I. Troch, F. Breitenecker, eds.   ISBN 978-3-901608-35-3

THE SIMULATION METHOD USED TO SOLVE ENGINEERING INVERSE PROBLEMS

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Abstract. In this paper a new numerical method used to solve engineering inverse problems is presented. This method named the Simulation Method because computer simulations are applied to aid the process of construction computational algorithms and to estimate coefficients in regularization procedure. Under consideration is taken a system appearing frequently in practice which is described by partial differential equation called the Poisson equation. To solve this equation the 2-D discrete models are used for the use of special computational tools constructed on bases of objects from modern combinatorics and principles standing in physics. From various inverse problems the identification of field sources was chosen to detail studies. It was shown that the simulation method may be applied to solve inverse problems for reduced measurement data, i.e. using only measurements results made on a bounder of investigated domain.

1 Introduction

From the engineering point of view inverse problems involve identification, system synthesis and controlling. To identification problems belong all problems aiming at the reconstruction of information on a mathematical model of the system from data come from measurements. Within various identification problems the identification of field sources was chosen to study because its solution is of importance in engineering and technology. It should be pointed that there is no general method which can solve the great number of parameter and source identification problems for 2-D systems. The choice of adequate method may depend on the kind of real system, its mathematical formulation, the number of data available for measurements, nature of the parameters (constant or space-varying) or nature of the measurements (noisy of noiseless).

2 System description

In the considerations is taken the system appears frequently in practice described by the following partial differential equation called the Poisson equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = f(x,y), \qquad u|_\Gamma = 0, \tag{1}$$

where: $x \in (0, \mathrm{r})$, $y \in (0, \mathrm{s})$, $u=u(x, y) \in R^2$ - potential function, $f= f(x, y) \in R^2$ - field sources' function, Γ - boundry of the system.

The zero boundary conditions is assumed to simplify notation without loss of generality it is assumed. In the aim to use numerical method a discretization of (1) was done by using a five point differences scheme and the method of separation of discrete variables $x = mh$, $m = 0, 1, 2, \dots , M$, $y = nh$, $n = 0, 1, 2, \dots , N$, $M = r/h$, $N = s/h$

$$\sum_{p\in P_{mn}} a_{mp}\, \mathrm{u}_{m,n} = \mathrm{f}_{m,n}, \qquad a_{mp} = \begin{cases} h^{-2} & if \quad n \neq m, \\ -4h^{-2} & if \quad n = m \end{cases}, \tag{2}$$

where: P_{mn} - five point differences operator.

3 Idea of the Simulation Method

The source function is approximated by discrete 2-D Fourier series

$$f_{m,n} = \sqrt{2}\sum_{k=1}^{N-1} F_m(k)\sin\frac{k\pi n}{N}, \quad f_{m,0} = f_{m,N} = 0, \quad m = 0, 1, 2, \dots , M, \tag{3}$$

where n is taken as a parameter, $F(k)$ mean amplitudes for $k = 1, 2, \dots N$-1.

Similarly the potential function is developed into discrete 2-D Fouries series

$$u_{m,n} = \sqrt{2}\sum_{k=1}^{N-1} U_m(k)\sin\frac{k\pi n}{N}, \quad u_{0,n} = u_{M,n} = 0, \quad n = 0, 1, \dots , N. \tag{4}$$

Equation (1) for the Fourier amplitudes has a following form

$$\frac{1}{h^2}\left[(U_{m+1}(k)–2U_m(k)+U_{m-1}(k))–(4\sin^2\frac{k\pi}{2N})U_m(k)\right]=F_m(k)\ ,\ m = 1, 2, \dots , M\text{-}1, \tag{5}$$

with the boundary condition $U_0(k) = 0$ and values $U_M(k)$ defined by $u_{M,n} = 0$, $n = 1, \dots , N$.

For the parameter $q_k = 4\cdot\sin^2\frac{k\pi}{2N}$ the special computational objects from modern combinatorial analysis may be applied. These objects are the non-zero monic power polynomials of the second kind $P(q)$ which coefficients generated by special modified numerical triangle. The $P(q)$ polynomials are defined by the recurrence

$$P_{n+2}(q) = (2+q)P_{n+1}(q) - P_n(q),\quad n = 0, 1, 2, \dots \qquad P_0(q) = 0,\ P_1(q) = 1, \tag{6}$$

therefore $P_n(x)=\sum_{r=0}^{n} b_{n,r}\,x^r$, n=0,1,2, ... ,$0 \le r \le n$, $b_{n,r}=2b_{n-1,r}+b_{n-1,r-1}-b_{n-2,r}$, $b_{0,0}=0$, $b_{1,0}=1$.

Using the polynomial approach the solution of direct problem (5) takes a form

$$U_m(k) = P_m(q_k)U_1(k)+\sum_{l=1}^{m-1}P_{m-l}(q_k)h^2F_l(k)\ ,\qquad m = 2, 3, \dots , M-1. \tag{7}$$

The inverse problem solution can be express by the formula

$$F_l(k) = \frac{U_{l+1}(k)–P_{l+1}(q_k)U_1(k)–\sum_{i=1}^{l-1}P_{l+1-i}(q_k)h^2F_i(k)}{P_l(q_k)h^2}\ . \tag{8}$$

Inverse problems are the ill problems according to Hadamard's definition of correctly posed problems [4]. Therefore, the special numerical procedures must be employed to stabilize the results of calculations. The correct regularization results were obtained for the approximation procedure elaborated on the basis of an inverse distance method named also the Shepard method [1]. In the inverse distance method in order to interpolate arbitrarily spaced data in the bivariate case the following interpolating function formula is defined

$$f(x,y) = \begin{cases} \left(\sum_{i=1}^{N}\frac{F_i}{r_i^{\mu}}\right)\Big/\left(\sum_{i=1}^{N}\frac{1}{r_i^{\mu}}\right) & \textit{when } r_i \ne 0 \textit{ for all } i \\ F_i & \textit{when } r_i = 0 \end{cases}, \tag{9}$$

where: $r_i = [(x - x_i)^2 + (y - y_i)^2]^{1/2}$ - distance in Euclidean metric, F_i – values at (x_i, y_i), $i = 1, 2, \dots , N$, and $0<\mu<\infty$.

To get smooth surfaces without cusps, the condition $1<\mu$ will be better. On the basis of experiments it was indicated that a choice of $\mu = 2$ is perhaps a good tradeoff [1].

The described above mathematical procedures were used to construct special numerical method for field sources' identification named Simulation Method. This name pointed the computer simulations play a main role within constructing the computational procedures. The computer simulations were applied in the preliminary part of construction of the method to test direct and inverse procedures with a use of special benchmark functions and during experimental verification. Also computer simulations were used to estimate the regularization parameters. This method was used to identity of sources in the heat path on a rail generating by rolling contact of wheel and rail as well as to study tension in a rail due its torsion [3, 4].

4 The reduced measurement data case

The problem of source identification with reduced measurement data consists in the use of only the data from measurements made on bounder of investigation domain. The proposed method applies the combinatorial approach in the matrix form. The numerical calculations are done for the discretization of arrays as follows

$$x= ih,\ i=0, 1, \dots, M+1,\ y = jh,\ j = 0,1, \quad N, \tag{10}$$

where h means a step of disretization, M, N defines the limit of the investigation domain.

The Dirichlet boundary conditions for the investigated domain are in the form

$$u(0, j) = U_{\mathbf{0}}(j), \qquad u(M+1, j) = U_{M+1}(j) \qquad j = 0, 1, \dots, N \tag{11}$$

$$u(i, 0) = U_{\mathbf{0}}(i), \quad u(i, N) = U_N(i) \qquad i = 0, 1, \dots, M, M+1.$$

After a discretization with the use of finite difference method and an introduction the quantity $\boldsymbol{U}(i)$ which means a vector of potentials in the i-th column of the net for j=1, 2, …, N-1, the equation (1) with the boundary conditions(11) can be written in the matrix form

$$\boldsymbol{U}(i+1)-(2\mathbf{E}+\mathbf{A})\boldsymbol{U}(i)+\boldsymbol{U}(i-1)=\boldsymbol{G}(i)-\boldsymbol{V}_0(i)-\boldsymbol{V}_N(i)\,, \qquad i=1,2,\ldots,M, \tag{12}$$

where: $\boldsymbol{G}(i)$ – the vector of values of sources' function
$\boldsymbol{V}_0(i)$ – the vector of potential values for j=0 in the i-th column of the net
$\boldsymbol{V}_N(i)$ – the Victor of potential values for j=N in the i-th column of the net
E - the unit matrix
A – the matrix of approximation with the finite difference method for array j.

Introducing the monic polynomials (6) for q = **A** to the equation (12) and doing the rejection from unknown values of potentials for i = 1, 2, …, M -1, the following equation can be written

$$P_{M-1}(\mathbf{A})\boldsymbol{U}_{M+1}-P_M(\mathbf{A})\boldsymbol{U}_M+\boldsymbol{U}_0+\boldsymbol{H}_{0\mathrm{N}}=\sum_{i=1}^{M}P_{i-1}(\mathbf{A})\boldsymbol{G}(i), \tag{13}$$

where: $\boldsymbol{U}_{M+1}$=$\boldsymbol{U}(M+1)$-a vector of known potentials on the border for i=M+1
$\boldsymbol{U}_M=\boldsymbol{U}(M)$ - a vector of potentials for i=M
$\boldsymbol{U}_0$ - a vector of known potentials for i=0
$\boldsymbol{H}_{0\mathrm{N}}=\boldsymbol{H}_0+\boldsymbol{H}_\mathrm{N}$, $\boldsymbol{H}_0$ – a vector of values of known potential for j=0, $\boldsymbol{H}_\mathrm{N}$ is a vector of values of known potential for j=N.

Values collected in vector $\boldsymbol{U}_M$ correspond with potentials in the nodes of the column neighbouring to the border column can be calculated by the calibration of boundary column $\boldsymbol{U}_M=\alpha\boldsymbol{U}_{M+1}$. Equation (13) can be solved in an another way by extending the domain by adding one more column of nodes and treating connected with them values of potentials as zero boundary conditions, i.e. $\boldsymbol{U}_{M+2}=\mathbf{0}$. Then using these values in the equation (13) and putting i=M+1 the values of potentials in a column of nodes neighbouring to the border column can be calculated

$$\boldsymbol{U}(M)=(2\mathbf{E}+\mathbf{A})\boldsymbol{U}(M+1)-\boldsymbol{V}_0(M+1). \tag{14}$$

Putting $\boldsymbol{U}_M=\boldsymbol{U}(M)$ to the equation (13), the solution of the identification problem can be established as follows

$$P_{M-1}(\mathbf{A})\boldsymbol{U}_{M+1}-P_M(\mathbf{A})\boldsymbol{U}_M[\,(2\mathbf{E}+\mathbf{A})\boldsymbol{U}_{M+1}-\boldsymbol{V}_0(M+1)]+\boldsymbol{U}_0+\boldsymbol{H}_{0\mathrm{N}}=\sum_{i=1}^{M}P_{i-1}(\mathbf{A})\boldsymbol{G}(i). \tag{15}$$

This solution uses only the data from measurements made on the bounder.

Example 1. The use of elaborated method in practice can be demonstrated if M = 3 and N = 3

$$\begin{aligned}
&\boldsymbol{U}(2)-(2\mathbf{E}+\mathbf{A})\boldsymbol{U}(1)+\boldsymbol{U}(0)=\boldsymbol{G}(1)-\boldsymbol{V}_0^{3}(1)\\
&\boldsymbol{U}(3)-(2\mathbf{E}+\mathbf{A})\boldsymbol{U}(2)+\boldsymbol{U}(1)=\boldsymbol{G}(2)-\boldsymbol{V}_0^{3}(2)\\
&\boldsymbol{U}(4)-(2\mathbf{E}+\mathbf{A})\boldsymbol{U}(3)+\boldsymbol{U}(2)=\boldsymbol{G}(3)-\boldsymbol{V}_0^{3}(3)
\end{aligned}$$

where

$$\boldsymbol{U}(0)=\begin{bmatrix}u(0,1)\\u(0,2)\end{bmatrix},\quad \boldsymbol{U}(1)=\begin{bmatrix}u(1,1)\\u(1,2)\end{bmatrix},\quad \boldsymbol{U}(2)=\begin{bmatrix}u(2,1)\\u(2,2)\end{bmatrix},\quad \boldsymbol{U}(3)=\begin{bmatrix}u(3,1)\\u(3,2)\end{bmatrix},\quad \boldsymbol{U}(4)=\begin{bmatrix}u(4,1)\\u(4,2)\end{bmatrix},$$

$$\boldsymbol{V}_0^{3}(1)=\boldsymbol{V}_0(1)-\boldsymbol{V}_3(1)=\begin{bmatrix}u(1,0)\\u(1,3)\end{bmatrix},\ \boldsymbol{V}_0^{3}(2)=\boldsymbol{V}_0(1)-\boldsymbol{V}_N(1)=\begin{bmatrix}u(2,0)\\u(2,3)\end{bmatrix},\ \boldsymbol{V}_0^{3}(2)=\boldsymbol{V}_0(1)-\boldsymbol{V}_N(1)=\begin{bmatrix}u(3,0)\\u(3,3)\end{bmatrix},$$

$$\mathbf{A}=\begin{bmatrix}2&-1\\-1&2\end{bmatrix},\quad \mathbf{E}=\begin{bmatrix}1&0\\0&1\end{bmatrix},\qquad \boldsymbol{G}(1)=\begin{bmatrix}g(1,1)\\g(1,2)\end{bmatrix},\ \boldsymbol{G}(2)=\begin{bmatrix}g(2,1)\\g(2,2)\end{bmatrix},\quad \boldsymbol{G}(3)=\begin{bmatrix}g(3,1)\\g(3,2)\end{bmatrix}.$$

After the substitution of $\boldsymbol{U}(1)$, $\boldsymbol{U}(2)$, a new equation can established

$$(3\mathbf{E}+4\mathbf{A}+\mathbf{A}^2)\boldsymbol{U}_4-(4\mathbf{E}+10\mathbf{A}+6\mathbf{A}^2+\mathbf{A}^3)\boldsymbol{U}_3+\boldsymbol{U}_0+(3\mathbf{E}+4\mathbf{A}+\mathbf{A}^4)\boldsymbol{U}_{0,3}{}^3+(2\mathbf{E}+\mathbf{A})\boldsymbol{U}_{0,1}{}^3=$$
$$=(3\mathbf{E}+4\mathbf{A}+\mathbf{A}^2)\boldsymbol{G}_3+(2\mathbf{E}+\mathbf{A})\boldsymbol{G}_2+\boldsymbol{G}_1,$$

$\boldsymbol{U}_1$=$\boldsymbol{U}(1)$, $\boldsymbol{U}_2$=$\boldsymbol{U}(2)$, $\boldsymbol{U}_3=\boldsymbol{U}(3)$, $\boldsymbol{U}_4$=$\boldsymbol{U}(4)$, $\boldsymbol{G}_1$=$\boldsymbol{G}(1)$, $\boldsymbol{G}_2$=$\boldsymbol{G}(2)$, $\boldsymbol{G}_3$=$\boldsymbol{G}(3)$, $\boldsymbol{V}_{0,1}{}^3$=$\boldsymbol{V}_0{}^3(1)$, $\boldsymbol{V}_{0,2}{}^3$=$\boldsymbol{V}_0{}^3(2)$, $\boldsymbol{V}_{0,3}{}^3$=$\boldsymbol{V}_0{}^3(3)$.

After the comparison of the coefficients behind the same power of **A**, the following formulas can be calculated

$3\boldsymbol{U}_4+4\boldsymbol{U}_3+\boldsymbol{U}_0+3\boldsymbol{V}_{0,3}{}^3+2\boldsymbol{V}_{0,2}{}^3+\boldsymbol{V}_{0,1}{}^3=3\boldsymbol{G}_3+2\boldsymbol{G}_2+\boldsymbol{G}_1$

$4\boldsymbol{U}_4-10\boldsymbol{U}_3+4\boldsymbol{V}_{0,3}{}^3+\boldsymbol{V}_{0,2}{}^3=4\boldsymbol{G}_3+\boldsymbol{G}_2$, $\qquad \boldsymbol{U}_4-6\boldsymbol{U}_3+\boldsymbol{V}_{0,3}{}^3=\boldsymbol{G}_3$.

The solution is obtained as follows $\boldsymbol{G}_1=14\boldsymbol{U}_3+\boldsymbol{U}_0+\boldsymbol{V}_{0,1}{}^3$, $\quad \boldsymbol{G}_2=14\boldsymbol{U}_3+\boldsymbol{V}_{0,2}{}^3$, $\quad \boldsymbol{G}_3=\boldsymbol{U}_4-6\boldsymbol{U}_3+\boldsymbol{V}_{0,3}{}^3$.

Considering $\boldsymbol{U}_3=\alpha\boldsymbol{U}_4$ the solution can be written in the form

$\boldsymbol{G}_1 = 14\alpha \boldsymbol{U}_4 + \boldsymbol{U}_0 + \boldsymbol{V}_{0,1}{}^3$, $\mathbf{G}_2 = 14\alpha \boldsymbol{U}_4 + \boldsymbol{V}_{0,2}{}^3$, $\boldsymbol{G}_3 = (1 - 6\alpha)\boldsymbol{U}_4 + \boldsymbol{U}_0 + \mathbf{V}_{0,1}{}^3$,

where: $\boldsymbol{G}_1 = \begin{bmatrix} g(1,1) \\ g(1,2) \end{bmatrix} = 14\alpha \begin{bmatrix} u(4,1) \\ u(4,2) \end{bmatrix} + \begin{bmatrix} u(0,1) \\ u(0,2) \end{bmatrix} + \begin{bmatrix} u(1,0) \\ u(1,3) \end{bmatrix}$,

$$\boldsymbol{G}_2 = \begin{bmatrix} g(2,1) \\ g(2,2) \end{bmatrix} = 14\alpha \begin{bmatrix} u(4,1) \\ u(4,2) \end{bmatrix} + \begin{bmatrix} u(2,0) \\ u(2,3) \end{bmatrix}, \qquad \boldsymbol{G}_3 = \begin{bmatrix} g(3,1) \\ g(3,2) \end{bmatrix} = (1\text{-}6\alpha) \begin{bmatrix} u(4,1) \\ u(4,2) \end{bmatrix} + \begin{bmatrix} u(3,0) \\ u(3,3) \end{bmatrix}.$$

5 Optimization with the power functional

The elaborated method of field sources identification with reduced measurement data can be applied to solve optimization problem using the power function in the form (16)

$$F = \sum_{i,j=1} F_{i,j} = \sum_{i,j=1} \left[\frac{1}{2} [(u_{i,j} - u_{i+1,j})^2 + (u_{i,jj} - u_{i-1,jj})^2 + (u_{i,j} - u_{i,j+1})^2 + (u_{i,j} - u_{i,j-1})^2] + u_{i,j} g_{i,j} \right], \; u_{i,j} = u(i,j), g_{i,j} = g(i,j).$$

The name power functional is used as an analogy to the problem of electrical circuit built with the net of resistors. The values of source function $g_{i,j}$ can be calculated in the process of finding the extremes of this function by comparison the first partial derivatives to the zeros, i.e. doing the calculations for the problem $\nabla F = 0$.

Example 2. In this example the discretization of the square investigated region was done with the square net for $N = M = 3$ where values V_1, V_2, V_3, V_4 mean indirect potentials in internal nodes used only in the process of calculation, values V_5, V_6, V_7, V_8, V_9, V_{10}, V_{11}, V_{12} - values of known potentials in the nodes placed at the bounder of the domain, q_1 , q_2 , q_3, q_4 - unknown values of source function placed in internal nodes. Doing a discretization with the finite differences method, the set of equations connecting the indirect potentials with potentials on the bounder and values of sources' function can be established in the matrix form

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} V_5 + V_{12} \\ V_{10} + V_{11} \\ V_6 + V_7 \\ V_8 + V_9 \end{bmatrix}, \; \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 4 & 0 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} V_5 + V_{12} \\ V_{10} + V_{11} \\ V_6 + V_7 \\ V_8 + V_9 \end{bmatrix}$$

The power function F on the basis of equation (16) can be written in the form

$$\mathrm{F} = \frac{1}{2} [(V_1 - V_5)^2 + (V_1 - V_{12})^2 + (V_1 - V_2)^2 + (V_1 - V_3)^2 + (V_3 - V_6)^2 + (V_3 - V_7)^2 + (V_3 - V_4)^2 + \ldots$$

$$\ldots + (V_4 - V_8)^2 + (V_4 - V_2)^2 + (V_4 - V_9)^2 + (V_2 - V_{10})^2 + (V_2 - \mathrm{V}_{11})^2] + V_1 q_1 + V_2 q_2 + V_3 q_3 + \mathrm{V}_4 q_4 \quad (17)$$

Then putting the values of indirect potentials, the power function can be written as a function of q_1 , q_2, q_3, q_4, i.e. $F = F(q_1, q_2, q_3, q_4)$. In this example the resulting power function has a square form and the operation of finding extremes may be done by matrix calculations like in the instance of grandient calculations in the stationary point or using the known optimization procedures.

6 Conclusions

In this paper there was presented special numerical method named the Simulation Method used to solve engineering inverse problems. This method uses special computational tools based on objects from combinatorics or on principles standing in physics. Especially the idea of construction computational algorithms was presented. Special computational tools and regularization procedures were described in detail. Also it was pointed that the Simulation Method may be applied to solve inverse problems for reduced measurement data as well as optimization problems with the use of the power functional.

7 References

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