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FRACTIONAL ORDER MODEL OF WIEN BRIDGE OSCILLATORS CONTAINING CPES

Z. Trzaska
Warsaw University of Technology, Electrical Engineering Faculty, Plac Politechniki 1, 00-661 Warsaw, Poland; zdzislaw.trzaska@netlandia.pl

Abstract. Nano- and micro-technologies change the nature of the majority of the human made structures, devices, and system. The real objects are generally fractional although, for many of them, the fractionality is very low. Main emphasis of this contribution is put on the mathematical description and analysis of fractional-order oscillators build up with constant phase elements and classical resistors and Op-Amp. Fractional Wien bridge oscillators are described in the time-domain by differential equation of fractional order 2α with $1<2\alpha \le 2$. Limits for α leading to no attenuate oscillations are established. Results of computer simulations are presented.

1 Introduction

Recent achievements in the area of nanotechnologies have involved many changes in the nature of the majority of the man made structures, devices, and systems. The diversity of materials produced in nano-engineering has led to spectacular progress in functionality of such electronic devices as oscillators, filters, antennas, etc.[1]. Among various possible approaches by means of which the structures and devices produced in cutting-edge technologies can be understand, described, analyzed and designed, a fractional modelling is much more appropriate rather than that based on using traditional tools of classic mathematics [4].Fractional calculus is concerned with the development and application of differential equations of non-integral order α with m-1 < α < m, where m denotes a nearest integer number. The attraction in using a constitutive description based on fractional calculus for modelling micro-structure devices is their potentially superior accuracy, and the possibility of correlating the hierarchical structure of physical systems to the fractional order α [2],[3]. Fractional calculus formulations have recently been applied to a number of practical structures and processes.

A number of applications where fractional models of nano-structured elements and devices has been used rapidly grow. They have found numerous applications in pattern formation in biology, chemistry, and physics. Fractional calculus formulations have recently been applied to a number of biological tissues and processes. Among the processes that have been probed are the mechanisms of many electrochemical processes and reactions, including the hydrogen electrode reaction, the oxygen electrode reaction, metal dissolution, and passivity, as well as various coupled processes, such as corrosion and electrocrystallization.

This contribution is a continuation of the author previous work [3] and deals with the mathematical description and analysis of fractional-order oscillation systems in the time domain. We present results of simulations of fractional-order Wien bridge oscillators with the aim to investigate the stability of such system.

2 Particular Models of Nano-structured Elements

To be able well interpreting most physical advantages of practically useful nanoelectronic devices and systems all essential properties of the nonlocal processes must be taken into account. They are related in different ways to the finite spatial extension of any real system. From now on we focus the attention on mathematical models of nano-structured elements and devices being in use recently.

- **Warburg diffusion element**

Diffusion of electric charges can create impedance known as the Warburg impedance. This impedance depends on the frequency of the potential perturbation. The fractional-order differential equation for the "infinite" Warburg diffusion element is

$$i(t) = \sigma \frac{d^{1/2}u(t)}{dt^{1/2}} \tag{1}$$

where $i(t)$ and $u(t)$ denote the output current and the input voltage, respectively. The coefficient σ is the Warburg capacitance defined as

$$\sigma = \frac{RT}{n^2 F^2 A\sqrt{2}} \left(\frac{1}{C_0^* \sqrt{D_0}} + \frac{1}{C_R^* \sqrt{D_R}}\right) \tag{2}$$

in which D_O = diffusion coefficient of the oxidant, D_R = diffusion coefficient of the reductant, A = surface area of the electrode, n = number of electrons transferred, C*= bulk concentration of the diffusing species (moles/cm^3).

On a Nyquist plot the infinite Warburg impedance appears as a diagonal line with a slope of 0.5. On a Bode plot, the Warburg impedance exhibits a phase shift of 45°.

The above form of the Warburg impedance is only valid if the diffusion layer has an infinite thickness. Quite often this is not the case. If the diffusion layer is bounded, the operator admittance at lower frequencies no longer obeys the equation above. Instead, we get the form:

$$Y_0 = \sigma \frac{s^{1/2}}{\tanh(\delta(\frac{s}{D})^{1/2})} \tag{3}$$

with δ = Nernst diffusion layer thickness, D = an average value of the diffusion coefficients of the diffusing species. This more general expression is called the "finite" Warburg admittance. For high frequencies where $\omega \to \infty$, or for an infinite thickness of the diffusion layer where $d \to \infty$, expression (3) simplifies to the "infinite" Warburg admittance.

- **Constant Phase Element**

Supercapacitor manufacturers use porous electrodes to circumvent limitations of the current output due to the fact that microscopic electrochemical process at the electrode-electrolyte interface has 'finite rate'. Practically, such type of disadvantages is overcome by way of increasing the electrode surface area of the super-capacitor. It has been experimentally established that metal-electrolyte surface interface does not exhibit pure capacitance behaviour, instead it acts like a constant phase element (CPE) governed as fractional order derivative equation, namely

$$i(t) = C_s \frac{d^{\alpha} u(t)}{dt^{\alpha}} \tag{4}$$

where $i(t)$ and $u(t)$ denote the output current and the input voltage. The exponent α for a CPE is less than one and fulfils the relation: $0 < \alpha < 1$. This order term approaches unity (CPE tending to ideal capacitor) as the smoothness of the interface is increased to infinity. The constant C_s in expression (4) depends on the relaxation time τ as follows

$$C_s = \tau_0^{\alpha-1}(\varepsilon_s - \varepsilon_\infty) \tag{5}$$

where ε_s and ε_∞ denote static and high-frequency limiting dielectric constant, respectively, and τ_0 is the mean relaxation time. The CPE name is given by this fact that the phase angle of the CPE impedance in the complex domain is independent of the frequency. The "double layer capacitor" on real metal-electrolyte interface often behaves like a CPE instead of like a classic capacitor. Several theories have been proposed to account for the non-ideal behavior of the double layer but none has been universally accepted. In most cases, we can safely treat α as an empirical constant and not worry about its physical basis. The symbol of a constant phase element is usually presented in the form shown in Fig.1b.

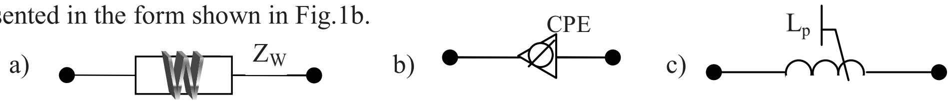


Fig.1. Symbols of particular elements: a) Warburg impedance, b) constant phase element, c) pseudo-inductance

The behavior of a CPE depends directly on an exponential distribution of activation energies and an exponential form for τ. The CPE exhibits no transition from intensive to extensive behavior as the frequency decreases. In fact, the CPE cannot be applied for all frequencies and becomes physically unrealizable for sufficiently low or high frequencies.

- **Pseudo-inductance**

When dealing with surface evolving interfaces with concomitant adsorbate relaxation, as in dissolution and some nucleation and growth processes an inductive behaviour must hence be formally related to the relaxation of the adsorbed species. Indeed, changes in the local or overall surface topology are effectively potential sources of inductive behaviour. This should be taken into account by an element L_p called pseudo-inductance. It is related to charge-transfer resistance R_{ct} by a time constant τ_c caused by certain sub-processes within the overall electrode-position mechanism. This time constant is a measure of how rapidly the parameter of the faradaic current component, being itself dependent on potential and time, relaxes to a new value after the potential is changed. Taking the above into account we get

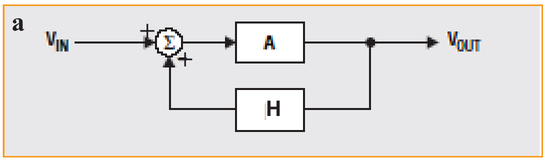
$$i(t) = L_p \frac{d^{\alpha} u(t)}{dt^{\alpha}} \tag{6}$$

where $L_p = \tau_c R_{ct}$ is a pseudo-inductance. It is worth noticing that pseudo-inductances should not be straightforwardly ascribed to surface relaxation only whenever the surface is expected to evolve in electrodeposition processes. In those cases, the possibility of inductive parameters being intrinsically electrochemical in nature should not be neglected. The symbol of a pseudo-inductance is usually presented in the form shown in Fig.1c.

3 Wien bridge oscillator with CPEs

3.1 Backgrounds

The most common signals used in electronic circuits are sine, square and triangle waves. Some of the applications of these signals are radio, TV, satellite, communication, AC-circuit response testing, switching circuits, digital clock pulse and raster displays, respectively. For oscillation to occur in an RC-(Op-Amp) circuit we need a positive feedback system (Fig.2a). If the loop gain satisfies the criterion $AH = 1$ at some frequency f_0 the circuit will oscillate sinusoidally at this frequency. In the Wien bridge oscillator the feedback voltage is derived from the output voltage through the voltage divider formed by Z_1 and Z_2 (Fig.2b). Because the amplifier does not invert, and because V_{FB} has the same polarity as V_{OUT}, this is a positive feedback circuit. Voltage feedback Op-Amps are limited to a few hundred *kHz* because they accumulate too much phase shift.



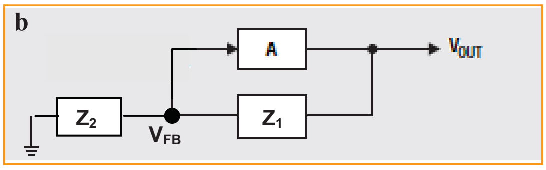
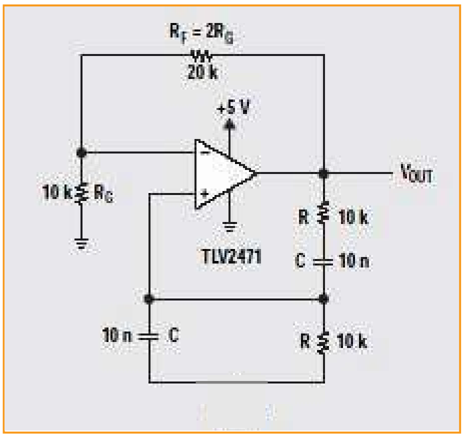


Fig.2. Positive feedback systems: a) feedback loop, b) feedback through voltage divider

Using two RC networks connected to the positive terminal of the Op-Amp we can form a frequency selective feedback network which causes oscillations to occur. One of possible realizations is shown in Fig.3a. In such a case we have H=1/3 and A=3. Generated oscillations for different A are presented in Fig.3.b. Observe that when A=3 oscillations occur at f =1/2πRC =1.59kHz but when A<3 oscillations attenuate and for A >3 oscillations amplify and the oscillator is unstable.



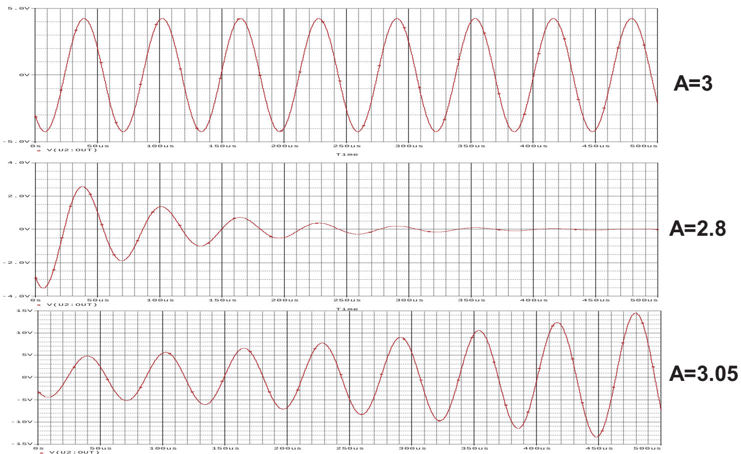


Fig.3. Basic Wien bridge: a) configuration, b) oscillations

By changing the resistor and capacitor values in the positive feedback network, the output frequency can be changed. With proper maintenance of the bridge configuration its oscillations can go on indefinitely.

3.2 Fractional Wien bridge oscillator

A fractional Wien bridge oscillator we can design and built on the principle clearly visible on Fig.2b. The basic element of this circuit is the appropriate impedances Z_1 and Z_2 – so called fractances in the feedback, which define the fractional-order (FO) of the differential operator describing the configuration. The function of the amplifier stage is to determine the required gain of the whole FO operator or to compensate the gain of the feedback stage (Fig.4a).Applying the Kirchhoff current and voltage laws and assuming identical CPEs represented by relation (4) we get the following mathematical model of the factorial Wien bridge oscillator as follows

$$\begin{aligned} C_s \frac{d^\alpha u_1}{dt^\alpha} + \frac{u_1}{R} + C_s \frac{d^\alpha u_2}{dt^\alpha} = 0, \\ RC_s \frac{d^\alpha u_2}{dt^\alpha} + u_2 = (1 - A) u_1 \end{aligned} \tag{7}$$

where *A* is a unknown gain of the amplifier stage which is to be determined.

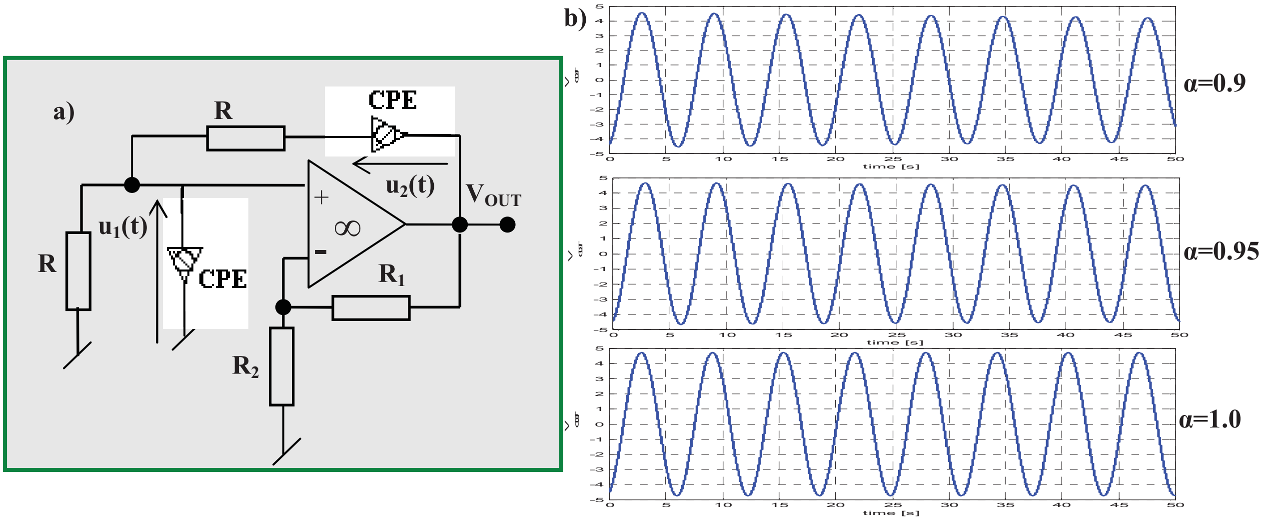


Fig.4. Fractional Wien bridge oscillator: a) configuration, b) oscillations

Performing appropriate manipulations on equations (7) and then rearranging terms yields

$$\frac{d^{2\alpha}u_2}{dt^{2\alpha}}+\frac{3-A}{RC_s}\cdot\frac{d^{\alpha}u_2}{dt^{\alpha}}+\frac{1}{R^2C_s^2}u_2=0 \tag{8}$$

It follows easily that taking $A = 3$ in equation (8) we get

$$\frac{d^{2\alpha}u_2}{dt^{2\alpha}}+\frac{1}{R^2C_s^2}u_2=0 \tag{9}$$

This equation demonstrates the fractional property of the Wien bridge oscillator build up with CPEs instead classic capacitors. We conclude from the above considerations that to determine the dynamics of the fractional Wien bridge oscillator we must consider the nature of solutions of fractional order differential equation (9). In order to do so, we notice that for practical reasons we study (9) with $1<2\alpha \le 2$. Next, for the sake of notation simplicity without lost of any generality we take into considerations the case in which $RC_s = 1$s. According to the outcome of the realm of the fractional calculus it is easy to show that the solution of the governing equation (9) takes the form

$$\begin{aligned} u_2(t) &= \frac{1}{\alpha}e^{t\cos(\pi/2\alpha)}\{u_2(0)\cos[t\sin(\pi/2\alpha)]+u_1(0)\cos[t\sin(\pi/2\alpha)-\pi/2\alpha]\} \\ &+u_2(0)\int_0^{\infty}e^{-rt}K_{2\alpha,0}(r)dr+u_1(0)\int_0^{\infty}e^{-rt}K_{2\alpha,1}(r)dr \end{aligned} \tag{10}$$

where $K_{2\alpha,0}(r)$ and $K_{2\alpha,1}(r)$ denote spectral functions of the respective solution. Substituting (10) into (7) yields the voltage V_{OUT} at the output of the fractional oscillator. Results of performed simulations are shown in Fig.4b for $\alpha \in \{0.9; 0.95; 1.0\}$. It is worth noting that for $\alpha \le 0.95$ the oscillations attenuate and for $\alpha \ge 0.95$ oscillations remain practically sinusoidal.

4 Concluding remarks

The analysis carried out in this short paper has attested possibilities of using supercapacitors as suitable CPEs for realization of fractional Wien bridge oscillators. The fractional oscillation equation was derived from the classical equations for linear dynamical circuits by introducing a fractional derivative of order 2α with $1<2\alpha<2$. Results of computer simulations are presented.

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