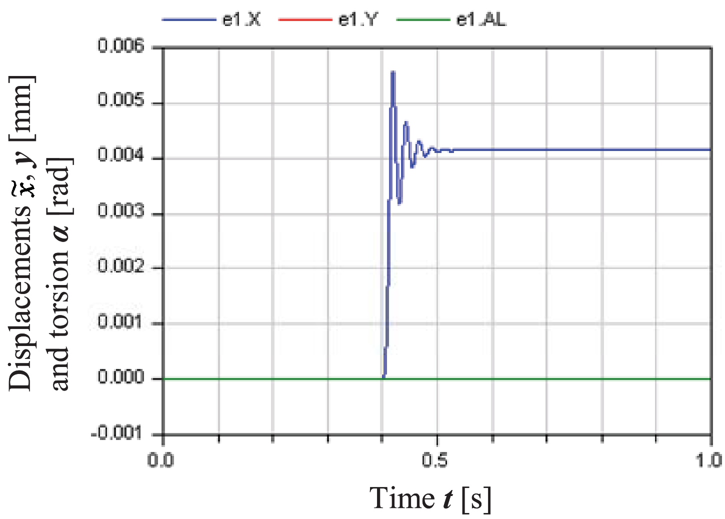

Figure 2. Nominal solution; e1.AL hides e1.Y (Dymola)

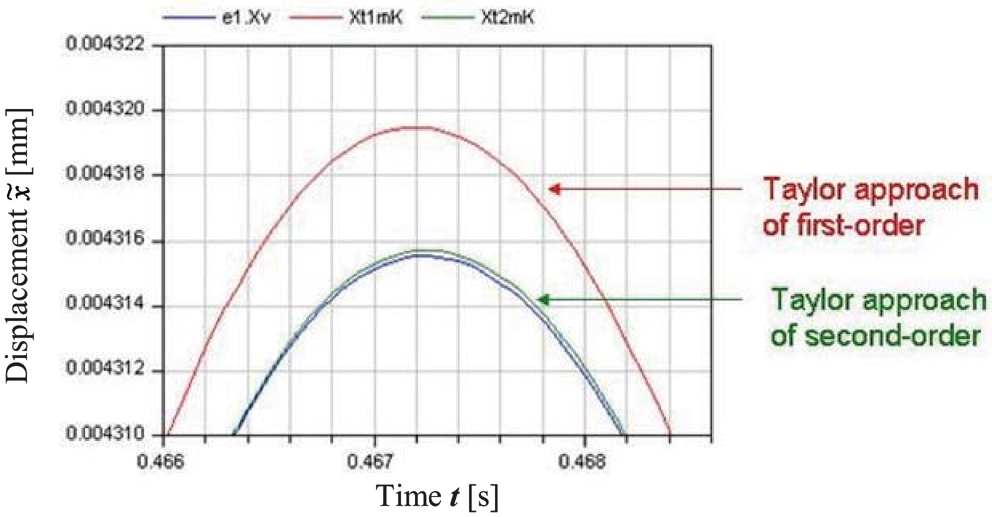

Figure 3. Comparison of first- and second-order Taylor approaches and parameter varied (+10%) nominal solution; detail (Dymola)

The second-order Taylor series based on sensitivity analysis can also be used to construct response surface models for MC simulation.

5 Conclusion

Sensitivities are well qualified to detect critical parameters of a system. In order to describe a nonlinear relation between parameters and system's characteristics at least second-order sensitivities should be considered. These sensitivities can be determined in parallel to the solution of the original system to reduce the computational effort. Sensitivities of second order allow a good approximation of the solution and yield a reduction of computing time compared to MC methods. The potential of this approach was investigated using the model of an existing test rig for automotive applications.

At the moment missing implementations to determine second-order sensitivities can be overcome by applying methods for first-order sensitivities on a combination of the original system and the first-order sensitivity equations.

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