LIMITING PERFORMANCE ANALYSIS OF SHOCK ISOLATION FOR A NON-RIGID OBJECT

D.V. Balandin[1], N.N. Bolotnik[2],
[1]Nizhny Novgorod State University, Russia,
[2]Institute for Problems in Mechanics of the Russian Academy of Sciences, Russia

Corresponding author: N.N. Bolotnik, Institute for Problems in Mechanics of the Russian Academy of Sciences
101 bld. 1 Vernadskii Ave, Moscow 119526, Russia, bolotnik@ipmnet.ru

Abstract. A limiting performance of shock isolation is studied for an object modeled by two rigid bodies connected by a viscoelastic element with a linear characteristic. The object is attached to a movable base by means of a shock isolator that produces a control force between the base and the object. The base and the object move along the same straight line. The base is subject to an external shock excitation, characterized by the time history of the acceleration of the base. A control law is determined for the shock isolator to minimize the maximum magnitude of the displacement of the object relative to the base, provided that the force of interaction between the components of the object does not exceed a prescribed value. An algorithm for constructing the solution of the problem is presented. The optimal control is shown to have impulse components. A two-component model similar to that considered in the paper was utilized to describe the mechanical response of a human body to a compressive shock load of the spine or thorax. Therefore, the problem under consideration can be regarded as a benchmark optimal control problem for a system that protects from injuries cased by shock loads. Solution of such problems is highly topical for development of safety systems for vehicles.

1 Introduction

This study is aimed at the development of the theory of optimal shock isolation. The major issues of this theory are the analysis of the limiting capabilities to protect an object from shock loads by means of partial isolation of the object from a movable base subject to shock excitations, and the development of optimal control strategies for shock isolators. To that end, a mathematical model of the system that consists of a base and an object attached to it is used. The base may be subjected to a transient (shock) external excitation of force or kinematic type. Between the base and the object there is a medium or a structure that allows the object to move relative to the base and reduces the shock load as compared with the case where the object is rigidly attached to the base. This medium is called a *shock isolator*. The basic problem is to define an optimal law of interaction of the object with the base by means of the isolator, which ensures a maximum reduction in the shock load, provided that the displacement of the object relative to the base remains within given limits or, vice versa, minimizes the maximum magnitude of the relative displacement of the body, provided that the maximum load does not exceed a prescribed value.

Solution of this problem is very important for engineers who deal with design of shock isolators for various objects. These objects may represent high-precision shock-sensitive devices for ships and aircraft, buildings and structures in seismic regions, as well as human beings moving in vehicles or producing work associated with an elevated risk of injuries caused by impacts (e.g., construction workers or personnel engaged in military or antiterrorist operations). Increasing safety of vehicles due to improvement of engineering equipment for protection of the occupants from shock loads in a crash is highly topical in view of extremely high level of trauma and death in road accidents.

The theory of optimal shock isolation began to be developed in the middle of the last century in connection with the design of effective shock isolation equipment for rockets and space objects, including military ones. Pioneering studies in this field are due to Sevin [1,2] and Guretskii [3] who were the first to associate the limiting performance analysis of shock isolation with an optimal control problem. The fundmentals and basic results of the theory are presented in 4 books [4–7]. It has been thoroughly developed for models in which the object to be protected is regarded as a rigid body. The issues of optimal shock isolation for multi-component systems and bodies that possess significant elastic compliance have not been studied with due profundity. Therefore, the theory of optimal shock isolation needs further development. New statements are required for optimal control problems for multi-component and elastic systems that would be adequate to engineering problems arising in design of shock isolation equipment. Solution of these problems should enable the limiting capabilities of shock isolation to be assessed and qualitative features of control laws to be identified.

The statement and the solution of an optimal control problem for a shock isolation system for an object that is modeled by two rigid bodies connected by a spring-and-dashpot element with a linear characteristic are given in the present paper. The object and the base move translationally along the same straight line. A shock isolator is located between the base and one of the object's bodies. For a given shock excitation (modeled by the time history of the acceleration of the base), it is required to find an optimal control law for the force produced by the isolator. This control law should minimize the maximum of the absolute value of the relative displacement of the body connected to the base by means of the shock isolator, provided that the maximum of the absolute value of the force of interaction between the object's components does not exceed a prescribed value. This problem generalizes the problem of optimal shock isolation considered by Guretskii [3] to the case of a non-rigid object. The solution of the problem for a compliant object qualitatively differs from the solution for a rigid object. The optimal control force for a compliant object may involve impulse components, i.e., on some infinitesimal time intervals, the theoretically optimal isolator acts with infinitely large force, despite the constraint on the force of interaction between the object's components. The optimal control force for a rigid object is always finite. The impulse components do not disappear as the stiffness of the spring increases without limit. The optimal control for the compliant model converges to the respective control for the rigid model only if both the stiffness and damping coefficients tend to infinity simultaneously in a certain coordinated way. Applying the control that is optimal for the rigid model of the object to a compliant object can lead to significant violation of the constraint on the force of interaction between the components.

The specific features of optimal shock isolation identified for a compliant visco-elastic object are important from the viewpoints of both theory and applications. The two-component model of the object that is considered in the paper is used in engineering biomechanics to describe the response of human body to some sorts of shock loads, in particular, to spine or thorax compression load [8–10]. Such loads, are typical, for example, for an aircraft pilot who is ejected from the aircraft with his seat or for the pilot of a helicopter that performs a hard emergency landing with high vertical velocity. To mitigate hazardous load on the vertebral column, the pilot's seat is equipped with a shock isolation system. In survey [11], devoted to shock isolation systems for helicopter seats, it is noticed that most shock isolation systems are designed so as to keep the force acting on the pilot close to constant during all time of deceleration of the motion of the pilot relative to the seat. This control strategy would have been close to the optimal strategy, if the response of the pilot to the shock load had been described by a rigid model. In accordance with the results of the present study, the optimal control force is essentially non-constant for the model in which elastic and viscous properties of the vertebral column are taken into account. At the beginning, a large force is necessary to be applied for a short time. Then this force rapidly decreases and, after decay of a transient, is kept on constant level that corresponds to the rigid model. The initial stage with a short-term spike enables longitudinal vibrations not to be excited in the pilot's body. Such vibrations substantially increase the load on the vertebral column. It is noticed [11] that the quality of protection of the pilot of a helicopter from the vertical shock load in a hard landing event can be improved by introducing a spike to the control force acting on the pilot at the beginning of operation of the shock isolator. However, rigorous proof of this conjecture is not given. In the present paper, this conjecture is proved by solving an optimal control problem.

An optimal shock isolation problem for a two-component object was considered by the authors previously [12]. An approximate near-optimal solution was constructed for systems with high stiffness coefficient. The near-optimal control was non-constant and involved impulse components. In the present paper, an exact solution of this problem is found under certain conditions that are typically satisfied for vehicle safety systems. Specific features of this solution are discussed.

2 Mathematical model of the system and statement of the optimal control problem

Consider a system that consists of a movable base and an object to be protected. The object is attached to the base by means of a shock isolator (Fig. 1). The base and the object move along the same straight line. The object is modeled by two bodies (1 and 2) connected to each other by a spring and a damper with linear characteristics. The shock isolator is located between the base and body 2 and acts on this body with force F directed along the line of motion of the system.

Motion of bodies 1 and 2 relative to the base is governed by the system of differential equations

$$\begin{aligned} &m_1(\ddot{x}+\ddot{z})+C(\dot{x}-\dot{y})+K(x-y)=0,\\ &m_2(\ddot{y}+\ddot{z})+C(\dot{y}-\dot{x})+K(y-x)=F, \end{aligned} \tag{1}$$

where m_1 and m_2 are the masses of bodies 1 and 2, respectively; x and y are the coordinates that measure the displacements of bodies 1 and 2 relative to the base; z is the coordinate that measures the displacement of the base relative to a fixed (inertial) reference frame; C is the damping coefficient ($C > 0$); K is the stiffness coefficient of the spring ($K > 0$). We assume that at the initial time instant $t = 0$, bodies 1 and 2 are resting in the positions corresponding to zero coordinates x and y, and, accordingly, the system of Eq. (1) is subjected to the initial conditions

$$x(0)=0,\quad \dot{x}(0)=0,\quad y(0)=0,\quad \dot{y}(0)=0. \tag{2}$$

The acceleration of the base $\ddot{z}$ is assumed to be a known function of time. This function characterizes an external shock disturbance (excitation) applied to the base.

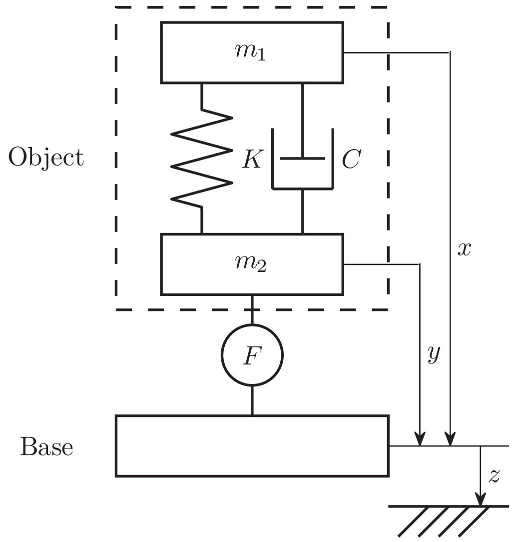

Figure 1: Two-body object attached to the base by a shock isolator

The system described can be interpreted as a simplified model of a viscoelastic object on a movable base subject to intensive impact loads that may destroy the object or cause serious damage to it. The isolator serves to reduce these loads. As the performance criteria of shock isolation we take the maximum of the absolute value of the displacement of body 2 relative to the base,

$$J_1 = \max_{t\in[0,\infty)} |y(t)|, \tag{3}$$

and the maximum magnitude of the force acting on body 1,

$$J_2 = \max_{t\in[0,\infty)} |m_1[\ddot{x}(t)+\ddot{z}(t)]| = \max_{t\in[0,\infty)} |C[\dot{x}(t)-\dot{y}(t)]+K[x(t)-y(t)]|. \tag{4}$$

The criterion J_1 characterizes the displacement of the viscoelastic object relative to the base, whereas J_2 characterizes the internal force (stress) acting between the components of the object.

The system under consideration can also serve as a simplified model of protection of occupants of vehicles from injuries caused by shock loads in extreme events. For example, a similar two-component model was used to evaluate the longitudinal deformation and the compression force in the spine of a pilot when ejecting from an aircraft with his seat [8, 9]. In this model, m_1 is the mass of the upper torso of the pilot (above the pelvis), m_2 is the mass of the lower torso, the coefficients C and K characterize the dissipative and elastic properties of the spine, $\ddot{z}$ is the vertical acceleration of the seat base, and F is the force applied to the pilot's lower torso by the seat cushion. The cushion plays the role of shock isolator and can be passive or active. The maximum magnitude of the spine compression force is used to assess the risk of spinal injuries of various degrees of danger. This model can also be used to assess the risk of spinal injury for a pilot of a helicopter in an event of hard landing with high vertical velocity. In this case, $\ddot{z}$ is the impact deceleration of the helicopter's body to which the seat base is rigidly attached.

Problem 1. For the system of Eq. (1), subject to the initial conditions (2) and a prescribed external disturbance $\ddot{z}(t)$, find the greatest lower bound for the criterion J_1 in the class of piecewise continuous control functions $F = F(t)$, provided that the criterion J_2 does not exceed a prescribed magnitude P. In mathematical terms, it is necessary to find

$$J_1^0 = \inf_F J_1(F), \tag{5}$$

provided that

$$J_2(F) \le P. \tag{6}$$

Problem 1 is a limiting performance problem for shock isolation of an object from a given external disturbance. The major target of the limiting performance analysis is to find an absolute minimum that can be attained in principle for a functional adopted as the performance index under imposed constraints. For the limiting performance analysis, the design of the isolator is unimportant and its action is characterized by a force represented as an explicit function of time. This force is treated as a control variable. Thus, the limiting performance problem for shock isolation for a prescribed external disturbance is an optimal control problem in which an open-loop (rather than feedback) control is to be found. By comparing the value of the performance index for an available system with the absolute minimum, one can judge the degree of perfection of the system and prospects for its improvement. The concept of limiting performance analysis is presented in more detail in [4, 6, 7].

In Problem 1, a constraint is imposed on the force acting between the components of the object to be protected, whereas no constraint is imposed on the control force F. This statement is reasonable at the initial stage of design of shock isolation systems and corresponds to the limiting performance analysis of isolation irrespective of the design characteristics of the isolator. Large internal forces acting between the object's components can destroy or damage the object and, hence, must be constrained. The magnitude of the force produced by a shock isolator depends on the parameters of actuators used in the system. Based on the solution of the problem under consideration, the designer can select an actuator with appropriate parameters or conclude that it is impossible to provide the desired quality of isolation using available hardware. In the latter case, it is reasonable to find out how much the protection quality that can be achieved with the best of the available actuators is inferior to the limiting performance and, possibly, to make a decision on the design of a new actuator.

3 Auxiliary problem

Introduce the auxiliary control variable

$$W = C(\dot{y} - \dot{x}) + K(y - x) \tag{7}$$

to represent system (1) as follows:

$$m_1(\ddot{x} + \ddot{z}) = W, \tag{8}$$

$$W = C(\dot{y} - \dot{x}) + K(y - x), \tag{9}$$

$$m_2(\ddot{y} + \ddot{z}) = F - W. \tag{10}$$

Constraint (6) on the maximum magnitude of the force acting on body 1 is equivalent to the constraint on the auxiliary control

$$|W| \le P. \tag{11}$$

Consider an optimal control problem for the system of Eqs. (8) – (10). Introduce the notation

$$\xi = x - y, \quad u = \frac{W}{m_1}, \quad c = \frac{C}{m_1}, \quad k = \frac{K}{m_1}, \quad v = -\ddot{z}, \quad U = \frac{P}{m_1} \tag{12}$$

to represent relations (8) and (9) as

$$\ddot{x} = u + v, \tag{13}$$

$$c\dot{\xi} + k\xi = -u. \tag{14}$$

From the initial conditions (2) it follows that

$$x(0) = 0, \quad \dot{x}(0) = 0, \quad \xi(0) = 0. \tag{15}$$

Although Eq. (2) implies also the relation $\dot{\xi}(0) = 0$, equation (14) cannot be subjected to this initial condition, since it is a first-order equation and the derivative $\dot{\xi}$ can be uniquely expressed in terms of ξ, u, k and c.

Problem 2. For the system of Eqs. (13) and (14), subject to the initial conditions (15) and a given external disturbance $v(t)$, find a piecewise continuous optimal control $u = u_0(t)$ that satisfies the constraint

$$|u| \le U, \tag{16}$$

and minimizes the functional

$$J_1(u) = \max_{t\in[0,\infty)} |y(t)| = \max_{t\in[0,\infty)} |x(t) - \xi(t)|. \tag{17}$$

Solution of Problem 2 enables one to determine the minimum of the functional $J_1(F)$ in Problem 1, the relationship $J_1(u_0) = J_1^0$ being valid. In addition, proceeding from the solution of Problem 2, one can readily calculate the optimal control $F = F_0(t)$ for Problem 1. Solution of the system of Eqs. (13) and (14) subject to the initial conditions (15) and the control $u = u_0(t)$ gives $x = x_0(t)$ and $\xi = \xi_0(t)$. The subscript 0 indicates that this solution corresponds to the optimal control $u_0(t)$. From Eq. (12) we obtain $y = y_0(t) = x_0(t) - \xi_0(t)$. By substituting $x_0(t)$, $y_0(t)$, $\ddot{z}(t) = -v(t)$, and $W(t) = m_1u_0(t)$ into Eq. (10) we find the optimal control $F_0(t)$:

$$F_0(t) = m_2(\ddot{y}_0(t) - v(t)) + m_1u_0(t). \tag{18}$$

At the instants of time when the optimal control $u_0(t)$ experiences jump discontinuities, the velocity $\dot{y}_0(t)$ also experiences jump discontinuities. Let $v(t)$ be a piecewise continuous function. Then the function $\dot{x}_0(t)$, defined by

$$\dot{x}_0(t) = \int_0^t [v(\tau) + u_0(\tau)]\, d\tau, \tag{19}$$

is continuous. The relation $y_0(t) = x_0(t) - \xi_0(t)$ implies that $\dot{y}_0(t) = \dot{x}_0(t) - \dot{\xi}_0(t)$. From Eq. (14) it follows that the function $\dot{\xi}_0(t)$ has jump discontinuities at the time instants when the function $u_0(t)$ has jump discontinuities. Therefore, the function $\dot{y}_0(t)$ also has jump discontinuities at these time instants. Accordingly, the optimal control (18) has impulse components that are represented by Dirac's delta functions concentrated at the points of discontinuity of the control u_0 for the auxiliary problem (Problem 2). Hence, it may occur that the greatest lower bound of the criterion J_1 to be determined in Problem 1 is not attained in the class of piecewise continuous functions $F(t)$.

4 Optimal control problem for the rigid model of the object

Consider the rigid model of the object as the limiting case of the two-component model in which bodies 1 and 2 are rigidly attached to one another. The motion of this object is governed by the differential equation

$$(m_1 + m_2)(\ddot{x} + \ddot{z}) = F \tag{20}$$

subject to the initial conditions

$$x(0) = 0, \quad \dot{x}(0) = 0, \tag{21}$$

where $\ddot{x}$ is the common acceleration of bodies 1 and 2 relative to the base. Without loss of generality we assume $x(t) \equiv y(t)$. The maximum magnitude of the relative displacement of the object is defined as

$$\widetilde{J}_1 = \max_{t\in[0,\infty)} |x(t)|, \tag{22}$$

while the maximum magnitude of the force acting on body 1 is given by

$$\widetilde{J}_2 = \max_{t\in[0,\infty)} |m_1[\ddot{x}(t) + \ddot{z}(t)]| = \max_{t\in[0,\infty)} \left| \frac{m_1}{m_1+m_2} F(t) \right|. \tag{23}$$

Here and in what follows, the tilde indicates the quantities calculated for the rigid model. An analogue of Problem 1 for the rigid model is formulated as

Problem 3. For system (20) subject to the initial conditions (21), find

$$\widetilde{J}_1^0 = \inf_F \widetilde{J}_1(F), \tag{24}$$

under the constraint

$$\widetilde{J}_2(F) \le P, \tag{25}$$

where P is a prescribed positive quantity. The greatest lower bound in Eq. (24) is sought in the class of piecewise continuous controls $F(t)$.

For the rigid model, the constraint (25) implies the constraint on the control force

$$|F| \le \frac{m_1 + m_2}{m_1} P. \tag{26}$$

Denote

$$u = \frac{F}{m_1 + m_2}, \quad v = -\ddot{z}, \quad U = \frac{P}{m_1}, \tag{27}$$

and formulate Problem 3 as follows:

Problem 4. For the system

$$\ddot{x} = u + v, \tag{28}$$

subject to the initial conditions

$$x(0) = 0, \quad \dot{x}(0) = 0 \tag{29}$$

and a prescribed external disturbance $v(t)$, find a piecewise continuous control $u = \widetilde{u}_0(t)$ such that

$$\widetilde{J}_1(\widetilde{u}_0) = \min_u \widetilde{J}_1(u), \tag{30}$$

under the constraint

$$|u| \leq U. \tag{31}$$

Problem 4 is known in the literature as the simplest (basic) problem of optimal shock isolation. Various techniques for solving this problem are presented in reasonable detail in monographs [4, 6, 7]. For some types of external disturbances, analytical solutions or numerical solutions depending on few dimensionless numerical parameters were constructed. In a number of cases, the optimal control $u_0(t)$ is constant in the time interval from the beginning of the motion to the time instant at which the displacement of the object relative to the base is a maximum. Proceeding from the solution of Problem 4, it is possible to construct an approximate solution and, in some cases, even an exact solution of Problem 2. In the next section, an exact solution of Problem 2 will be constructed under the assumption that the optimal control in Problem 4 is constant. It was shown [11] that for large stiffness coefficient K of the spring that connects bodies 1 and 2, the solution of Problem 4 provides good approximation to the solution of Problem 2 in terms of the functional to be minimized. Accordingly, the solution of Problem 3 gives good approximation to the solution of Problem 1.

5 Construction of the optimal control for the two-component model on the basis of the optimal control for the rigid model

5.1 Notation and preliminary calculations

. Let $\widetilde{u}_0(t)$ and $\widetilde{x}_0(t)$ be the optimal control and the corresponding time history of the coordinate x for the rigid model. Solution of Eq. (14) subject to the initial condition $\xi(0) = 0$, which follows from (15), gives

$$\xi(t) = -\frac{1}{c}\int_0^t \exp\left(-\frac{k}{c}(t-\tau)\right) u(\tau)\, d\tau. \tag{32}$$

Denote the function $\xi(t)$ for the control $\widetilde{u}_0(t)$ by $\widetilde{\xi}_0(t)$ and introduce the variable

$$\widetilde{y}_0(t) = \widetilde{x}_0(t) - \widetilde{\xi}_0(t). \tag{33}$$

Equation (28) that governs the dynamics of the rigid model coincides with Eq. (13) that governs the motion of body 1 in the two-component model. The variables x, y, and ξ in the two-component model are related by

$$y = x - \xi. \tag{34}$$

This relation coincides with relation (33) for the variables $\widetilde{x}_0$, $\widetilde{y}_0$, and $\widetilde{\xi}_0$. The variable y in (34) denotes the coordinate of body 2 of the two-component model relative to the base. Therefore, the function $\widetilde{y}_0$ can be regarded as the time history of the coordinate y of the two-component model, provided that the motion of body 1 coincides with the optimal motion of the rigid model of the object.

It is assumed in what follows that the rigid object in the optimal motion decelerates with constant acceleration $-U$ in an interval $[0, T]$ until it comes to a complete stop at a time instant $t = T$ and that the action of the external disturbance has ended by this time. Then $\widetilde{u}_0(t) \equiv -U$ for $t \in [0, T]$ and relation (32) implies

$$\widetilde{\xi}_0(t) = \frac{U}{k}\left[1 - \exp\left(-\frac{k}{c}t\right)\right], \quad 0 \leq t \leq T. \tag{35}$$

5.2 Basic assumptions and solution of Problem 2

To construct the optimal control for Problem 2 we will make a number of assumptions.

Assumption 1. The external disturbance (shock pulse) $v(t)$ has a finite duration τ, i.e., $v(t) \equiv 0$ for $t > \tau$.

Assumption 2. The optimal control for the rigid-body model of the object is constant and is given by

$$\widetilde{u}_0(t) \equiv -U, \quad 0 \leq t \leq T, \quad T > \tau, \tag{36}$$

where T is the instant at which the object comes to a complete stop and, hence,

$$\dot{\widetilde{x}}_0(T) = 0. \tag{37}$$

Assumption 3. The maximum absolute value of the function $\widetilde{y}_0(t)$ in the interval $0 \leq t \leq T$ satisfies the relation

$$\widetilde{y}_0(T_*) = \max_{t\in[0,\, T]} |\widetilde{y}_0(t)|, \tag{38}$$

where T_* is the instant of time at which the maximum is attained. This implies, in particular, that $\widetilde{y}_0(T_*) \geq 0$.

P r o p o s i t i o n 1. The function

$$u_0(t) = \begin{cases} -U, & \text{for} \quad 0 \leq t \leq T, \\ \bar{u}(t), & \text{for} \quad t > T, \end{cases} \tag{39}$$

where $\bar{u}(t)$ is the solution of the differential equation

$$\ddot{\bar{u}} + c\dot{\bar{u}} + k\bar{u} = 0 \tag{40}$$

subject to the initial conditions

$$\bar{u}(T) = -k\widetilde{\xi}_0(T), \quad \dot{\bar{u}}(T) = ck\widetilde{\xi}_0(T), \tag{41}$$

is an optimal control for Problem 2. The behavior of the coordinate y under this control is characterized by the function

$$y_0(t) = \begin{cases} \widetilde{y}_0(t), & \text{for} \quad 0 \leq t \leq T, \\ \widetilde{y}_0(T), & \text{for} \quad t > T, \end{cases} \tag{42}$$

and the minimum of the performance index is defined by

$$J_1(u_0) = \widetilde{y}_0(T_*). \tag{43}$$

This proposition provides an algorithm for solving Problem 2. The algorithm involves the following steps.

Step 1. Solve Problem 4 for the rigid-body model of the object by using an appropriate method [3–7].

Step 2. Check relations (36) and (37).

Step 3. Use relations (33) and (35) to form the function

$$\widetilde{y}_0(t) = \widetilde{x}_0(t) - \frac{U}{k}\left[1 - \exp\left(-\frac{k}{c}t\right)\right], \quad 0 \leq t \leq T, \tag{44}$$

where the function $\widetilde{x}_0(t)$ has resulted from solution of Problem 4 at step 1.

Step 4. Check relation (38).

Step 5. Solve the initial-value problem of (40) and (41) to find the optimal control $\bar{u}(t)$ for $t > T$.

P r o o f. To prove Proposition 1, we will first show that the inequality

$$y_0(t) \leq J_1(u) \tag{45}$$

holds for any $u(t)$ that satisfies the constraint

$$|u(t)| \leq U. \tag{46}$$

Then we will prove that the control $\bar{u}(t)$ provides the identity $y_0(t) \equiv \widetilde{y}_0(T)$ for $t > T$. With reference to (38) and (42), this means that the control $u_0(t)$ defined by expressions (39)–(41) provides the value of (43) for the criterion J_1. Finally, we will prove the constraint (46) for the control $\bar{u}(t)$.

To prove relation (45) it suffices to show that the inequality

$$y(t) \geq \widetilde{y}_0(t), \quad 0 \leq t \leq T \tag{47}$$

holds for any control u that satisfies constraint (46), since $J_1(u) \geq y(t)$ for $t \in [0, T]$ and, with reference to (42), $y_0(t) \equiv \widetilde{y}_0(T)$ for $t > T$. Expression (32) and the inequality $u \geq -U$ imply the relation

$$-\xi(t) \geq -\widetilde{\xi}_0(t), \quad 0 \leq t \leq T, \tag{48}$$

since the function $\widetilde{\xi}_0(t)$ in the interval $0 \leq t \leq T$ is defined as $\xi(t)$ for $u = -U$. In a similar way, the inequality

$$x(t) \geq \widetilde{x}_0(t), \quad 0 \leq t \leq T, \tag{49}$$

can be derived from Eq. (13), the initial conditions of (15) for x and $\dot{x}$, the inequality $u \geq -U$, and the function $\widetilde{x}_0$ defined for $0 \leq t \leq T$ as the solution of the initial value problem of (28) and (29) for $u = -U$. Add inequalities (48) and (49) and use expressions (33) and (34) for $\widetilde{y}_0$ and y, respectively, to obtain relation (47).

To derive Eq. (40) for the control $\overline{u}$ we will proceed from expression (42). From this expression, it follows that $y = \text{const}$ and, hence, $\dot{y} = 0$ and $\ddot{y} = 0$ for $t > T$. Using these relations, with reference to (34), we obtain

$$\dot{\xi} = \dot{x}, \quad \ddot{\xi} = \ddot{x}, \quad t > T. \tag{50}$$

Then Eqs. (13) and (14) imply

$$\ddot{\xi} + c\dot{\xi} + k\xi = 0, \quad t > T. \tag{51}$$

This equation takes into account Assumptions 1 – 3, according to which $v(t) = 0$ for $t > T$. Use Eqs. (14) and (51) and the definition of the quantity $\overline{u}$ as the control u for $t > T$ to obtain

$$\overline{u} = \ddot{\xi}. \tag{52}$$

Differentiate Eq. (51) twice with respect to time and then substitute $\overline{u}$ for $\ddot{\xi}$ into the resulting relation to arrive at Eq. (40).

The initial conditions (41) for the variable $\overline{u}$ follow from the conditions for the variables ξ and $\dot{\xi}$ at the instant T:

$$\xi(T) = \widetilde{\xi}_0(T), \quad \dot{\xi}(T) = 0. \tag{53}$$

The first of these conditions follows from the relation $\xi(t) = \widetilde{\xi}_0(t)$, which is valid for $0 \leq t < T$, and continuity of the function $\xi(t)$. The second condition follows from relation $\dot{\widetilde{x}}(T) = 0$, which is valid in view of (37), continuity of the function $\dot{x}(t)$, and the first relation of (50). To obtain the condition $\dot{\xi}(T) = 0$, proceed in the relation $\dot{\xi}(t) = \dot{x}(t)$ to the limit as $t \to T + 0$.

To obtain the first relation of (41), substitute the initial conditions of (53) for the variables ξ and $\dot{\xi}$ into Eq. (51) and take into account the definition of (52) for the variable $\overline{u}$. To obtain the second relation, differentiate Eq. (51) with respect to time and use Eq. (52) to arrive at the relation

$$\dot{\overline{u}} = -k\overline{u} - c\dot{\xi} \tag{54}$$

and then use the conditions $\dot{\xi}(T) = 0$ and $\overline{u}(T) = -k\widetilde{\xi}_0(T)$.

It remains to prove inequality (46) for the control $\overline{u}(t)$. Solving Eq. (40) subject to the initial conditions (41) gives

$$\overline{u}(t) = k\widetilde{\xi}_0(T) \exp\left(-\frac{c}{2}(t - T)\right) \Psi(t - T), \tag{55}$$

where

$$\Psi(t) = \begin{cases} \dfrac{c}{2\omega} \sinh \omega t - \cosh \omega t, & \text{for } \dfrac{c^2}{4} - k > 0, \\ \dfrac{c}{2} t - 1, & \text{for } \dfrac{c^2}{4} - k = 0, \\ \dfrac{c}{2\omega} \sin \omega t - \cos \omega t, & \text{for } \dfrac{c^2}{4} - k < 0, \end{cases} \qquad \omega = \sqrt{\left| \frac{c^2}{4} - k \right|}. \tag{56}$$

Equation (40) coincides with the equation of a damped linear oscillator. Due to dissipation of energy, the maximum of $|\overline{u}|$ occurs either at the initial instant of time ($t = T$ in the case under consideration) or at the instant when the function $\overline{u}(t)$ has its first local extremum and the derivative $\dot{\overline{u}}(t)$ vanishes.

The constraint $|\overline{u}| \leq U$ holds at the instant $t = T$, since, in accordance with (41) and (35),

$$|\overline{u}(T)| = k\widetilde{\xi}_0(T) = U\left[1 - \exp\left(-\frac{k}{c}T\right)\right]. \tag{57}$$

The analysis of the function $\overline{u}(t)$, defined by Eq. (55), for extremum shows that the first local extremum occurs at the instant $t_* = T + \tau_*$, where τ_* satisfies the relations

$$\begin{aligned} &\sinh \omega\tau_* = \frac{4\omega c}{c^2 - 4\omega^2}, \quad \cosh \omega\tau_* = \frac{c^2 + 4\omega^2}{c^2 - 4\omega^2}, \quad \text{for } \frac{c^2}{4} - k > 0, \\ &\tau_* = \frac{4}{c}, \qquad\qquad \text{for } \frac{c^2}{4} - k = 0, \\ &\sin \omega\tau_* = \frac{4\omega c}{c^2 + 4\omega^2}, \quad \cos \omega\tau_* = \frac{c^2 - 4\omega^2}{c^2 + 4\omega^2}, \quad \text{for } \frac{c^2}{4} - k > 0. \end{aligned} \tag{58}$$

We choose to omit a detailed derivation of these equations. Substitute the relations of (58) into those of (55) and (56) to obtain

$$|\overline{u}(t_*)| = |\overline{u}(T)| \exp\left(-\frac{c}{2}\tau_*\right). \tag{59}$$

It is apparent that $|\overline{u}(t_*)| < U$, since $|\overline{u}(T)| < U$. This completes the proof of Proposition 1.

5.3 Construction of an optimal control for Problem 1

To construct the optimal control $F_0(t)$ for Problem 1 we will use Eq. (18). Differentiate expression (42) with reference to (44) to obtain

$$\dot{y}_0(t) = \begin{cases} \dot{\widetilde{x}}_0(t) - \dfrac{U}{c}\exp\left(-\dfrac{k}{c}t\right) & \text{for} \quad 0 \le t \le T, \\ 0, & \text{for} \quad t > T, \end{cases} \tag{60}$$

$$\ddot{y}_0(t) = \begin{cases} \ddot{\widetilde{x}}_0(t) + \dfrac{Uk}{c^2}\exp\left(-\dfrac{k}{c}t\right) & \text{for} \quad 0 < t < T, \\ 0, & \text{for} \quad t > T, \end{cases} \tag{61}$$

The function $\widetilde{x}_0(t)$ corresponds to the solution of Problem 4. With reference to Eqs. (28), (29), (36), and (37), this function satisfies the differential equation

$$\ddot{\widetilde{x}}_0 = -U + v(t), \quad 0 \le t \le T \tag{62}$$

and the boundary conditions

$$\widetilde{x}_0(0) = 0, \quad \dot{\widetilde{x}}_0(0) = 0, \quad \dot{\widetilde{x}}_0(T) = 0. \tag{63}$$

Substitute expression (62) into (61) to obtain

$$\ddot{y}_0(t) = \begin{cases} -U + v(t) + \dfrac{Uk}{c^2}\exp\left(-\dfrac{k}{c}t\right) & \text{for} \quad 0 < t < T, \\ 0, & \text{for} \quad t > T. \end{cases} \tag{64}$$

In accordance with (2), the variable $\dot{y}_0(t)$ satisfies the initial condition

$$\dot{y}_0(0) = 0. \tag{65}$$

On the other hand, Eqs. (60) and (63) imply

$$\dot{y}_0(0) = -\frac{U}{c}. \tag{66}$$

From these two relations it follows that the function $\dot{y}_0(t)$ instantaneously changes from 0 to $-\frac{U}{k}$ at the time instant $t = 0$ and, hence, experiences a jump discontinuity

$$\dot{y}_0(+0) - \dot{y}_0(0) = -\frac{U}{c}, \tag{67}$$

where $+0$ stands for the instant "just after" $t = 0$. In terms of mathematics, $\dot{y}_0(+0)$ is the right-hand limit of the function $\dot{y}_0(t)$ at the point $t = 0$. Similar reasoning leads to the conclusion that the function $\dot{y}_0(t)$ has a jump discontinuity at the instant $t = T$:

$$\dot{y}_0(T+0) - \dot{y}_0(T) = \frac{U}{c}\exp\left(-\frac{k}{c}T\right), \tag{68}$$

where $T+0$ is the instant of time "just after" T.

Since the function $\dot{y}_0(t)$ is discontinuous at the instants $t = 0$ and $t = T$, it does not have classical derivatives at these points. To take into account the jumps it is necessary to add to the expression of Eq. (64) for $\ddot{y}_0(t)$ two impulse functions, $I_1(t)$ and $I_2(t)$, defined by

$$I_1(t) = -\frac{U}{c}\delta(t), \quad I_2(t) = \frac{U}{c}\exp\left(-\frac{k}{c}T\right)\delta(t-T), \tag{69}$$

where $\delta(t)$ and $\delta(t-T)$ are Dirac's delta functions. Finally, the expression for $\ddot{y}_0(t)$ becomes

$$\ddot{y}_0(t) = -\frac{U}{c}\delta(t) + \frac{U}{c}\exp\left(-\frac{k}{c}T\right)\delta(t-T) + v(t) + w_0(t), \tag{70}$$

where

$$w_0(t) = \begin{cases} -U + \dfrac{Uk}{c^2}\exp\left(-\dfrac{k}{c}t\right) & \text{for } 0 < t < T, \\ 0, & \text{for } t > T. \end{cases} \tag{71}$$

Substitute expression (70) for $\ddot{y}_0(t)$ and expression (39) for $u_0(t)$ into relation (18) to obtain

$$F_0(t) = -\frac{m_2 U}{c}\delta(t) + \frac{m_2 U}{c}\exp\left(-\frac{k}{c}T\right)\delta(t-T) + G_0(t), \tag{72}$$

where

$$G_0(t) = \begin{cases} -(m_1+m_2)U + \dfrac{m_2 Uk}{c^2}\exp\left(-\dfrac{k}{c}t\right) & \text{for } 0 < t < T, \\ m_1\overline{u}(t), & \text{for } t > T. \end{cases} \tag{73}$$

Thus, the optimal control force has two impulse components applied to body 2 at the instants $t = 0$ and $t = T$. The first impulse has the magnitude m_2U/c and instantaneously reduces the velocity of body 2 by an amount of U/c. The second impulse has the magnitude $\dfrac{m_2U}{c}\exp\left(-\dfrac{k}{c}T\right)$ and instantaneously increases the velocity of body 2 by an increment of $\dfrac{U}{c}\exp\left(-\dfrac{k}{c}T\right)$.

6 Example: Optimal protection of an object from an instantaneous shock

For an instantaneous shock, the function $v(t)$ is defined by

$$v(t) = V\delta(t), \tag{74}$$

where $-V$ is the change in the velocity of the base resulting from an external disturbance. An instantaneous shock can be interpreted, for example, as a perfectly inelastic impact of the base that moves at the velocity V against a fixed obstacle. As a result of this impact, the base comes to an instantaneous complete stop. The disturbance of (74) can be used as a simplified model for the shock pulse applied to a vehicle in a frontal crash of two identical automobiles that move with the same speed or when a vehicle hits a fixed obstacle. In this case, V is the impact velocity of the vehicle. We assume the following numerical values for the parameters of the object (m_1, m_2, C, K) and for the maximum magnitude P of the force allowed to act between bodies 1 and 2:

$$m_1 = 10\,\text{kg}, \quad m_2 = 5\,\text{kg}, \quad C = 1\,\frac{\text{kN}\cdot\text{s}}{\text{m}}, \quad K = 1000\,\frac{\text{kN}}{\text{m}}, \quad P = 4\,\text{kN}. \tag{75}$$

This example has a relation to the analysis of limiting capabilities of protection of the right leg of the driver of an automobile from fracture in a frontal crash. Just before the crash, the driver usually presses on the brake pedal with his or her right leg. The impact in a crash can cause inadmissibly high compressive load on the leg and lead to its fracture. To mitigate this load, the floor pan of the automobile near the brake pedal can be made compliant by introducing a shock isolator between the pan and the automobile's body. The solution of Problem 1 enables one to estimate the minimum displacement of the floor pan (minimum stroke of the isolator) for which the magnitude of the axial load on the leg does not exceed the value beyond which the risk of fracture of the shank (especially, the tibia) becomes unacceptably high.

The parameters m_1, C, K, and P specified by (75) are close to the parameters of a model of the mechanical response of a human leg to an axial impact load [13]. The quantity m_1 represents an effective mass of both the thigh and the shank, while the coefficients K and C characterize viscous and elastic properties of the shank. The shank (especially, the tibia) is most vulnerable to fracture in frontal crashes at low and moderate velocities. The quantity P assesses the maximum axial force that the human shank can tolerate in shock loading. The parameter m_2 represents the mass of the floor pan to which the control force F is applied. The parameters c, k, and U of (12) that correspond to the values of (75) are

$$c = 100\,\text{s}^{-1}, \quad k = 10^5\,\text{s}^{-2}, \quad U = 400\,\text{m/s}^2. \tag{76}$$

The optimal control for Problem 4 that corresponds to the rigid model of the object is defined as follows:

$$\widetilde{u}_0(t) = -U \quad \text{if} \quad 0 \le t \le T = V/U. \tag{77}$$

The corresponding time history of the coordinate x is given by

$$\widetilde{x}_0(t) = Vt - Ut^2/2, \quad \text{for} \quad 0 \le t \le T. \tag{78}$$

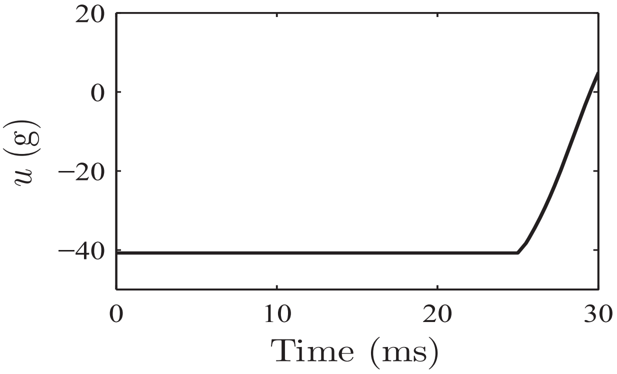

Figure 2: Optimal control $u_0(t)$ for Problem 2

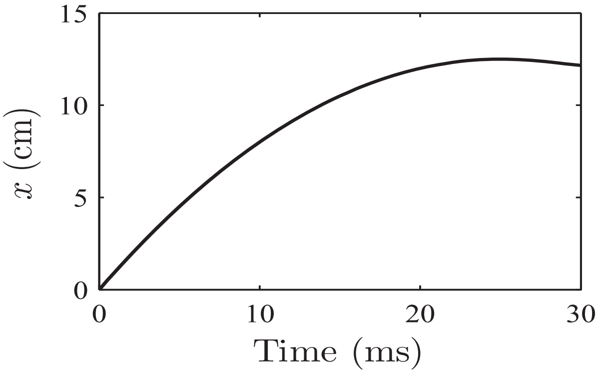

Figure 3: Optimal time history of the displacement of body 1 relative to the base

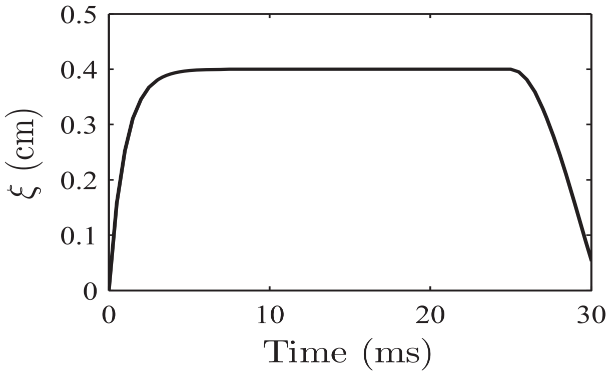

Figure 4: Optimal time history of the displacement of body 1 relative to body 2

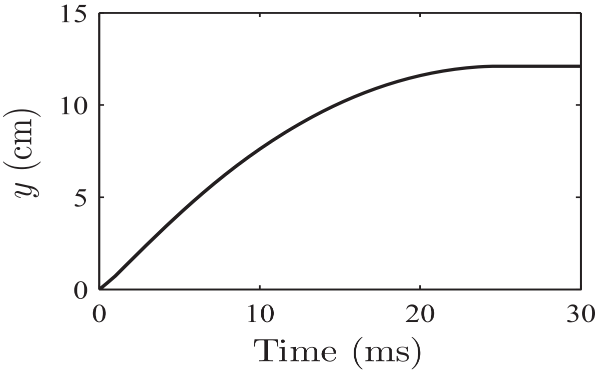

Figure 5: Optimal time history of the displacement of body 2 relative to the base

The numerical results will be presented below for

$$V = 10\,\mathrm{m/s}\ (= 36\,\mathrm{km/h})\,. \tag{79}$$

This velocity is typical of vehicle crashes. For this velocity, the deceleration time of the object in the rigid body model is identified as

$$T = 25\,\mathrm{ms}. \tag{80}$$

In the case under consideration, the response characteristics satisfy Assumptions 1 to 3.

The behavior of the system under the optimal control is shown in Figs. 2 – 5. Figure 2 shows the optimal control $u_0(t)$ for Problem 2. The optimal time histories of the coordinates $x = x_0(t)$, $\xi = \xi_0(t)$, and $y = y_0(t)$ are depicted in Figs. 3, 4, and 5, respectively. For the interval $0 \le t \le T$, the function $x_0(t)$ is defined by expression (78), while the functions $\xi_0(t)$ and $y_0(t)$ are defined by expressions (35) and (44), respectively. To determine the functions $x_0(t)$ and $\xi_0(t)$ for $t > T$, one should solve the system of differential equations

$$\ddot{x} = \overline{u}(t), \quad c\dot{\xi} + k\xi = -\overline{u}(t) \tag{81}$$

subject to the initial conditions

$$x(T) = \frac{V^2}{2U}, \quad \dot{x}(T) = 0, \quad \xi(T) = \frac{U}{k}\left[1 - \exp\left(-\frac{k}{c}T\right)\right], \tag{82}$$

where the control $\overline{u}(t)$ is defined by expressions (55) and (56). Equations (81) are those of (13) and (14) specified for $t \ge T$. This initial value problem can be solved analytically in closed form. The details of the solution are omitted here.

Figure 4 plots the time history of the regular part ($G_0(t)$) of the optimal control $F_0(t)$, defined by relation (72), for Problem 1. The minimal value of the criterion (43) is

$$J_1(u_0) = 12.1\,\mathrm{cm}. \tag{83}$$

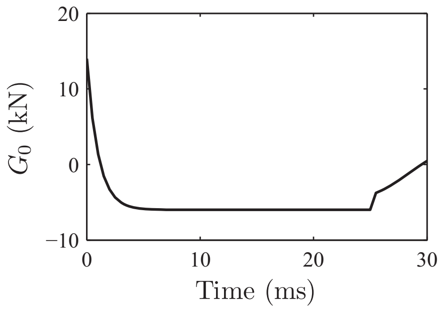

Figure 6: Regular part of the optimal control force for Problem 1

Optimal control (72) involves two impulse components applied at the instants $t = 0$ and $t = T$. The first impulse has a magnitude of $20\,\mathrm{kg}\cdot\mathrm{m/s}$ and instantaneously reduces the velocity of body 2 by $4\,\mathrm{m/s}$. The second impulse has an order of magnitude of $10^{-9}\,\mathrm{kg}\cdot\mathrm{m/s}$ and can be ignored.

7 Conclusions

. The limiting performance analysis of shock isolation for a two-component viscoelastic object demonstrates qualitative distinctive features of the optimal control that are not developed for a rigid object. The control force for a two-component object involves impulse components, whereas the optimal control for a rigid object does not have such components. Apparently, impulse components will occur in the optimal control of a shock isolation system for a multibody object the bodies of which are connected by viscoelastic elements. If a control law optimal for a rigid model is applied to a viscoelastic object, the displacement of the object relative to the base will change insignificantly, if the stiffness of the object is high enough. However, the maximum magnitude of the force acting between the constituent bodies of the object may turn out to be unacceptably large because of relative vibrations of these bodies. The impulse components do not allow these vibrations to be excited.

8 References

[1] Sevin E. *Min-max solutions for the linear mass-spring system*. J. Applied Mechanics, **24** (1957), 131–136.

[2] Sevin E, *On the design of shock isolated floor systems*. Shock and Vibration Bulletin (1960), 22-35.

[3] Guretskii V.V.: *On one optimal control problem (in Rissian)*. Izvestiya AN SSSR. Mekhanika [Proceedings of the USSR Academy of Sciences. Mechanics], No 1 (1965), 159–162.

[4] Sevin E. and Pilkey W.D.: *Optimum Shock and Vibration Isolation*. Government Print Office, Washington D.C., 1971.

[5] Kolovskii M.Z.: *Automatic Control of Vibration Isolation Systems (in Russian)*. Nauka, Moscow, 1976.

[6] Bolotnik N.N.: *Optimization of Shock and Vibration Isolation Systems (in Russian)*. Nauka, Moscow, 1983.

[7] Balandin D.V., Bolotnik N.N., and Pilkey W.D.: *Optimal Protection from Impact, Shock, and Vibration*. Gordon and Breach, Amsterdam, 2001.

[8] Stech E.L. and Payne P.R.: *Dynamic models of the human body*. Aerospace Medical Research Laboratory Report AMRL-TR-66-157, Wright-Patterson Air Force Base, Ohio, 1969.

[9] Brinkley J.W. and Shaffer J.T.: *Dynamic simulation techniques for the design of escape systems: Current applications and future Air Force requirements*. Aerospace Medical Research Laboratory Symposium on Biodynamic Models and Their Applications, AMRL-TR-71-29, Wright-Patterson Air Force Base, Ohio, 1970.

[10] Lobdell T.E., Kroell C.K., Schneider D.C., Hering W.E., and Nahum A.M.: *Impact response of the human thorax*. In: Human Impact Response Measurement and Simulation, (Eds.: King, W. F. and , Mertz, H.J.), Plenum Press, New York-London, 1973, 201–245.

[11] Desjardins S. P.: *The evolution of energy absorption systems for crashworthy helicopter seats*. Presented at the American Helicopter Society 59th Annual Forum, Phoenix, Arizona, May 6–8, 2003.

[12] Balandin D.V. and Bolotnik N.N.: *Limiting performance analysis of shock isolation for a two-degree-of-freedom system*. Mechanics of Solids, **36**(6) (2001), 43–51.

[13] Kuppa S.M., Klopp G.S., Crandall J.R., Hall G., Yoganamdan N., Pintar F.A., Eppinger R.H., Sun E., Khaewpong N., and Kleinberger M.: *Axial impact characteristics of dummy and cadaver lower limbs*. SAE paper 98-S7-O-10 (1998), 1608–1614.