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MIXED CONFORMING ELEMENTS FOR THE LARGE-BODY LIMIT IN MICROMAGNETICS

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Abstract. The macroscopic behavior of stationary micromagnetic phenomena can be modeled by a relaxed version of the Landau-Lifshitz minimization problem first introduced by DESIMONE in 1993, [5], where the magnetic potential u is linked to the magnetization $\mathbf{m}$ through the magnetostatic Maxwell equation. In the case of large and soft magnets Ω, one neglects the exchange energy and convexifies the remaining energy densities. In our discretization we enforce the pointwise side constraint $|\mathbf{m}| \leq 1$ by a penalization strategy and we replace the entire space $\mathbb{R}^d$ in the energy functional E and in the potential equation by a bounded Lipschitz domain $\widehat{\Omega}$ containing Ω. Since conforming elements appear to be unstable for the pure Galerkin discretization (cf. [3]) we append to the Galerkin discretization a consistent stabilization term. We reformulate the minimization problem in terms of the augmented Lagrangean, which leads us to a so called saddle-point formulation for the corresponding Euler-Lagrange equations.

In this paper we discuss the well-posedness of the discrete problem and the *a priori* error analysis. Furthermore we introduce a residual-based *a posteriori* error estimator and comment on adaptive strategies.

1 Introduction

In rigid ferromagnetic bodies which cover a domain $\Omega \subset \mathbb{R}^d$, $d = 2,3$, the mathematical description of magnetization states $\mathbf{m} : \Omega \to \mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ goes back to the classical model by Landau and Lifshitz [2], where the magnetization solves the minimizing problem (MP_α)

$$\min_{|\mathbf{m}|=1 \text{ a.e.}} E(\mathbf{m}) \tag{1}$$

of the energy functional

$$E(\mathbf{m}) := \alpha \int_\Omega |\nabla \mathbf{m}|^2 + \int_\Omega \phi(\mathbf{m}) - \int_\Omega \mathbf{f} \cdot \mathbf{m} + \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2. \tag{2}$$

Here, the magnetic potential $u : \mathbb{R}^d \to \mathbb{R}$ is related to $\mathbf{m}$ by the magnetostatic Maxwell equation

$$\operatorname{div}(\nabla u - \chi_\Omega \mathbf{m}) = 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^d). \tag{3}$$

The four terms in (2) favor different properties of the minimizer. The first term favors non-oscillating structures, where $\alpha > 0$ denotes a (typically very small) exchange parameter (length scale). The non-convex anisotropy $\phi \in \mathscr{C}^\infty(\mathbb{R}^d, \mathbb{R}_{\geq 0})$ in the second term models crystallographic properties of the ferromagnet. The third term with the applied exterior field $\mathbf{f} \in L^2(\Omega, \mathbb{R}^d)$ favors magnetizations aligned with $\mathbf{f}$ and the last term is a measure of $\operatorname{div} \mathbf{m}$.

In applications, one often considers “macroscopic” bodies, where $\operatorname{diam}(\Omega)$ is large compared with the length scale α. On the one hand, features on small scales cannot be resolved numerically, but on the other hand many quantities of interest do not depend on these small scale features. This fact justifies eliminating the first term in the energy functional (2) by setting $\alpha = 0$. By doing this, we end up by the so called *large-body limit* of micromagnetics with the goal to find minimizers $\mathbf{m} : \Omega \to \mathbb{S}^{d-1}$ of the problem (MP_0):

$$\min_{|\mathbf{m}|=1 \text{ a.e.}} E^{\ell b}(\mathbf{m}) \tag{4}$$

with

$$E^{\ell b}(\mathbf{m}) := \int_\Omega \phi(\mathbf{m}) - \int_\Omega \mathbf{f} \cdot \mathbf{m} + \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2. \tag{5}$$

under the side constraint (3).

2 Relaxation

Because of the non-convex side constraint given by the pointwise length condition

$$|\mathbf{m}(x)| = 1 \quad \text{for almost every } x \in \Omega$$

and the non-convex anisotropy density ϕ, (MP_0) turns out to be a non-convex minimization problem. Then the energy functional $E^{\ell b}$ may not have minimizers but only infimizing sequences [7]. To overcome this difficulty, one strategy is to relax the problem (MP_0) by convexification [5]. We consider the relaxed energy minimization (RMP): Find minimizer(s) $\mathbf{m} : \Omega \to \mathbb{B}^d := \{x \in \mathbb{R}^d : |x| \leq 1\}$ of

$$\min_{|\mathbf{m}|\leq 1 \text{ a.e.}} E^{**}(\mathbf{m}) \tag{6}$$

with

$$E^{**}(\mathbf{m}) := \int_\Omega \phi^{**}(\mathbf{m}) - \int_\Omega \mathbf{f}\cdot\mathbf{m} + \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 \tag{7}$$

under the side constraint (3).

Here, ϕ^{**} denotes the lower convex envelope of ϕ defined by

$$\phi^{**}(x) := \sup\{\varphi(x) | \varphi : \mathbb{R}^d \to \mathbb{R} \text{ convex and } \varphi|_{\mathbb{S}^{d-1}} \leq \phi\} \quad \text{for } |x| \leq 1,$$

so that $E^{**}(\mathbf{m})$ turns out to be a convex energy functional. Following [3], we replace the entire space $\mathbb{R}^d$ in (3) and (7) by a bounded Lipschitz domain $\widehat{\Omega}$ containing Ω. Then the integral over $\mathbb{R}^d$ in (7) is replaced by an integral over $\widehat{\Omega}$ and the full - space equation (3) simplifies to a PDE on the finite domain $\widehat{\Omega}$. In other words, (RMP) is replaced with (RMP_{bd}): Find minimizer(s) $\mathbf{m} : \Omega \to \mathbb{B}^d$ of

$$\min_{|\mathbf{m}|\leq 1 \text{ a.e.}} E^{**}_{bd}(\mathbf{m}) \tag{8}$$

with

$$E^{**}_{bd}(\mathbf{m}) := \int_\Omega \phi^{**}(\mathbf{m}) - \int_\Omega \mathbf{f}\cdot\mathbf{m} + \frac{1}{2}\int_{\widehat{\Omega}} |\nabla u|^2 \tag{9}$$

under the side constraint

$$\operatorname{div}(\nabla u - \chi_\Omega \mathbf{m}) = 0 \quad \text{in } H^{-1}(\widehat{\Omega}), \tag{10}$$

where $H^{-1}(\widehat{\Omega})$ denotes the dual space of $H^1_0(\widehat{\Omega})$. We refer to [5] for a justification of the following facts concerning solvability of (RMP):

- E^{**} is convex and weakly lower semicontinuous. Therefore, the direct method in the calculus of variations provides the existence of solutions.
- $\min\{E^{**}(\mathbf{m}) | \mathbf{m} \in L^2(\Omega, \mathbb{B}^d)\} = \inf\{E^{\ell b}(\mathbf{m}) | \mathbf{m} \in L^2(\Omega, \mathbb{S}^{d-1})\}$
- For each solution $\mathbf{m} \in L^2(\Omega, \mathbb{B}^d)$ of (RMP), there exists an infimizing sequence $(\mathbf{m}_k)_{k\in\mathbf{N}}$ of $E^{\ell b}$, which converges weakly to $\mathbf{m}$ in $L^2(\Omega, \mathbb{B}^d)$.
- For each infimizing sequence $(\mathbf{m}_k)_{k\in\mathbf{N}}$ of $E^{\ell b}$, there exists a weakly convergent subsequence $(\mathbf{m}_{k_\ell})$ whose weak limit solves (RMP).
- The minimizers of E^{**} contain physically relevant informations.

Remark: Mutatis mutandis, the above 5 statements also hold for (RMP_{bd}).

Furthermore, the energy functional E^{**}_{bd} is Gâteaux differentiable as a function of $(u, \mathbf{m})$, so the corresponding Euler-Lagrange equations can be formulated. Introducing a further Lagrange multiplier λ for the side constraint $|\mathbf{m}| \leq 1$, we get the problem (RP_{bd}): Find $u \in H^1_0(\widehat{\Omega})$, $\mathbf{m} \in L^2(\Omega, \mathbb{R}^d)$, and $\lambda \in L^2(\Omega)$ such that

$$\int_{\widehat{\Omega}} \nabla u \cdot \nabla v + \int_\Omega D\phi^{**}(\mathbf{m})\cdot\mathbf{n} + \int_\Omega \lambda\,\mathbf{m}\cdot\mathbf{n} = \int_\Omega \mathbf{f}\cdot\mathbf{n} \quad \text{for all} \quad v \in H^1_0(\widehat{\Omega}),\ \mathbf{n} \in L^2(\Omega, \mathbb{R}^d), \tag{11}$$

$$\operatorname{div}(\nabla u - \chi_\Omega \mathbf{m}) = 0 \qquad \text{in} \quad H^{-1}(\widehat{\Omega}), \tag{12}$$

together with

$$0 \leq \lambda, \quad |\mathbf{m}| \leq 1, \quad \text{and} \quad \lambda(1 - |\mathbf{m}|)_+ = 0 \quad \text{almost everywhere in } \Omega. \tag{13}$$

Here, we denote by $(x)_+ := \max\{x, 0\}$ the non-negative part of x.

Example. Uniaxial materials such as cobalt can be modelled with the aid of the *uniaxial* anisotropy density

$$\phi : \mathbb{S}^{d-1} \to \mathbb{R}, \quad \phi(\mathbf{x}) = \frac{1}{2}(1-(\mathbf{e}\cdot\mathbf{x})^2) \quad \text{for all } |x| = 1,$$

where $\mathbf{e} \in \mathbb{R}^d$ is a given fixed unit vector, called *easy axis*. ϕ favors magnetizations $\mathbf{m}$ aligned with $\mathbf{e}$. In this case, the lower convex envelope can easily be computed and reads, e.g. for the case $d = 2$,

$$\phi^{**}(\mathbf{x}) = \frac{1}{2}\begin{cases} (\mathbf{x}\cdot\mathbf{z})^2 & \text{if } |\mathbf{x}| \le 1 \\ \infty & \text{else,} \end{cases}$$

where $(\mathbf{e},\mathbf{z})$ is an orthonormal basis of $\mathbb{R}^2$, c.f. [8, 5].

We refer to [5] and [3] for the following statements about solvability of (RMP_{bd}) and (RP_{bd}) :

- Any solution of (RMP_{bd}) solves (RP_{bd}) and vice versa.
- Problem (RMP_{bd}) has solutions.
- In the uniaxial case (c.f. the example above), there exists only one solution of problems (RMP_{bd}) and (RP_{bd}), c.f. [3] for the two-dimensional case and [8] for the three dimensional one.

Following the idea of Yosida regularization, we enforce the side constraint $|\mathbf{m}| \le 1$ by a penalization procedure. The resulting minimization problem $(RMP_{bd,\varepsilon})$ now reads as follows: Given $\varepsilon > 0$, find minimizer $(u,\mathbf{m}) \in X := H_0^1(\widehat{\Omega}) \times L^2(\Omega)$ of

$$E^{**}_{bd,\varepsilon}(u,\mathbf{m}) := \int_\Omega \phi^{**}(\mathbf{m}) - \int_\Omega \mathbf{f}\cdot\mathbf{m} + \frac{1}{2}\int_{\widehat{\Omega}} |\nabla u|^2 + \frac{1}{2\varepsilon}\int_\Omega (|\mathbf{m}|-1)_+^2 \tag{14}$$

under the side constraint

$$\operatorname{div}(\nabla u - \chi_\Omega \mathbf{m}) = 0 \quad \text{in } H^{-1}(\widehat{\Omega}). \tag{15}$$

Later on, the penalization parameter ε will be related to the local mesh-size in the discrete version of (14). We mention that $E^{**}_{bd,\varepsilon}$ is convex, continuous and coercive but now on the entire space X. In particular, $(RMP_{bd,\varepsilon})$ has solutions, and can equivalently be stated by the corresponding Euler-Lagrange equations $(RP_{bd,\varepsilon})$, see e.g. [4]. We omit the details.

Saddle-point formulation. Now, our approach for solving the energy minimizing problem is to formulate (14) and (15) in terms of the augmented Lagrangean. We define

$$\mathscr{L}(u,\mathbf{m};p) := E^{**}(u,\mathbf{m}) + b(u,\mathbf{m};p) \quad \text{for} \quad (u,\mathbf{m}) \in X := H_0^1(\widehat{\Omega}) \times L^2(\Omega) \quad \text{and} \quad p \in H_0^1(\widehat{\Omega}), \tag{16}$$

where

$$b(u,\mathbf{m};p) := \int_{\widehat{\Omega}} (\nabla u - \chi_\Omega \mathbf{m})\cdot\nabla p = \langle \operatorname{div}(\nabla u - \chi_\Omega \mathbf{m}), p\rangle_{H^{-1}(\widehat{\Omega})\times H_0^1(\Omega)}$$

denotes the usual weak formulation of (15). The saddle-point formulation $(SPP_{bd,\varepsilon})$ of (14) and (15) is

$$\frac{\partial \mathscr{L}}{\partial(u,\mathbf{m})} = 0, \quad \frac{\partial \mathscr{L}}{\partial p} = 0 \tag{17}$$

and reads in detail: Find $(u,\mathbf{m}) \in X$ and $p \in H_0^1(\widehat{\Omega})$ such that

$$a(u,\mathbf{m};v,\mathbf{n}) + b(v,\mathbf{n};p) = \int_\Omega \mathbf{f}\cdot\mathbf{n} \quad \text{for all} \quad (v,\mathbf{n}) \in X, \tag{18}$$

$$b(u,\mathbf{m};q) = 0 \quad \text{for all} \quad q \in H_0^1(\widehat{\Omega}), \tag{19}$$

where (for the uniaxial case and $d = 2$)

$$a(u,\mathbf{m};v,\mathbf{n}) := \int_{\widehat{\Omega}} \nabla u\cdot\nabla v + \int_\Omega (\mathbf{m}\cdot\mathbf{z})(\mathbf{n}\cdot\mathbf{z}) + \int_\Omega \lambda_\varepsilon \mathbf{m}\cdot\mathbf{n}, \tag{20}$$

$$\lambda_\varepsilon := \frac{1}{\varepsilon}\frac{(|\mathbf{m}|-1)_+}{|\mathbf{m}|}, \tag{21}$$

$$b(u,\mathbf{m};q) := \int_{\widehat{\Omega}} (\nabla u - \chi_\Omega \mathbf{m})\cdot\nabla q. \tag{22}$$

It is easy to show that every solution of $(SPP_{bd,\varepsilon})$ solves $(RP_{bd,\varepsilon})$. To see that $(SPP_{bd,\varepsilon})$ is, in fact, equivalent to $(RP_{bd,\varepsilon})$, let $(u,\mathbf{m})$ be a solution of $(RP_{bd,\varepsilon})$. Existence and uniqueness of the Lagrange multiplier p follows from the fact that the bilinear form b in (22) satiesfies an inf-sup or LBB condition. Indeed, for arbitrary $q \in H_0^1(\widehat{\Omega}) \setminus \{0\}$ we obtain

$$\sup_{(u,\mathbf{m}) \in X\setminus\{0\}} \frac{|b(u,\mathbf{m};q)|}{\|(u,\mathbf{m})\|_X \, \|\nabla q\|_{L^2(\widehat{\Omega})}} \geq \frac{|b(q,0;q)|}{\|(q,0)\|_X \, \|\nabla q\|_{L^2(\widehat{\Omega})}} = 1,$$

hence

$$\inf_{q \in H_0^1(\widehat{\Omega})\setminus\{0\}} \sup_{(u,\mathbf{m}) \in X\setminus\{0\}} \frac{|b(u,\mathbf{m};q)|}{\|(u,\mathbf{m})\|_X \, \|\nabla q\|_{L^2(\widehat{\Omega})}} = 1 > 0,$$

where we define $\|(u,\mathbf{m})\|_X := \left(\|\nabla u\|^2_{L^2(\widehat{\Omega})} + \|\mathbf{m}\|^2_{L^2(\Omega,\mathbb{R}^d)} \right)^{1/2}$. The triple $(u,\mathbf{m},p)$ then solves $(SPP_{bd,\varepsilon})$. For uniqueness of $(u,\mathbf{m})$ in the uniaxial case, we refer to the discussion of (RMP_{bd}) and (RP_{bd}) above. We want to mention that the main difficulties in the uniqueness proof arise from the fact that (20) provides only partial control over $\mathbf{m}$, c.f. [3, 8].

3 Discretization

Let $\widehat{\mathscr{T}} := \{T_1, \dots, T_M\}$ denote a regular triangulation of $\widehat{\Omega}$ and $\mathscr{T} := \widehat{\mathscr{T}}|_\Omega := \{K \in \mathscr{T} | K \subset \overline{\Omega}\}$ a subtriangulation exactly covering Ω. Let

- $S_0^1(\widehat{\mathscr{T}}) = \{u \in H_0^1(\widehat{\Omega}) : u|_K \in \mathscr{P}_1, \ \forall K \in \widehat{\mathscr{T}}\}$ be the space of all $\widehat{\mathscr{T}}$-piecewise affine, globally continuous and
- $S^0(\mathscr{T}) = \{\mathbf{m} \in L^2(\Omega) : \mathbf{m}|_K \in \mathscr{P}_0, \ \forall K \in \mathscr{T}\}$ be the space of all $\mathscr{T}$-piecewise constant functions.

In addition we use the abbreviation $X_h := S_0^1(\widehat{\mathscr{T}}) \times S^0(\mathscr{T})$. We are now in the position to formulate the discrete saddle-point problem $(SPP^h_{bd,\varepsilon})$: Find $(u_h,\mathbf{m}_h) \in X_h$ and $p_h \in S_0^1(\widehat{\mathscr{T}})$ such that

$$a_h(u_h,\mathbf{m}_h;v,\mathbf{n}) + b_h(v,\mathbf{n};p_h) = \int_\Omega \mathbf{f}\cdot\mathbf{n} \quad \text{for all } (v,\mathbf{n}) \in X_h, \tag{23}$$

$$b_h(u_h,\mathbf{m}_h;q) = 0 \quad \text{for all } q \in S_0^1(\widehat{\mathscr{T}}), \tag{24}$$

where a_h and b_h are the restrictions of a and b to $X_h \times X_h$ and $X_h \times S_0^1(\widehat{\mathscr{T}})$, respectively.

Now, we use the fact that the according discrete formulations $(RMP^h_{bd,\varepsilon})$ and $(RP^h_{bd,\varepsilon})$ are equivalent and provide a solution $(u_h,\mathbf{m}_h)$ in the uniaxial case. Following the same ideas as in the continuous case, we can show the existence of a solution $(u_h,\mathbf{m}_h,p_h)$ of the saddle-point formulation $(SPP^h_{bd,\varepsilon})$. Furthermore, the Lagrange parameter p_h is unique. Uniqueness of $(u_h,\mathbf{m}_h)$ does not follow, since $\mathscr{N}(b_h) \not\subset \mathscr{N}(b)$, where we define the kernels

$$\begin{aligned} \mathscr{N}(b) &:= \{(u,\mathbf{m}) \in X \mid b(u,\mathbf{m};q) = 0 \quad \text{for all} \quad q \in H_0^1(\widehat{\Omega})\} \\ \mathscr{N}(b_h) &:= \{(u_h,\mathbf{m}_h) \in X \mid b(u_h,\mathbf{m}_h;q) = 0 \quad \text{for all} \quad q \in S_0^1(\widehat{\mathscr{T}})\}. \end{aligned}$$

To ensure uniqueness of a solution $(u_h,\mathbf{m}_h)$, we add a stabilizing bilinear form $\langle .,. \rangle_\sigma$ to the augmented Langrangean for the discrete problem

$$\mathscr{L}^{aug}(u,\mathbf{m};p) := E^{**}_{bd,\varepsilon}(u,\mathbf{m}) + b(u,\mathbf{m};p) + \frac{1}{2}\langle (u,\mathbf{m}),(u,\mathbf{m})\rangle_\sigma$$

and stipulate the following three properties:

$$\left.\begin{array}{l} \bullet \ \text{non negativity: } \langle (u,\mathbf{m}),(u,\mathbf{m})\rangle_\sigma \geq 0 \\ \bullet \ \text{consistency: } \langle (u,\mathbf{m}),.\rangle_\sigma = 0 \text{ for the exact solution } (u,\mathbf{m}) \\ \bullet \ \text{for all } (u,\mathbf{m}) \in \mathscr{N}(b_h) \text{ there holds } \|\mathrm{div}(\nabla u - \chi_\Omega \mathbf{m})\|^2_{H^{-1}(\widehat{\Omega})} \lesssim \langle (u,\mathbf{m}),(u,\mathbf{m})\rangle_\sigma + \|\nabla u\|^2_{L^2(\widehat{\Omega})} \end{array}\right\} \tag{25}$$

Remark: The last property will ensure uniqueness of the discrete problem (see Theorem 1 ahead).
We now formulate the stabilized saddle-point problem $(SPP^{h,\sigma}_{bd,\varepsilon})$: Find $u_h \in S_0^1(\widehat{\mathscr{T}})$, $\mathbf{m}_h \in S^0(\mathscr{T})$, $p_h \in S_0^1(\widehat{\mathscr{T}})$ such that

$$a_\sigma(u_h,\mathbf{m}_h;v,\mathbf{n}) + b_h(v,\mathbf{n};p_h) = \int_\Omega \mathbf{f}\cdot\mathbf{n} \quad \text{for all } (v,\mathbf{n}) \in X_h, \tag{26}$$

$$b_h(u_h,\mathbf{m}_h;q) = 0 \quad \text{for all } q \in S_0^1(\widehat{\mathscr{T}}), \tag{27}$$

where

$$a_\sigma(u_h,\mathbf{m}_h;v,\mathbf{n}) = a(u_h,\mathbf{m}_h;v,\mathbf{n}) + \langle (u_h,\mathbf{m}_h),(v,\mathbf{n})\rangle_\sigma .$$

Theorem 1 (unique solvability)
For the uniaxial case the discrete problem $(SPP^{h,\sigma}_{bd,\varepsilon})$ *is uniquely solvable, if* $\langle .,. \rangle_\sigma$ *satisfies the above properties* (25)*.*

Proof: Let $(u_{h,i}, \mathbf{m}_{h,i}, p_{h,i})$, for $i = 1,2$ be the solutions of $(SPP^{h,\sigma}_{bd,\varepsilon})$. We use the abbreviations $e := u_{h,2} - u_{h,1}$, $\delta := \mathbf{m}_{h,2} - \mathbf{m}_{h,1}$, and $\eta := p_{h,2} - p_{h,1}$. From (27), we obtain

$$\langle \nabla e - \boldsymbol{\delta}, \nabla q \rangle_{L^2(\widehat{\Omega})} = 0 \quad \text{for all } q \in S^1_0(\widehat{\mathscr{T}}),$$

and hence $(e, \boldsymbol{\delta}) \in \mathscr{N}(b_h)$. Equation (26) shows

$$\begin{aligned}&\langle \nabla e, \nabla v \rangle_{L^2(\widehat{\Omega})} + \langle (\boldsymbol{\delta} \cdot \mathbf{z})\mathbf{z}, \mathbf{n} \rangle_{L^2(\Omega)} + \left\langle \lambda_\varepsilon(\mathbf{m}_{h,2})\mathbf{m}_{h,2} - \lambda_\varepsilon(\mathbf{m}_{h,1})\mathbf{m}_{h,1}, \mathbf{n} \right\rangle_{L^2(\Omega)} \\ &+ \langle \nabla v - \mathbf{n}, \nabla \eta \rangle_{L^2(\widehat{\Omega})} + \langle (e, \boldsymbol{\delta}), (v, \mathbf{n}) \rangle_\sigma = 0 \quad \text{for all } (v, \mathbf{n}) \in X_h.\end{aligned}$$

For $v := e$ and $\mathbf{n} := \boldsymbol{\delta}$, this yields

$$\|\nabla e\|^2_{L^2(\widehat{\Omega})} + \|(\boldsymbol{\delta} \cdot \mathbf{z})\mathbf{z}\|^2_{L^2(\Omega)} + \left\langle \lambda_\varepsilon(\mathbf{m}_{h,2})\mathbf{m}_{h,2} - \lambda_\varepsilon(\mathbf{m}_{h,1})\mathbf{m}_{h,1}, \boldsymbol{\delta} \right\rangle_{L^2(\Omega)} + \langle (e, \boldsymbol{\delta}), (e, \boldsymbol{\delta}) \rangle_\sigma = 0.$$

In [3], it is shown that

$$\left\langle \lambda_\varepsilon(\mathbf{m}_{h,2})\mathbf{m}_{h,2} - \lambda_\varepsilon(\mathbf{m}_{h,1})\mathbf{m}_{h,1}, \boldsymbol{\delta} \right\rangle_{L^2(\Omega)} \geq 0.$$

Non negativity of the bilinear form $\langle .,. \rangle_\sigma$ now leads to

$$e = 0 \quad \text{and} \quad \boldsymbol{\delta} \cdot \mathbf{z} = 0 \quad \text{almost everywhere.}$$

Furthermore,

$$\begin{aligned}\|\nabla \cdot (\nabla e - \boldsymbol{\delta})\|_{H^{-1}(\widehat{\Omega})} &= \sup_{q \in H^1_0(\widehat{\Omega}) \setminus \{0\}} \frac{\langle \nabla \cdot (\nabla e - \boldsymbol{\delta}), q \rangle}{\|\nabla q\|_{L^2(\widehat{\Omega})}} = \sup_{q \in H^1_0(\widehat{\Omega}) \setminus \{0\}} \frac{\langle \nabla e - \boldsymbol{\delta}, \nabla q \rangle_{L^2(\widehat{\Omega})}}{\|\nabla q\|_{L^2(\widehat{\Omega})}} \\ &\lesssim \langle (e, \boldsymbol{\delta}), (e, \boldsymbol{\delta}) \rangle_\sigma + \|\nabla e\|^2_{L^2(\widehat{\Omega})} = 0\end{aligned}$$

shows $(e, \boldsymbol{\delta}) \in \mathscr{N}(b)$. In particular, we get together with $e = 0$

$$\operatorname{div} \boldsymbol{\delta} = 0 \quad \text{in } H^{-1}(\widehat{\Omega}). \tag{28}$$

Uniqueness of $\mathbf{m}$ in the two-dimensional case now follows in the same way as in [3] by use of the 2D Helmholtz decomposition.. Finally, the discrete inf-sup condition of the bilinear form b_h ensures uniqueness of the Lagrange multiplier p.

□

Lemma 2 (stabilizing bilinear form) *Let* $\overline{\mathscr{E}}$ *be the edges of* $\Omega \cup \partial\Omega$*. Define*

$$\langle (u, \mathbf{m}), (v, \mathbf{n}) \rangle_\sigma := \sum_{\overline{E} \in \mathscr{E}} h_E \int_E [(\nabla u - \mathbf{m}) \cdot \boldsymbol{\nu}][(\nabla v - \mathbf{n}) \cdot \boldsymbol{\nu}], \tag{29}$$

where $[\cdot]$ *denotes the jump over an edge and* $\boldsymbol{\nu}$ *the according normal vector. Then the bilinear form* $\langle \cdot, \cdot \rangle_\sigma$ *has the properties* (25)*.*

Proof: The first two statements are clear. The third statement follows by integration by parts and the applications of standard finite element techniques.

□

Remark: Furthermore (28) shows $[\boldsymbol{\delta} \cdot \boldsymbol{\nu}] = 0$ for all edges in $\overline{\mathscr{E}}$ and this implies $\boldsymbol{\delta} \in H(\operatorname{div}, \mathbb{R}^d)$. Therefore, we have $\operatorname{div} \boldsymbol{\delta} = 0$ in $L^2(\mathbb{R}^d)$ and $\boldsymbol{\delta}$ has compact support. This observation enables us to prove uniqueness of $\mathbf{m}_h$ with smoothing techniques as were first used in [8]. For the uniaxial case this allows one to prove uniqueness of $(SPP^{h,\sigma}_{bd,\varepsilon})$ for 2D and 3D problems, whereas the proof along the lines of [3] is restricted to 2D.

4 A priori error estimate

Theorem 3 (optimal convergence, 2007) *Under the regularity assumption* $(u, \mathbf{m}) \in H^2_0(\widehat{\Omega}) \times H^1(\Omega)$*, there holds the a priori estimate*

$$\|\nabla(u - u_h)\|_{L^2(\widehat{\Omega})} + \|(\mathbf{m} - \mathbf{m}_h) \cdot \mathbf{z}\|_{L^2(\Omega)} + \|\nabla(p - p_h)\|_{L^2(\widehat{\Omega})} = O(h + \varepsilon). \tag{30}$$

Proof: see [1] □

Note that estimate (30) is optimal with respect to the local mesh-size h and favors the choice of $\varepsilon = O(h^\alpha)$ for $\alpha = 1$. Numerical experiments in [1] reveal that the choice $\alpha \in (0,1)$ dominates the error in the sense that, for smooth exact solution $(u, \mathbf{m})$, one observes experimental convergence $O(h^\alpha)$. Empirically, the estimate (30) thus is even optimal with respect to ε and $\varepsilon = O(h)$ leads in this case to optimal convergence $O(h)$. Throughout the following experiments, we thus choose the $\mathscr{T}$-piecewise constant penalization function $\varepsilon = h$, where $h \in L^\infty(\Omega)$ is defined by $h|_K := \mathrm{diam}(K)$.

Numerical experiment 1. In the first numerical experiment we choose a constant exterior field $f = [0.6, 0]$ parallel to the easy axis $\mathbf{e} = [1,0]$. Therefore we have $\mathbf{z} = [0,1]$. Furthermore we choose the magnetic rod $\Omega = (-0.5, 0.5) \times (-2.5, 2.5)$ and the surrounding area $\widehat{\Omega} = (-5.5, 5.5)^2$. This example is suggested in [3]. Fig 1 shows two plots of the computed magnetic potential u. On the left-hand-side one sees the magnetic rod together with some isolines of u, whereas on the right-hand-side a 3D version of u is plotted. As can be seen in Fig 2, the theoretical prediction for the convergence rates are verified. Although we do not control the error $\|(\mathbf{m} - \mathbf{m}_h) \cdot \mathbf{e}\|_{L^2(\Omega)}$, as can be seen in (30), we observe convergence $O(h)$ in the full norm $\|\mathbf{m} - \mathbf{m}_h\|_{L^2(\Omega)}$.

In the continuous case, the Lagrange multiplier p turns out to be exactly $-u$. This relation does not hold in the discrete case. Here, we only observe convergence of ∇p towards $-\nabla u$ with almost the same rate as for the error in ∇u and ∇p.

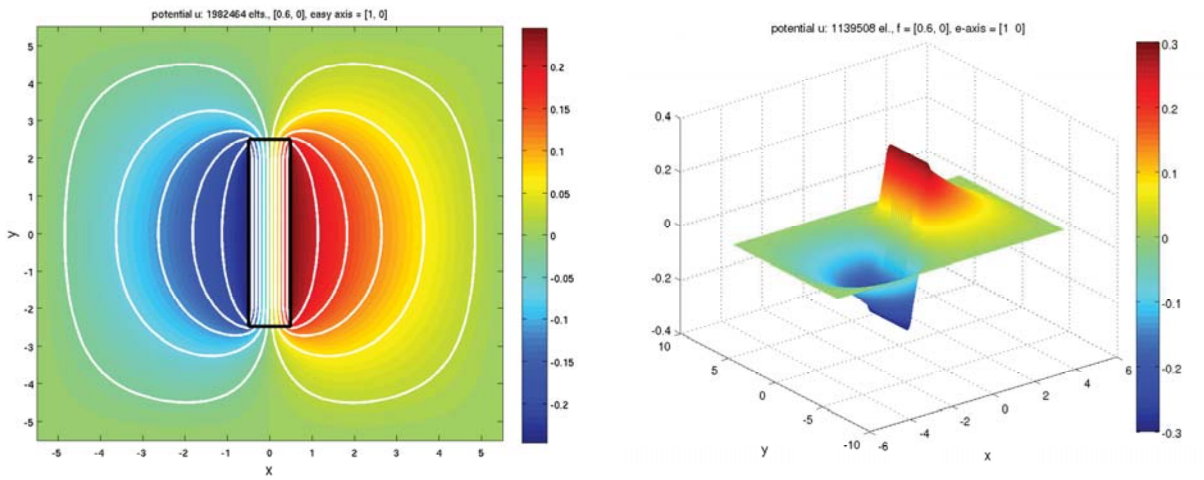

Figure 1: The magnetic potential u with some isolines (left)

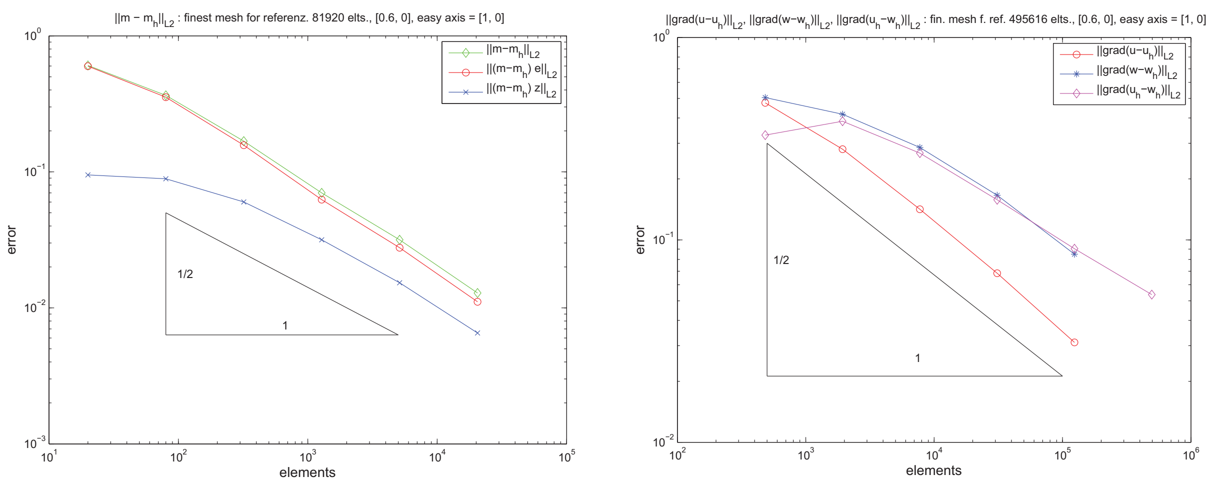

Figure 2: Error in $\mathbf{m}$ (left), ∇u and ∇p (right), where $w \equiv p$

5 A posteriori error estimate

For the next theorem we use the following notation:

- $\mathbf{e}_j$ denotes the j-th canonical unit vector
- K is a triangle of the finite element mesh with area |K|
- E an edge of $\mathscr{T}$ with length h_E
- $(\cdot)_\Pi$ denotes the L^2 - orthogonal projection onto $S^0(\mathscr{T})$

Furthermore we need a *Riesz* representative for the stabilizing bilinear form $\langle\cdot,\cdot\rangle_\sigma$. This is an element $(\widetilde{u}_h, \widetilde{\mathbf{m}}_h) \in X_h$ such that for fixed $(u_h, \mathbf{m}_h) \in X_h$

$$\langle(\widetilde{u}_h, \widetilde{\mathbf{m}}_h), (v_h, \mathbf{n}_h)\rangle_{X_h \times X_h} := \langle\nabla\widetilde{u}_h, \nabla v_h\rangle_{L^2(\widehat{\Omega})} + \langle\widetilde{\mathbf{m}}_h, \mathbf{n}_h\rangle_{L^2(\Omega)} = \langle(u_h, \mathbf{m}_h), (v_h, \mathbf{n}_h)\rangle_\sigma$$

holds, for all $(v_h, \mathbf{n}_h) \in X_h$, where $\langle\cdot,\cdot\rangle_{X_h \times X_h}$ denotes the scalar product in the Hilbert space X_h. For the a posteriori estimate (31), we only need $\widetilde{\mathbf{m}}_h$, which can be computed as

$$\widetilde{\mathbf{m}}_h \cdot \mathbf{e}_j|_K = \frac{1}{|K|} \sum_{E \subset \overline{K}} h_E \left\langle [(\nabla u_h - \mathbf{m}_h) \cdot \boldsymbol{\nu}], \mathbf{e}_j \cdot \boldsymbol{\nu} \right\rangle_{L^2(E)}.$$

After this preparation we formulate

Theorem 4 (residual based error estimator, 2008) *Let $(u, \mathbf{m}, p)$ be the exact solution of $(SPP_{bd,\varepsilon})$ and $(u_h, \mathbf{m}_h, p_h)$ be the computed solution of $(SPP^{h,\sigma}_{bd,\varepsilon})$. Then there holds*

$$\begin{aligned} &\|\nabla(u-u_h)\|^2_{L^2(\widehat{\Omega})} + \|(\mathbf{m}-\mathbf{m}_h)\cdot\mathbf{z}\|^2_{L^2(\Omega)} + \|\nabla(p-p_h)\|^2_{L^2(\widehat{\Omega})} \\ &\quad \lesssim \|\nabla(u_h+p_h)\|^2_{L^2(\widehat{\Omega})} + \|\widetilde{\mathbf{m}}_h\|^2_{L^2(\widehat{\Omega})} + \|\widetilde{\mathbf{m}}_h\|_{L^1(\widehat{\Omega})} + \|\varepsilon\lambda_h\mathbf{m}_h\|^2_{L^2(\Omega)} + \sum_{E\subset\widehat{\Omega}} h_E \,\|[(\nabla u_h - \mathbf{m}_h)\cdot\nu]\|^2_{L^2(E)} \qquad (31) \\ &\quad + \sum_{E\subset\widehat{\Omega}} h_E \,\|[\nabla(u_h+p_h)\cdot\nu]\|^2_{L^2(E)} + \langle\varepsilon|\lambda_h\mathbf{m}_h|, |f - f_\Pi + \nabla(u_h+p_h) + \widetilde{\mathbf{m}}_h|\rangle_{L^2(\widehat{\Omega})} + \langle f - f_\Pi, \mathbf{m} - \mathbf{m}_\Pi\rangle_{L^2(\Omega)}. \end{aligned}$$

Proof: see [1] □

Adaptive Algorithm. We use a common adaptive mesh-refining algorithm of the type

`solve` → `estimate` → `mark` → refine

For error estimation, we use the reliable (upper) error bound from Theorem 4, which is written in the form

$$\eta^2 = \sum_{K\in\widehat{\mathscr{T}}} \eta_K^2$$

with local refinement indicators

$$\begin{aligned} \eta_K^2 := &\|\nabla(u_h+p_h)\|^2_{L^2(K)} + \|\widetilde{\mathbf{m}}_h\|^2_{L^2(K)} + \|\widetilde{\mathbf{m}}_h\|_{L^1(K)} + \|\varepsilon\lambda_h\mathbf{m}_h\|^2_{L^2(K)} \\ &+ \sum_{E\subset K} h_E \,\|[(\nabla u_h - \mathbf{m}_h)\cdot\nu]\|^2_{L^2(E)} + \sum_{E\subset K} h_E \,\|[\nabla(u_h+p_h)\cdot\nu]\|^2_{L^2(E)} \\ &+ \langle\varepsilon|\lambda_h\mathbf{m}_h|, |f - f_\Pi + \nabla(u_h+p_h) + \widetilde{\mathbf{m}}_h|\rangle_{L^2(K)} + \langle f - f_\Pi, \mathbf{m} - \mathbf{m}_\Pi\rangle_{L^2(K)}. \end{aligned}$$

For element marking, we use the strategy proposed by Dörfler, c.f. [6]. For given $\theta \in (0,1]$, we seek the minimal subset $\mathscr{M} \subset \widehat{\mathscr{T}}$ such that

$$\theta \sum_{K\in\widehat{\mathscr{T}}} \eta_K^2 \le \sum_{K\in\mathscr{M}} \eta_K^2,$$

where we use $\theta = 0.25$ in the numerical experiment below.

Numerical experiment 2. In the second numerical experiment we choose a constant exterior field $f = [0.5, 0.5]$. We choose the easy axis $\mathbf{e}$, the magnetic rod Ω, and the surrounding area $\widehat{\Omega}$ as in the first numerical experiment. Again we refer to [3].

In our numerical computation, we observe a strong mesh-refinement in $\widehat{\Omega} \setminus \Omega$ towards the four corners of Ω. This is probably due to the corner singularities of u and p with respect to the exterior domain induced by the four re-entrant corners, see left-hand-side of Fig 3. Moreover, we observe some mesh-refinement of Ω at the left lower and right upper corner of Ω. Although the exact solution in this experiment is unknown, this may indicate some singular behavior of $\mathbf{m}$, c.f. right-hand-side Fig 3. Altogether, the proposed adaptive algorithm seems to recover the optimal order of convergence $O(h)$ at least for $\|(\mathbf{m}-\mathbf{m}_h)\cdot\mathbf{z}\|_{L^2(\Omega)}$, which is not observed for uniform mesh-refinement, see Fig 4.

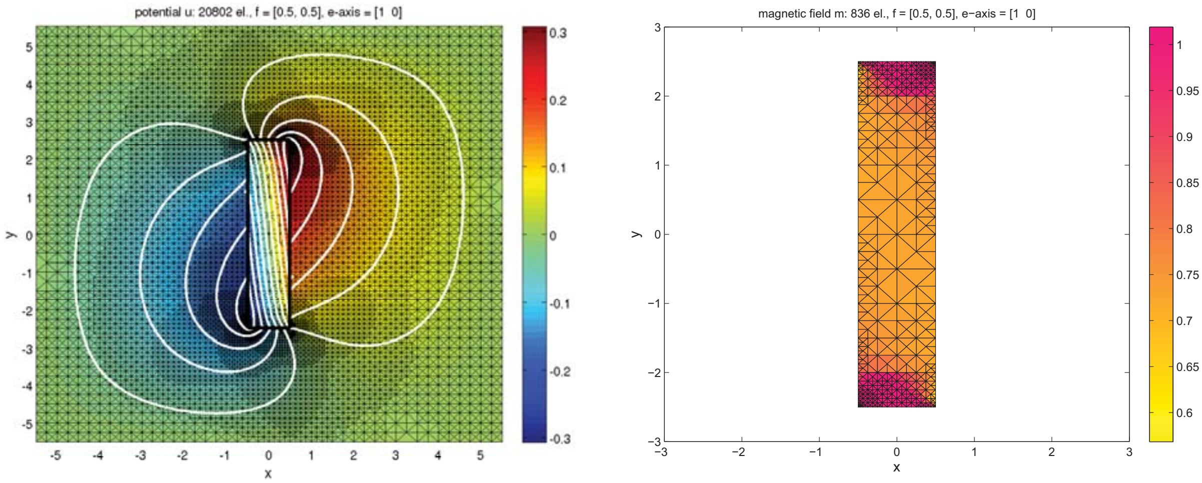

Figure 3: adaptively generated mesh for $\widehat{\Omega}$ and Ω

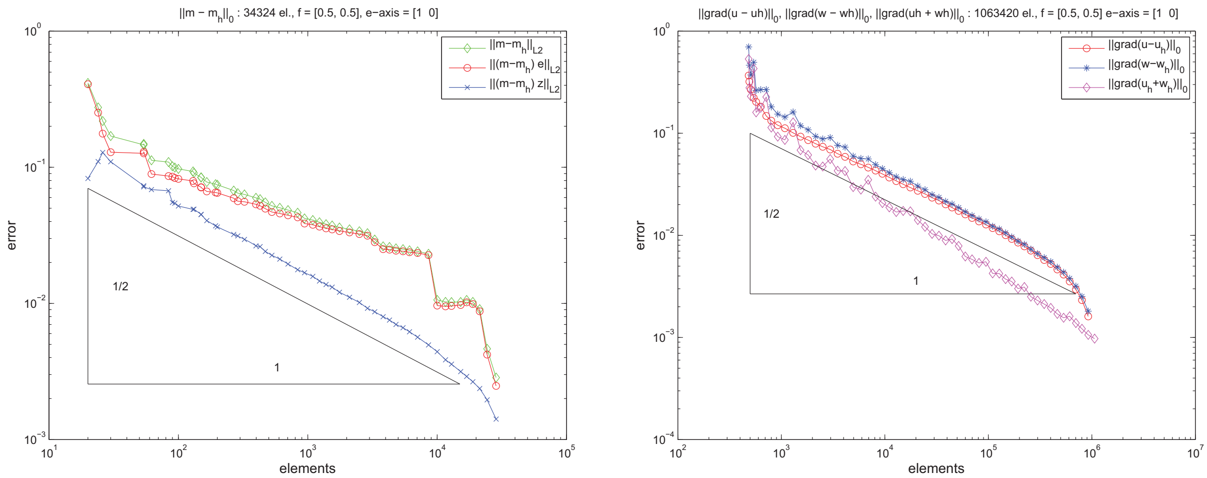

Figure 4: Error in **m** (left), ∇u and ∇p (right), where $w \equiv p$

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