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On certain continual and discrete convolution operators

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Abstract. One considers discrete Calderon-Zygmnd operators constructed on given continual kernel for uniform lattice points in m-dimensional space. It's shown the spectrums of continual and discrete Calderon-Zygmund operators are the same in square integrable function spaces and don't depend on lattice parameter.

1 Calderon-Zygmund operators

Let $K(x)$ be singular Calderon-Zygmund kernel, i.e. the function defined on $\mathbf{R}^m\setminus\{0\}$ and homogeneous of order $-m$ and infinitely differentiable with vanishing mean value on unit sphere S^{m-1}

$$\int\limits_{S^{m-1}} K(\omega)d\omega = 0.$$

Given kernel $K(x)$ construct continual (classical) Calderon-Zygmund operator

$$(Ku)(x) = \text{v.p.}\int\limits_{\mathbf{R}^m} K(x-y)u(y)dy =$$

$$= \lim_{\substack{\varepsilon\to 0\\ N\to+\infty}} \int\limits_{\varepsilon<|x-y|<N} K(x-y)dy, \quad x\in\mathbf{R}^m. \tag{1}$$

For such operators the boundedness theorem in $L_p(\mathbf{R}^m)$, $1<p<+\infty$, were obtained by Calderon-Zygmund [1].

For the operator K one defines its symbol $\sigma(\xi)$, $\xi\in S^{m-1}$, as Fourier transform in principal value sense of kernel $K(x)$:

$$\sigma(\xi) = \lim_{\substack{\varepsilon\to 0\\ N\to+\infty}} \int\limits_{\varepsilon<|x|<N} K(x)e^{ix\cdot\xi}\,dx \tag{2}$$

$x\cdot\xi = x_1\xi_1+\ldots+x_m\xi_m$ denotes inner product.

Its is well-known spectra of operator K coincides with image of its symbol $\sigma(\xi)$.

Indeed, if we apply the Fourier transform (2) to singular convolution operator (1) in $L_2(\mathbf{R}^m)$- space, then we obtain multiplication operator instead of convolution ones (multiplication by its symbol), and the spectra for such operator is easily shown. Obviously, the analogue situation we must obtain in discrete case.

2 Discrete variant

Let $\mathbf{Z}_h^m$ – lattice in $\mathbf{R}^m$ with uniform step h for each variable. We'll define "lattice kernel" $u_h(\tilde{x})$, $\tilde{x}\in\mathbf{Z}_h^m$, and given kernel $K(x)$ we"ll construct "lattice kernel" $K_h(\tilde{x})$ restricting $K(x)$ on $\mathbf{Z}_h^m\setminus\{0\}$, and define discrete Calderon-Zygmund operator

$$(K_hu_h)(\tilde{x}) = \lim_{N\to+\infty}\sum_{\tilde{y}\in Q_h^N\setminus\{0\}} K_h(\tilde{x}-\tilde{y})u_h(\tilde{y})h^m \tag{3}$$

where Q_h^N is a "discrete cube" of lattice $\mathbf{Z}_h^m$ with side Nh.

The symbol $\sigma_h(\xi)$ of such operator can be treated as multivariable discrete Fourier transform [2] of its kernel $K_h(\tilde{x})$ in principal value sense

$$\sigma_h(\xi) = \lim_{N\to+\infty}\sum_{\tilde{y}\in Q_h^N\setminus\{0\}} e^{i\tilde{x}\cdot\xi}K_h(\tilde{x})h^m. \tag{4}$$

Evidently, the symbol $\sigma_h(\xi)$ is defined and continuous on $\left[\frac{-\pi}{h},\frac{\pi}{h}\right]^m\setminus\{0\}$ and periodical function on $\mathbf{R}^m$. Let's denote by $L_2(\mathbf{Z}_h^m)$ the discrete analogue of $L_2(\mathbf{R}^m)$. Applying the Fourier transform (4) to operator (3) leads to multiplication operator

$$\tilde{u}_h(\xi)\longmapsto\sigma_h(\xi)\tilde{u}_h(\xi), \tag{5}$$

in Fourier images, and the spectra of operator (3) acting $L_2(\mathbf{Z}_h^m) \to L_2(\mathbf{R}^m)$ (or operator (5) acting $L_2(Q_h) \to L_2(Q_h)$, where $Q_h = \left[\frac{-\pi}{h}, \frac{\pi}{h}\right]^m$) will coincide with image $\sigma_h(\xi)$.

What we can say if h tends to 0 ? It's fully evidently that $Q_h \setminus \{0\}$ will transform to $\mathbf{R}^m \setminus \{0\}$, but what about $\sigma_h(\xi)$? No less evidently that

$$\sigma_h(\xi) \to \sigma(\xi) \tag{6}$$

in pointwise convergence sense.

Some simple consideration can help us explain (6). Take $u \in S(\mathbf{R}^m)$ (Schwartz class of infinitely differentiable rapidly decreasing at infinity functions) and write (1) in Fourier images

$$\sigma(\xi)\tilde{u}(\xi).$$

"Discrete variant" of such expression to "lattice function" $u_h(\tilde{x})$ constructing on $u(x)$ leads to

$$\sigma_h(\xi)\tilde{u}_h(\xi).$$

So, as (1) and (3) under consideration will be bounded, then we have $\exists c > 0$,

$$|\sigma_h(\xi)| \leq c, \quad |\tilde{(u)}(\xi)| \leq c.$$

If so, we consider the difference

$$(\sigma(\xi) - \sigma_h(\xi))\tilde{u}(\xi) =$$
$$= (\sigma(\xi)\tilde{u}(\xi) - \sigma_h(\xi)\tilde{u}(\xi)) - \sigma_h(\xi)(\tilde{u}(\xi) - \tilde{u}_h(\xi)),$$

which have tend to 0 as $h \to 0$, because $\sigma_h u_h$ and $\tilde{u}_h$ are "cubature formulas" for corresponding integrals (if they exist).

So, (6) holds exactly.

Let $\mathbf{Z}^m$ be "integer number" lattice in $\mathbf{R}^m$, $Q = [-\pi, \pi]$ be corresponding "unit" cube in dual space in Fourier sense. One of easily seen properties of symbol of operator (3) is the following.

Proposition 1. $\sigma_1(h\xi) = \sigma_h(\xi)$.

Let's note that σ_1 is defined on $Q \setminus \{0\}$ (period 2π,) and σ_h is defined on $Q_h \setminus \{0\}$ (period $2\pi h^{=1}$).

Further, because σ_h isn't defined at 0, but under $\xi \neq 0$ the limit $\sigma_h(\xi)$ exists as $h \to 0$, then, it follows the limit depends on direction of vector $\xi \neq 0$ in which the limit exists. Thus, this direction can be determined by point of unit sphere S^{m-1} in which this radius-vector ξ pierses the S^{m-1}. If one fixes point ξ, then has

$$\sigma(\xi) = \lim_{h \to 0} \sigma_h(\xi) = \lim_{h \to 0} \sigma_1(h\xi),$$

and it follows

Proposition 2. *For $\forall \xi \neq 0$ the $\sigma_h(\xi)$ doesn't depend on h.*

The last may be useful for justifying some approximate methods for solution of multidimensional singular integral equations with Calderon-Zygmund operators. Bur we have take into account the hard works related to change of "infinite discrete" object by "finite discrete" ones; here are different possibilities, and the author thinks to conside their separately.

3 Conclusion

At large author interested the relations between four (possible philosophical) things, but he is a mathematician in education, he has tried to express some (very simple things) in mathematical language. These philosophical points are the following:

a) continuous + finite (integral);

b) continuous + infinite (improper integral);

c) discrete + finite (sum);

d) discrete + infinite (series).

So, particularly, the classical quadrature (cubature) formulas describe the limit from c) to a) with corresponding error estimate.

Unfortunately, it is very little, as all point of views lead to the same result, but here I would like to extract "the mathematical situation", in which b) and d) will give the same.

4 References

[1] Mikhlin S.G., Prössdorf S.: *Singular integral operators,* Akademie – Verlag, Berlin, 1986.
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