

ON EXACT/APPROXIMATE REDUCTION OF DYNAMICAL SYSTEMS LIVING ON PIECEWISE LINEAR PARTITION

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Abstract. Order reduction problem for dynamical systems living on piecewise linear partitions is addressed in this paper. This problem is motivated by analysis and control of hybrid systems. The technique presented is based on the transformation of affine dynamical systems inside the cells into a new structure and it can be applied for both exact reduction and also approximate model reduction. In this method both controllability and observability of the affine system inside the polytopes are considered for the reduction purpose. The framework is illustrated with a numerical example.

1 Introduction

Over the past two decades model reduction has become an ubiquitous tool in a variety of application areas and, accordingly, a research focus for many mathematicians and engineers [1]. Most of the methods that are proposed so far for control and analysis of hybrid and switched systems are suffering from high computational burden when dealing with large-scale dynamical systems. This fact has motivated the researchers in hybrid systems to study model reduction. Because of the weakness of nonlinear model reduction techniques and also pronounced needs for efficient analysis and control of large-scale dynamical hybrid and switched systems; it is essential to study model reduction of hybrid and switched systems in particular. One of the most important classes of hybrid systems which has been studied extensively in the literature is a class of piecewise affine systems. This class is equivalent to many other hybrid system classes such as mixed logical dynamical systems, linear complementary systems, and maxmin-plus-scaling systems and thus form a very general class of linear hybrid systems. To our knowledge the only available study in the context of reduction of affine systems in the literature is the work done by Habets and Schuppen [2] which has considered the problem of the exact reduction due to non-observability. Model reduction problem for dynamical systems which are defined on piecewise linear partitioning is addressed in this paper. Our presented work is generalization and modification of the method in [2]. It is easy to show that in our method if we restrict our attention just to reduction due to non-observability the method also provides the same results as [2]. The technique presented is based on the transformation of affine dynamical systems inside the cells to a new structure and it can be applied to both exact reduction and also approximate model reduction. In this framework both controllability and observability of the affine system inside the polytopes are considered for reduction purpose. The paper is organized as follows: In the next section we review some definitions and notions which clarify our problem formulation. Section 3 presents the main contribution of this paper. In this section we show the technique to transform affine dynamical systems inside the cells to a new structure in which switching information and input/output relation information are embedded. This section ends up with some remarks on reduction which is the step after transformation. Section 4 presents our numerical results followed by a brief discussion. Section 5 concludes the paper.

2 Linear Partitions, Affine Systems and Reduction

Let J be a finite index set and cardinality of J is $\#J$. A polyhedral set P in \mathbb{R}^n is the intersection of a family of closed half spaces $H_j^- = \{x \in \mathbb{R}^n \mid \langle x, N_j \rangle \leq a_j\}$ for $N_j \in \mathbb{R}^n$ and $a_j \in \mathbb{R}$, where $j \in J$ and $\langle \cdot, \cdot \rangle$ is scalar product in \mathbb{R}^n , i.e. $P := \bigcap_{j \in J} H_j^-$. The polyhedral set P can be expressed by the inequality (1) to be understood components wise:

$$P = \{x \in \mathbb{R}^n \mid Nx \leq a\} \tag{1}$$

where $N = [N_1^T \dots N_{\#J}^T]^T$, $a = [a_1 \dots a_{\#J}]^T$.

Let $K = \{P_j \mid j \in I\}$ be a polyhedral Complex with the index set I .

$$|K| := \bigcup_{j \in I} P_j \subseteq \mathbb{R}^n$$

Let E be any polyhedral set (\mathbb{R}^n inclusively). A piecewise linear partition of E is a polyhedral complex K such that $E = |K|$. The elements of K will be called cells.

We define $K_i := \{P \in K \mid \dim(P) = i\}$. The class of dynamical systems that we deal with in this paper is the class of affine dynamical systems living on full dimensional cells K_n of linear partition associated to a quadruple (E, K, U, S) , where E is a polyhedral set (a polytope) in \mathbb{R}^n , K is a piecewise affine partition of E , U is a polyhedral set (of admissible inputs) in \mathbb{R}^m , and $S = \{s_p : P \in K_n\}$ is a family of piecewise affine systems:

$$s_p : \begin{cases} \dot{x} = A_p x + B_p u + a_p \\ y = C_p x + D_p u \end{cases} \quad (2)$$

The problem that we address is the reduction of this class of dynamical systems. In model reduction the goal is to reduce the order of dynamical systems, input/output behaviour must be preserved when the reduction is in the exact sense. Approximate reduction keeps the input/output behaviour close to the original system while we reduce the order of dynamical system.

3 Reduction Framework for Affine Systems on Linear Partitions

Our framework has two main steps. First, the system should be transformed to a new structure which contains the switching information and is suitable for reduction. Second main step is the reduction part. In this step we can check if the system is reducible in exact sense and if it is we can reduce it. We can also apply linear model reduction techniques at this point for approximate reduction. The system can be retransformed to the structure (2) at the end.

3.1 Transformation

In the following we first transform s_p into a new structure. In this structure input/output information and also switching information is embedded. We can apply linear reduction methods easily to the new structure and it can be retransformed to the original structure after reduction. If we introduce the new input vector:

$$W_p := \begin{bmatrix} u \\ a_p \end{bmatrix} \quad (3)$$

the transformed system will be:

$$s_p : \begin{cases} \dot{x} = A_p x + \bar{B}_p W_p \\ y = C_p x + \bar{D}_p W_p \end{cases} \quad (4)$$

where :

$$\begin{aligned} \bar{B}_p &= [B_p \quad I] \\ \bar{D}_p &= [D_p \quad O] \end{aligned} \quad (5)$$

Transformation to this structure makes sense because the reduction procedure has nothing to do with the vector of inputs and it is obvious that based on the dimension of a_p we can recover the new constant vector in the reduced system.

The next step is to find a way to embed the switching information to the structure; in other words information of the cell in which our affine system is defined (1), this will help us to pay attention to the importance of the states which are probably not important from local input/output maps but they are actively involved the switching conditions. The idea is to define new output and using the advantage of exact/approximate preservation of input-output behaviour in model reduction.

In other words, for the structure (4) we define a new output vector:

$$Y_{new} := \begin{bmatrix} y \\ Nx \end{bmatrix} \quad (6)$$

Hence we have:

$$Y_{new} = \begin{bmatrix} C_p \\ N \end{bmatrix} x + \begin{bmatrix} \bar{D}_p \\ O \end{bmatrix} W_p \quad (7)$$

which gives us new C, D matrices. This new structure can be retransformed to the original structure after reduction by partitioning based on the length of the vector Nx and the original output.

Transformed LTI systems contain state contribution in local input/output behaviour and their contribution to the switching actions.

3.2 Order Reduction

At this point, we are in position to use several results from linear system theory regarding conditions for exact reduction and also methods to find appropriate projection for exact/ approximate reduction. In the case of exact reduction, applying ordinary controllability/observability tests for LTI systems on the aforementioned transformed system provides us with conditions for exact reducibility. In these propositions for exact reduction we have conditions on the rank of controllability/observability matrices of the transformed system and consequently conditions on affine system matrices and N . One can also approach the problem using Grammians which leads the same results. It is also straightforward to find appropriate projection to remove the states due to non-observability or non-cotrollability [3]. In the case of approximate reduction after transformation of the affine system to the aforementioned structure one can use different reduction techniques such as balanced reduction techniques and then it is possible to recover the original structure of the system by partitioning the system based on original output and input. Although this method provides satisfactory approximate results but in approximate reduction a lot of other issues arise which needs more investigations and further research in this context.

4 Illustrative Example

In this section we illustrate the proposed framework with a numerical example. Consider a randomly generated dynamical system:

$$\begin{cases} \dot{x} = \begin{bmatrix} -1.119 & 0.2557 & -0.01542 \\ 0.2557 & -1.892 & 0.1438 \\ -0.01542 & 0.1438 & -1.889 \end{bmatrix} x + \begin{bmatrix} 1.444 \\ 0 \\ 0.6232 \end{bmatrix} u \\ y = [0.799 \quad 0.9409 \quad -0.9921]x \end{cases}$$

which is defined on the cell:
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

This dynamical system is linear i. e. $a_p = 0$ therefore defining new input (3) is not needed in this case and we can skip this step. The transformed system will be:

$$\dot{x} = \begin{bmatrix} -1.119 & 0.2557 & -0.01542 \\ 0.2557 & -1.892 & 0.1438 \\ -0.01542 & 0.1438 & -1.889 \end{bmatrix} x + \begin{bmatrix} 1.444 \\ 0 \\ 0.6232 \end{bmatrix} u, \quad y_{new} = \begin{bmatrix} 0.799 & 0.9409 & -0.9921 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u \quad (8)$$

If we calculate the observability matrix, we can see that it is a full column rank matrix therefore the system is not reducible in the exact sense due to non-observability. The rank of controllability matrix is 3 which shows, the associated system is not reducible in the exact sense due to non-controllability. In order to apply balanced truncation for approximate reduction we first should transform (8) to the balanced realization.

The associated singular values are: $\{1.2594, 0.0920, 0.0014\}$. If we reduce the system to the second order system and retransform the original structure we have:

$$\begin{cases} \dot{x} = \begin{bmatrix} -1.034 & 0.01587 \\ -0.7875 & -2.046 \end{bmatrix} x + \begin{bmatrix} 1.614 \\ 0.6135 \end{bmatrix} u \\ y = [0.5064 \quad -0.46]x \end{cases} \quad (9)$$

with the switching inequality:

$$\begin{bmatrix} 0.8679 & 0.0655 \\ 0.0892 & -0.2289 \\ 0.2731 & 0.2993 \\ -1.2302 & -0.1359 \end{bmatrix} x \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad (10)$$

In Fig.1 the states for both reduced and original systems are shown in terms of time. The solid line shows the switching signal which is 1 when the dynamical system hits the facet and switching occurs. As we can, both systems hit the facet and switch almost simultaneously at 0.0495. This Figure confirms that approximation provides us with accurate results regarding the switching time. Fig.2 shows that approximate reduction also preserve input/output behaviour quite well. In general for exact reduction this framework works very well but in the case of approximate reduction some other issues should be taken into account such as stability preservation. It might happen that the framework can not keep the stability of original hybrid system. Although the accuracy of the method inside the cell is quite well depending on the dynamics outside of the cell it might also happen that the approximation in the neighbourhood and outside of the cell is not satisfactory. These problems need further investigation and research to be done.

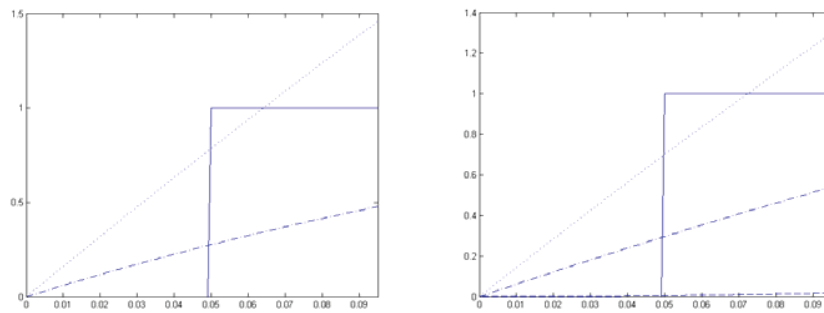


Figure 1. Left: reduced system(x_1 :dotted, x_2 :dash dotted, switch: solid) Right: original system(x_1 :dotted, x_2 :dash dotted, x_3 :dashed ,switch: solid)

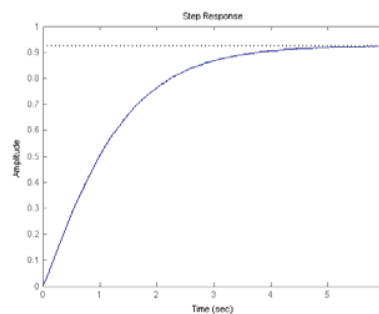


Figure 2. Step response of reduced system(dotted) and original system(solid)

5 Conclusion

Model reduction problem for dynamical systems which are defined on piecewise linear partitioning was addressed in this paper. The method compromises generalization and modification of [2]. The technique presented is based on the transformation of affine dynamical systems inside the cells to a new structure and it can be applied for both exact reduction and also approximate model reduction. In the case of exact reduction the method works very well. Although in the approximate reduction numerical results are satisfactory but still several issues like stability preservation and approximation error in the neighbourhood of the cell will arise that needs more research to be carried out in this context.

6 References

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