Reduced Model in Thin-Film Micromagnetics

S. Ferraz-Leite, J. M. Melenk, D. Praetorius
Vienna University of Technology, Austria

Corresponding author: S. Ferraz-Leite, Vienna Univ. of Technology, Inst. f. Analysis and Scientific Computing
1040 Wien, Wiedner Hauptstraße 8-10, Austria, samuel.ferraz-leite@tuwien.ac.at

Abstract. The full minimization problem in micromagnetics due to Landau and Lifschitz is, from a numerical point of view, very complex. In [7] a reduced model in thin-film micromagnetics has been proposed and analyzed with focus on a distributional point of view. In contrast, we present a functional analytic framework which is more suitable for numerical analysis. Well-posedness of the model problem in this setting is studied and also certain uniqueness results are proven. A conforming discretization is proposed and some numerical examples are performed.

1 Introduction

The steady state of a magnetization $\mathbf{M}$ of a ferromagnetic sample is described by a minimization problem due to Landau and Lifschitz, which is nowadays accepted as the relevant model in micromagnetics. The bulk energy reads

$$E(\mathbf{M}) = d^2 \int_\Omega |\nabla \mathbf{M}|^2\,dx + Q \int_\Omega (\mathbf{M}_2^2 + \mathbf{M}_3^2)\,dx + \int_{\mathbb{R}^3} |\nabla U|^2\,dx - 2\int_\Omega \mathbf{F}\cdot\mathbf{M}\,dx. \tag{1}$$

The problem of micromagnetics is to find a local minimizer $\mathbf{M}^*$ of E which satisfies the non-convex constraint $|\mathbf{M}^*| = 1$. In this formulation, $\nabla\mathbf{M}$ denotes the Jacobian of $\mathbf{M}$, and $|\cdot|$ denotes the Euclidean norm of the matrix $\nabla\mathbf{M}$ and the vector ∇U, respectively. The constants $d, Q > 0$ are known material parameters, and $\mathbf{F}$ is a given applied exterior field.

The first energy term in (1) is referred to as the exchange energy and penalizes variations of $\mathbf{M}$ in space. Typically, the crystalline structure of a ferromagnetic material favors certain alignments of the magnetization field. For simplicity, we restrict ourselves to the case of uniaxial materials, where only one direction is energetically favored. This is accounted for in the second energy contribution, and we consider a uniaxial material with so-called easy axis parallel to the first standard unit vector e_1. The third term, called the magnetostatic energy, is the result of interactions between magnetic dipoles. It is determined through the magnetostatic Maxwell equation

$$\int_{\mathbb{R}^3} \nabla U\cdot\nabla V\,dx = \int_\Omega \mathbf{M}\cdot\nabla V\,dx, \quad \text{for all} \quad V\in\mathscr{D}(\mathbb{R}^3) := \mathscr{C}_c^\infty(\mathbb{R}^3). \tag{2}$$

The last energy contribution finally favors magnetizations, which are well aligned with the applied exterior field.

Geometry and scales. We will focus on the simulation of thin ferromagnetic films. We thus restrict our attention to a simple, yet interesting geometric set-up. We consider the sample to be cylindrical with basis ω and thickness t. Therefore Ω can be written in the form

$$\Omega = \omega \times (0,t).$$

We furthermore assume that the crystalline anisotropy favors the first in-plane axis. Since our interest is focused on thin ferromagnetic films, we assume

$$t \ll \ell,$$

where ℓ denotes the diameter of ω. Our model problem is therefore governed by four different scales, namely t, ℓ, d and $d/Q^{1/2}$, where the latter two scales stem from material properties. We stress, that these scales may typically vary by orders of magnitude.

Numerical challenges. The full micromagnetic problem (1) is, from a numerical point of view, quite complex. Besides the non-convex constraint $|\mathbf{M}| = 1$, the minimization problem (1) is also non-local since the magnetostatic potential U has to be computed and evaluated in the entire space $\mathbb{R}^3$. Additionally, the presence of various length scales, which have to be resolved, make the simulation very expensive from a computational point of view. For these reasons, various reduced models have been introduced and studied lately, which focus on certain phenomena. The aim is always to simplify the full problem in order to improve the mathematical understanding and the numerical tractability. In the following, we present a reduced model for thin-film micromagnetics which has been introduced and mathematically studied in [7] and which is consistent with the prior works [4] and [17]. In contrast to [7] where the focus is on a distributional point of view, our emphasis is layed on a functional analytic framework, which is fundamental to develop a sophisticated numerical analysis and discuss numerical discretization schemes.

2 A reduced model in thin-film micromagnetics

We consider the reduced model of [7] which describes micromagnetic phenomena in thin films: Let $\omega \subseteq \mathbb{R}^2$ denote a bounded Lipschitz domain with diameter $\ell = 1$. This domain represents our ferromagnetic sample, whose thickness t is neglected for simplicity. Throughout, we embed $\mathbb{R}^2$ into $\mathbb{R}^3$ by identifying vectors $x \in \mathbb{R}^2$ with $(x,0) \in \mathbb{R}^3$. With an in-plane applied exterior field $\mathbf{f} : \omega \to \mathbb{R}^2$, we seek a minimizer $\mathbf{m}^* : \omega \to \mathbb{R}^2$ of the reduced energy

$$e(\mathbf{m}) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\, dx + \frac{q}{2}\int_\omega \mathbf{m}_2^2\, dx - \int_\omega \mathbf{f}\cdot\mathbf{m}\, dx, \tag{3}$$

under the constraint $|\mathbf{m}^*| \leq 1$. The magnetic potential $u : \mathbb{R}^3 \to \mathbb{R}$ satisfies

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla v\, dx = \int_\omega \mathbf{m}\cdot\nabla v(x,0)\, dx \quad \text{for all} \quad v \in \mathscr{D}(\mathbb{R}^3). \tag{4}$$

We assume that $\mathbf{m}$ satisfies $\mathbf{m}\cdot\mathbf{n} = 0$ on the boundary $\gamma := \partial\omega$, where $\mathbf{n}$ denotes the outer normal vector of ω in $\mathbb{R}^2$. Provided that the functions involved are sufficiently smooth, integration by parts on the right-hand side of (4) yields

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla v\, dx = -\int_\omega \nabla\cdot\mathbf{m} v\, dx \quad \text{for all} \quad v \in \mathscr{D}(\mathbb{R}^3). \tag{5}$$

We stress that we solve, in fact, the Maxwell equation stated in the latter form in order to compute the magnetostatic potential u.

An extensive analysis has been performed in [7], where the Γ-convergence of the full problem (1) to the reduced problem (3) is proven for vanishing thickness $t \to 0$ in a certain regime. This reduced model describes the behaviour of sufficiently large and thin samples, i.e.

$$t \ll q \ll \ell.$$

First numerical experiments in [6] and [8] illustrate the success of the reduced model, since the numerically simulated behaviour of the magnetization coincides well with measured data from experiments with thin permalloy films.

We stress that in the reduced problem the exchange energy contribution vanishes. The consequences of this are twofold: First, the non-convex constraint of the full problem relaxes to the convex constraint $|\mathbf{m}| \leq 1$, which from a practical point of view makes the problem easier to solve. On the other hand, the absence of the exchange energy makes the reduced model highly degenerate.

3 Functional setting

In order to analyze the problem from the point of view of numerical analysis, it is essential to determine the precise setting, i.e. to define function spaces for all involved quantities. The applied exterior field $\mathbf{f}$ as well as the magnetic field $\mathbf{m}$ could, e.g., be chosen as $\mathbf{f},\mathbf{m} \in L^2(\omega)^2$ to ensure the well-posedness of the energy functional $e(\mathbf{m})$. This space, however, is still to large in the sense that the well-posedness of the magnetostatic potential equation (5) needs some further regularity of $\mathbf{m}$ to define $\nabla\cdot\mathbf{m}$ in an appropriate way. We first recall the definitions of certain fractional order Sobolev spaces and then take a closer look at the magnetostatic Maxwell equation.

3.1 Sobolev spaces

The treatment of $\nabla\cdot\mathbf{m}$ from (5) needs the definition of $\widetilde{H}^{-1/2}$ which is done in several steps.

We interpret ω as a (relatively) open surface piece embedded in the space $\mathbb{R}^3$. As usual we define the Sobolev spaces

$$H^0(\omega) = L^2(\omega) \quad \text{and} \quad H^1(\omega) := \{f \in L^2(\omega)\,|\,\nabla f \in L^2(\omega)\},$$

where the latter is associated with the norm $\|f\|^2_{H^1(\omega)} = \|f\|^2_{L^2(\omega)} + \|\nabla f\|^2_{L^2(\omega)}$. Here, ∇f denotes the weak surface gradient. Moreover, we define $H^1_0(\omega)$ to be the completion of $\{f \in \mathrm{Lip}(\omega)\,|\,f = 0 \text{ on } \partial\omega\}$ in $H^1(\omega)$.

Next, we introduce fractional order Sobolev spaces by interpolation: For $0 \leq s \leq 1$, we define

$$\widetilde{H}^s(\omega) := [L^2(\omega); H^1_0(\omega)]_s \quad \text{and} \quad H^s(\omega) := [L^2(\omega); H^1(\omega)]_s,$$

where $[X_0, X_1]_s$ denotes the complex interpolation of Banach spaces X_0 and $X_1 \subseteq X_0$, cf. [2].

Finally, we define the negative order Sobolev spaces

$$\widetilde{H}^{-s}(\omega) := (H^s(\omega))^* \quad \text{and} \quad H^{-s}(\omega) := (\widetilde{H}^s(\omega))^*,$$

as the dual spaces of $\widetilde{H}^s(\omega)$ and $H^s(\omega)$, respectively, where duality is understood with respect to the extended L^2-scalar product.

Remark. There are other equivalent definitions of the Sobolev spaces $\widetilde{H}^s(\omega)$ and $H^s(\omega)$, e.g., as trace spaces or by real interpolation, a Fourier norm, or Sobolev-Slobodeckij norms [16]. Here, equivalence means that all of these definitions lead to the same sets of functions but only to equivalent norms. We stress that $\widetilde{H}^s(\omega)$ and $H^s(\omega)$ are, in fact, Hilbert spaces. □

The treatment of the magnetic potential u from (5) needs the definition of the so-called Beppo-Levi space $B_1^2(\mathbb{R}^3)$. To that end, we first define the Sobolev space

$$H^1_{\ell oc}(\mathbb{R}^3) := \{f \,|\, f \text{ is a linear functional on } \mathscr{C}_c^\infty(\mathbb{R}^3) \text{ with } \varphi f \in H^1(\mathbb{R}^3) \text{ for all } \varphi \in \mathscr{C}_c^\infty(\mathbb{R}^3)\}.$$

Note that the evaluation of the energy (3) demands $\nabla u \in L^2(\mathbb{R}^3)$. We introduce the space

$$\widetilde{B}_1^2(\mathbb{R}^3) := \{u \in H^1_{\ell oc}(\mathbb{R}^3) \,|\, \nabla u \in L^2(\mathbb{R}^3)^3\} \tag{6}$$

associated with the seminorm $\|u\|_{B_1^2(\mathbb{R}^3)} := \|\nabla u\|_{L^2(\mathbb{R}^3)^3}$. Finally, the Beppo-Levi space is defined as

$$B_1^2(\mathbb{R}^3) := \widetilde{B}_1^2(\mathbb{R}^3)/\mathbb{R} \tag{7}$$

where one factors out the constant functions. Note that $\|\cdot\|_{B_1^2(\mathbb{R}^3)}$ then gives a norm on $B_1^2(\mathbb{R}^3)$. Moreover, $B_1^2(\mathbb{R}^3)$ is a Hilbert space and $\mathscr{D}(\mathbb{R}^3)$ is a dense subspace of $B_1^2(\mathbb{R}^3)$, cf. [12]. Finally, there is a linear and continuous lifting operator $L: H^{1/2}(\omega) \to B_1^2(\mathbb{R}^3)$, i.e. $v \in H^{1/2}(\omega)$ with $v \neq 0$, then

$$v = (Lv)|_\omega \quad \text{and} \quad \|\nabla(Lv)\|_{L^2(\mathbb{R}^3)^3} \le C\|v\|_{H^{1/2}(\omega)} \tag{8}$$

with $C > 0$ the operator norm of L [12]. Throughout, estimates like (8) are written in the form $\|\nabla(Lv)\|_{L^2(\mathbb{R}^3)^3} \lesssim \|v\|_{H^{1/2}(\omega)}$, where the symbol $\lesssim$ indicates a bound up to a certain multiplicative but generic constant $C > 0$.

3.2 The magnetostatic Maxwell equation in thin-film micromagnetics

Strong form of the magnetostatic Maxwell equation.

The magnetic potential is given as the solution of the variational formulation (4). The mathematical understanding of the potential u is crucial to define the appropriate function space $\mathscr{H}$ for the magnetization $\mathbf{m}$. To that end, we write (4) in strong form.

Lemma 1. *With given data* $\mathbf{m} \in \mathscr{C}^1(\omega)$ *that satisfies*

$$\mathbf{m}\cdot\mathbf{n} = 0 \quad on \quad \gamma = \partial\omega \subseteq \mathbb{R}^2,$$

with $\mathbf{n}$ *the outer normal in* $\mathbb{R}^2$ *of* ω*, every weak solution* $u \in \mathscr{C}^2(\mathbb{R}^2 \setminus \omega)$ *of the magnetostatic Maxwell equation* (4) *solves the strong form*

$$\begin{aligned} \Delta u = 0 \quad &in \quad \mathbb{R}^3 \setminus \overline{\omega}, \\ [\partial u/\partial x_3] = \nabla\cdot\mathbf{m} \quad &on \quad \omega, \end{aligned} \tag{9}$$

where $[\varphi] := \lim_{x\to(x_1,x_2,0^+)} \varphi(x) - \lim_{x\to(x_1,x_2,0^-)} \varphi(x)$ *denotes the jump across the surface piece* ω.

Proof. Let $v \in \mathscr{D}(\mathbb{R}^3)$. We split the left-hand side of (4) into two integrals

$$\int_{\mathbb{R}^3} \nabla u\cdot\nabla v\,dx = \int_{\mathbb{R}^3_-} \nabla u\cdot\nabla v\,dx + \int_{\mathbb{R}^3_+} \nabla u\cdot\nabla v\,dx$$

over domains $\mathbb{R}^3_+ := \{x \in \mathbb{R}^3 | x_3 > 0\}$ and $\mathbb{R}^3_- := \{x \in \mathbb{R}^3 | x_3 < 0\}$, respectively. We use integration by parts to see

$$\int_{\mathbb{R}^3_-} \nabla u\cdot\nabla v\,dx + \int_{\mathbb{R}^3_+} \nabla u\cdot\nabla v\,dx = -\int_{\mathbb{R}^3} \Delta u v\,dx - \int_{\mathbb{R}^2} \frac{\partial u(x,0+)}{\partial x_3} v\,dx + \int_{\mathbb{R}^2} \frac{\partial u(x,0-)}{\partial x_3} v\,dx.$$

On the other hand, integration by parts on the right-hand side of (4) shows

$$\int_\omega \mathbf{m}\cdot\nabla v(x,0)\,dx = -\int_\omega \nabla\cdot\mathbf{m} v\,dx + \int_\gamma (\mathbf{m}\cdot\mathbf{n}) v\,ds.$$

These observations result in

$$-\int_{\mathbb{R}^3} \Delta u v\,dx - \int_{\mathbb{R}^2} \frac{\partial u(x,0+)}{\partial x_3} v\,dx + \int_{\mathbb{R}^2} \frac{\partial u(x,0-)}{\partial x_3} v\,dx = -\int_\omega \nabla\mathbf{m}\cdot v\,dx + \int_\gamma (\mathbf{m}\cdot\mathbf{n}) v\,ds \quad \text{for all } v\in\mathscr{D}(\mathbb{R}^3).$$

A closer look at the boundary term on the left-hand side reveals

$$-\int_{\mathbb{R}^2} \frac{\partial u(x,0+)}{\partial x_3} v\,dx + \int_{\mathbb{R}^2} \frac{\partial u(x,0-)}{\partial x_3} v\,dx = -\int_{\mathbb{R}^2} \left[\frac{\partial u}{\partial x_3}\right] v\,dx.$$

Since u is smooth in $\mathbb{R}^3 \setminus \overline{\omega}$, the jump $[\partial u/\partial x_3]$ can only be nonzero on ω. Altogether, u solves the strong form (9). $\square$

Solution of the magnetostatic Maxwell equation.

For $v \in L^1(\omega)$, we define the simple-layer potential $Sv \in \mathscr{C}^\infty(\mathbb{R}^3 \setminus \overline{\omega})$ by

$$Sv(x) = \frac{1}{4\pi}\int_\omega \frac{v(y)}{\|x-y\|}\,ds_y. \tag{10}$$

From the context of the boundary element method [16], it is well known that the simple-layer potential may be extended to a continuous linear operator $S \in L(\widetilde{H}^{-1/2}(\omega); H^1_{\ell oc}(\mathbb{R}^3))$, which satisfies $\Delta Sv = 0$ in an appropriate sense.

Theorem 1 (Jump conditions)**.** *The simple-layer potential satisfies the jump conditions*

$$[S\varphi] = 0 \in H^{1/2}(\omega), \qquad \left[\frac{\partial S\varphi}{\partial x_3}\right] = -\varphi \in \widetilde{H}^{-1/2}(\omega). \tag{11}$$

The jump conditions are usually proven for the simple-layer potential defined over the whole boundary $\Gamma = \partial G$ of a Lipschitz-domain $G \subseteq \mathbb{R}^3$. This, however, is not the case here, where we only consider an open subset $\omega \subseteq \partial G$. Basically, one can still follow the lines of the proof in, e.g., [16, Theorem 3.3.1]. Regularity of the simple-layer potential Sv away from $\overline{\omega}$ and a density argument then show that the jump conditions even hold in our case.

Corollary 1. *For $\nabla\cdot\mathbf{m} \in \widetilde{H}^{-1/2}(\omega)$, the simple-layer potential $-S(\nabla\cdot\mathbf{m})$ is a solution of the Maxwell equation* (5)*. It additionally satisfies $[-S(\nabla\cdot\mathbf{m})] = 0$ on ω.* $\square$

3.3 The energy space $\mathscr{H}$

In this section, we aim to construct an appropriate Hilbert space $\mathscr{H}$ so that the thin-film micromagnetic energy $e(\mathbf{m})$ in (3) is meaningful for $\mathbf{m} \in \mathscr{H}$. To that end, we recall our observations: First, to explain the anisotropic and exterior contributions of $e(\mathbf{m})$, it is sufficient to require $\mathbf{m} \in L^2(\omega)^2$. Second, the (appropriately defined) divergence of $\mathbf{m}$ should satisfy $\nabla\cdot\mathbf{m} \in \widetilde{H}^{-1/2}(\omega)$. Finally, we aim to include the constraint $\mathbf{m}\cdot\mathbf{n} = 0$ on $\gamma = \partial\omega$ in an appropriate way into the definition of $\mathscr{H}$.

To that end, we define $\mathscr{D}(X) := \mathscr{C}^\infty_c(X)$ to be the test space of smooth functions with compact support over the domain X. Furthermore, we define $\mathscr{D}(\overline{X}) := \{\varphi|_X \,|\, \varphi \in \mathscr{D}(\mathbb{R}^n)\}$.

In a first step, we consider the weak divergence and the spaces $H^1(\nabla\cdot;\omega))$ and $H^1_0(\nabla\cdot;\omega)$. For the convenience of the reader and for the sake of completeness, we recall these definitions.

Definition 1. *A function $v \in L^2(\omega)$ is called weak the divergence of* $\mathbf{m} \in L^2(\omega)^2$ *if there holds*

$$\int_\omega v\varphi\,dx = -\int_\omega \mathbf{m}\cdot\nabla\varphi\,dx \quad \textit{for all} \quad \varphi \in \mathscr{D}(\omega). \tag{12}$$

In this case and according to the fundamental theorem of calculus of variations, the weak divergence of $\mathbf{m}$ *is unique, and we simply write $\nabla\cdot\mathbf{m} := v$.*

Definition 2. *We define the space*

$$H^1(\nabla\cdot;\omega) := \{\mathbf{m} \in L^2(\omega)^2 \,|\, \nabla\cdot\mathbf{m} \in L^2(\omega)\} \tag{13}$$

with the canonical graph norm $\|\mathbf{m}\|_{H^1(\nabla\cdot;\omega)} := (\|\mathbf{m}\|^2_{L^2(\omega)^2} + \|\nabla\cdot\mathbf{m}\|^2_{L^2(\omega)})^{1/2}$. We further define the space of functions with vanishing normal at the boundary $H^1_0(\nabla\cdot;\omega) := \overline{\mathscr{D}(\omega)^2}^{\|\cdot\|_{H^1(\nabla\cdot;\omega)}}$.

One can easily show that $H^1(\nabla\cdot;\omega)$ is a Hilbert space. Furthermore it is known that the space of test functions $\mathscr{D}(\overline{\omega})^2$ is dense in $H^1(\nabla\cdot;\omega)$, see [10, Theorem 2.4]. The linear mapping $f_n : \mathbf{v} \mapsto \mathbf{v}\cdot\mathbf{n}|_\gamma$ defined on $\mathscr{D}(\overline{\omega})^2$ can be extended to a continuous linear operator from $H^1(\nabla\cdot;\omega)$ to $H^{-1/2}(\gamma)$, cf. [10, Theorem 2.5]. We are particularly interested in functions which satisfy $\mathbf{m}\cdot\mathbf{n} = 0$. According to [10, Theorem 2.6] there holds $H^1_0(\nabla\cdot;\omega) = \ker(f_n) = \{\mathbf{m} \in H^1(\nabla\cdot;\omega) \,|\, \mathbf{m}\cdot\mathbf{n} = 0\}$. This leads to the following definition.

Definition 3. *The energy space for the magnetization is defined by*

$$\mathscr{H} = \overline{H^1_0(\nabla\cdot;\omega)}^{\|\cdot\|_{\mathscr{H}}} \tag{14}$$

with the graph norm

$$\|\mathbf{m}\|^2_{\mathscr{H}} := \|\mathbf{m}\|^2_{L^2(\omega)^2} + \|\nabla\cdot\mathbf{m}\|^2_{\widetilde{H}^{-1/2}(\omega)}. \tag{15}$$

Lemma 2. *The energy space $\mathscr{H}$ is a Hilbert space, and $\mathscr{D}(\omega)^2 \subseteq \mathscr{H}$ is a dense subspace.*

Proof. By definition, $\mathscr{D}(\omega)^2$ is a dense subspace of $H^1_0(\nabla\cdot;\omega)$, and $H^1_0(\nabla\cdot;\omega)$ is a dense subspace of $\mathscr{H}$. It thus only remains to prove $\|\mathbf{m}\|_{\mathscr{H}} \le \|\mathbf{m}\|_{H^1_0(\nabla\cdot;\omega)}$ for all $\mathbf{m} \in H^1_0(\nabla\cdot;\omega)$. This, however, follows from

$$\begin{aligned}
\|\mathbf{m}\|^2_{\mathscr{H}} = \|\mathbf{m}\|^2_{L^2(\omega)^2} + \|\nabla\cdot\mathbf{m}\|^2_{\widetilde{H}^{-1/2}(\omega)} &= \|\mathbf{m}\|^2_{L^2(\omega)^2} + \left(\sup_{w\in H^{1/2}(\omega)} \frac{\langle\nabla\cdot\mathbf{m}\,,w\rangle_{\widetilde{H}^{-1/2}(\omega)\times H^{1/2}(\omega)}}{\|w\|_{H^{1/2}(\omega)}}\right)^2 \\
&= \|\mathbf{m}\|^2_{L^2(\omega)^2} + \left(\sup_{w\in H^{1/2}(\omega)} \frac{(\nabla\cdot\mathbf{m},w)_{L^2(\omega)}}{\|w\|_{H^{1/2}(\omega)}}\right)^2 \\
&\le \|\mathbf{m}\|^2_{L^2(\omega)^2} + \left(\sup_{w\in L^2(\omega)} \frac{(\nabla\cdot\mathbf{m},w)_{L^2(\omega)}}{\|w\|_{L^2(\omega)}}\right)^2 \\
&= \|\mathbf{m}\|^2_{H^1_0(\nabla\cdot;\omega)},
\end{aligned}$$

where we used $\|w\|_{L^2(\omega)} \le \|w\|_{H^{1/2}(\omega)}$ as well as $H^{1/2}(\omega) \subseteq L^2(\omega)$. □

The following simple observation will be crucial for the analysis below.

Corollary 2. *For all functions $\mathbf{m} \in \mathscr{H}$, there holds $\langle\nabla\cdot\mathbf{m}\,,1\rangle_{\widetilde{H}^{-1/2}(\omega)\times H^{1/2}(\omega)} = 0$.*

Proof. For $\mathbf{m} \in \mathscr{D}(\omega)^2$, the Gauss divergence theorem yields

$$\langle\nabla\cdot\mathbf{m}\,,1\rangle_{\widetilde{H}^{-1/2}(\omega)\times H^{1/2}(\omega)} = \int_\omega \nabla\cdot\mathbf{m}\,dx = \int_\gamma \mathbf{m}\cdot\mathbf{n}\,ds = 0.$$

According to density of $\mathscr{D}(\omega)^2$ in $\mathscr{H}$, continuity arguments conclude the proof. □

4 Well-posedness of the thin-film problem in micromagnetics

Definition 4. *We define the operator*

$$V\varphi(x) = \frac{1}{4\pi}\int_\omega \frac{\varphi(x)}{\|x-y\|}ds_y \quad \textit{for } x\in\omega, \tag{16}$$

as the trace of S on ω. The operator V is also called simple-layer potential. It is well known [16] that V may be extended to a continuous linear operator $V \in L(\widetilde{H}^{-1/2}(\omega);H^{1/2}(\omega))$.

Moreover, V turns out to be elliptic and symmetric on $\widetilde{H}^{-1/2}(\omega)$ so that $(\phi,\psi)_V := \langle\phi\,,V\psi\rangle_{\widetilde{H}^{-1/2}(\omega)\times H^{1/2}(\omega)}$ turns out to be a scalar product. Therefore, the induced norm $\|\phi\|_V := (\phi,\phi)_V^{1/2}$ is an equivalent norm on $\widetilde{H}^{-1/2}(\omega)$.

As discussed above, $\mathbf{m} \in \mathscr{H}$ allows for $\nabla\cdot\mathbf{m} \in \widetilde{H}^{-1/2}(\omega)$. Therefore, the variational formulation (5) of the static Maxwell equation reads

$$(\nabla u,\nabla v)_{L^2(\mathbb{R}^3)^3} = -\langle\nabla\cdot\mathbf{m}\,,v\rangle_{\widetilde{H}^{-1/2}(\omega)\times H^{1/2}(\omega)} \quad \text{for all} \quad v\in\mathscr{D}(\mathbb{R}^3). \tag{17}$$

First, we state existence and uniqueness of a magnetostatic potential u in the Beppo-Levi space $B^2_1(\mathbb{R}^3)$.

Theorem 2. *(i) Given $\mathbf{m} \in \mathscr{H}$, there exists a unique $u \in B^2_1(\mathbb{R}^3)$ with* (17)*, and there holds (up to additive constants) $u = -S(\nabla\cdot\mathbf{m})$ with the simple-layer potential S of* (10)*.*

(ii) Equation (17) *holds with $\mathscr{D}(\mathbb{R}^3)$ replaced by the full space $B^2_1(\mathbb{R}^3)$.*

(iii) The stray-field operator $\mathscr{P} : \mathscr{H} \to L^2(\mathbb{R}^3)^3$, which maps $\mathbf{m}$ to the corresponding stray-field $\mathscr{P}(\mathbf{m}) := -\nabla u$, is a linear and continuous operator.

(iv) For $\mathbf{m}, \widetilde{\mathbf{m}} \in \mathscr{H}$*, there holds* $(\mathscr{P}\mathbf{m}, \mathscr{P}\widetilde{\mathbf{m}})_{L^2(\mathbb{R}^3)^3} = (\nabla\cdot\mathbf{m}, \nabla\cdot\widetilde{\mathbf{m}})_V$.

(v) In particular, there holds $\|\mathscr{P}\mathbf{m}\|_{L^2(\mathbb{R}^3)^3} = \|\nabla\cdot\mathbf{m}\|_V \sim \|\nabla\cdot\mathbf{m}\|_{\widetilde{H}^{-1/2}(\omega)}$.

Proof. Let $\mathbf{m} \in \mathscr{H}$ be fixed. We first consider $F_\mathbf{m}(v) := \langle \nabla\cdot\mathbf{m}\,, v\rangle_{\widetilde{H}^{-1/2}(\omega)\times H^{1/2}(\omega)}$ for $v \in \mathscr{D}(\mathbb{R}^3) \subseteq B_1^2(\mathbb{R}^3)$. According to Lemma 2, there holds

$$F_\mathbf{m}(v) := \langle \nabla\cdot\mathbf{m}\,, v\rangle_{\widetilde{H}^{-1/2}(\omega)\times H^{1/2}(\omega)} = \langle \nabla\cdot\mathbf{m}\,, v-\lambda\rangle_{\widetilde{H}^{-1/2}(\omega)\times H^{1/2}(\omega)} \quad \text{for all constants} \quad \lambda \in \mathbb{R}.$$

We consider the cylindrical domain $\widehat{\omega} := \omega \times (0,1) \subseteq \mathbb{R}^3$. For the particular choice $\lambda := (1/|\widehat{\omega}|)\int_{\widehat{\omega}} v\,dx$, a trace inequality and the Poincaré inequality show

$$\|v-\lambda\|_{H^{1/2}(\omega)} \lesssim \|v-\lambda\|_{H^1(\widehat{\omega})} \lesssim \|\nabla v\|_{L^2(\widehat{\omega})^3} \le \|\nabla v\|_{L^2(\mathbb{R}^3)^3}.$$

This proves that $F_\mathbf{m}$ defines a linear and continuous functional $F_\mathbf{m} : \mathscr{D}(\mathbb{R}^3) \to \mathbb{R}$ with respect to $\|\cdot\|_{B_1^2(\mathbb{R}^3)}$, where the operator norm satisfies $\|F_\mathbf{m}\| \lesssim \|\nabla\cdot\mathbf{m}\|_{\widetilde{H}^{-1/2}(\omega)}$. Since $\mathscr{D}(\mathbb{R}^3)$ is dense in $B_1^2(\mathbb{R}^3)$, the functional $F_\mathbf{m}$ may be uniquely extended to a continuous and linear functional on the entire Beppo-Levi space $B_1^2(\mathbb{R}^3)$, and the extension satisfies $\|F_\mathbf{m}\| \lesssim \|\nabla\cdot\mathbf{m}\|_{\widetilde{H}^{-1/2}(\omega)}$. Since the left-hand side of (17) is the scalar product of the Hilbert space $B_1^2(\mathbb{R}^3)$, continuity allows to replace the dense subspace $\mathscr{D}(\mathbb{R}^3)$ by $B_1^2(\mathbb{R}^3)$. This proves (ii).
In particular, (17) reads

$$(u,v)_{B_1^2(\mathbb{R}^3)} = F(v) \quad \text{for all} \quad v \in B_1^2(\mathbb{R}^3),$$

and the Riesz representation theorem proves existence and uniqueness of a potential $u \in B_1^2(\mathbb{R}^3)$.
To verify (iii) and (iv), we stress that the Riesz theorem implies equality of norms

$$\|\mathscr{P}\mathbf{m}\|_{L^2(\mathbb{R}^3)^3} = \|u\|_{B_1^2(\mathbb{R}^3)} = \|F_\mathbf{m}\| \lesssim \|\nabla\cdot\mathbf{m}\|_{\widetilde{H}^{-1/2}(\omega)}.$$

In particular the mapping $\mathscr{P} : \mathscr{H} \to L^2(\mathbb{R}^3)^3$ is well defined and continuous. Linearity is obvious.
Next, we prove the converse estimate $\|\nabla\cdot\mathbf{m}\|_{\widetilde{H}^{-1/2}(\omega)} \lesssim \|F_\mathbf{m}\|$. The properties of the lifting operator (8) and equation (17) imply

$$\frac{|\langle \nabla\cdot\mathbf{m}\,, v\rangle_{\widetilde{H}^{-1/2}(\omega)\times H^{1/2}(\omega)}|}{\|v\|_{H^{1/2}(\omega)}} \lesssim \frac{|(\mathscr{P}\mathbf{m}, \nabla(Lv))_{L^2(\mathbb{R}^3)^3}|}{\|\nabla(Lv)\|_{L^2(\mathbb{R}^3)^3}} \lesssim \|\mathscr{P}\mathbf{m}\|_{L^2(\mathbb{R}^3)^3}$$

for arbitrary $v \in H^{1/2}(\omega)$. Taking the supremum over all $v \in H^{1/2}(\omega)$ we obtain $\|\nabla\cdot\mathbf{m}\|_{\widetilde{H}^{-1/2}(\omega)} \lesssim \|\mathscr{P}\mathbf{m}\|_{L^2(\mathbb{R}^3)^3}$. Note that $S\varphi \in B_1^2(\mathbb{R}^3)$ follows from [16, Equation (3.1.23)] for arbitrary $\varphi \in \widetilde{H}^{-1/2}(\omega)$. According to Corollary 1 the uniquely determined magnetostatic potential $u \in B_1^2(\mathbb{R}^3)$ therefore reads $u = -S(\nabla\cdot\mathbf{m})$. For $\mathscr{P}\widetilde{\mathbf{m}} = -\nabla\widetilde{u}$, this representation and the variational form (17) for $\mathscr{P}\mathbf{m} = -\nabla u$ imply

$$(\mathscr{P}\mathbf{m}, \mathscr{P}\widetilde{\mathbf{m}})_{L^2(\mathbb{R}^3)^3} = -\langle \nabla\cdot\mathbf{m}\,, \widetilde{u}\rangle_{\widetilde{H}^{-1/2}(\omega)\times H^{1/2}(\omega)} = \langle \nabla\cdot\mathbf{m}\,, V(\nabla\cdot\widetilde{\mathbf{m}})\rangle_{\widetilde{H}^{-1/2}(\omega)\times H^{1/2}(\omega)}.$$

The choice of $\widetilde{\mathbf{m}} = \mathbf{m}$ proves $(\mathscr{P}\mathbf{m}, \mathscr{P}\widetilde{\mathbf{m}})_{L^2(\mathbb{R}^3)^3} = \|\nabla\cdot\mathbf{m}\|_V^2$, which finally concludes the proof. □

We now use our functional setting to state the thin-film problem in micromagnetics. We define the set of admissible magnetizations

$$\mathscr{A} := \{\mathbf{m} \in \mathscr{H} \,|\, |\mathbf{m}| \le 1 \text{ almost everywhere in } \omega\}. \tag{18}$$

We aim to find a minimizer $\mathbf{m}^* \in \mathscr{A}$ of the energy $e(\mathbf{m})$ in (3), which is now rewritten in the form

$$e(\mathbf{m}) = \frac{1}{2}\|\nabla\cdot\mathbf{m}\|_V^2 + \frac{q}{2}\|\mathbf{m}_2\|_{L^2(\omega)}^2 - (\mathbf{f},\mathbf{m})_{L^2(\omega)^2}. \tag{19}$$

To show that this minimization problem is well-posed, we aim to apply the direct method of the calculus of variations. This will be prepared by the following two lemmata.

Our first observation follows immediately from the representation (19):

Lemma 3. *The energy functional* $e : \mathscr{H} \to \mathbb{R}$ *is well-defined, convex, and continuous. Moreover, it is strictly convex with respect to* $\nabla\cdot\mathbf{m}$ *and* $\mathbf{m}_2$. □

Our second observation reads as follows:

Lemma 4. *The set* $\mathscr{A}$ *is non-empty, convex, and closed with respect to* $\mathscr{H}$.

Proof. Clearly $\mathscr{A}$ is convex and non-empty. It therefore only remains to prove that $\mathscr{A}$ is closed. To that end, let $(\mathbf{m}_k)_{k\in\mathbb{N}} \subseteq \mathscr{A}$ be a convergent sequence with limit $\mathbf{m} \in \mathscr{H}$. In particular, convergence in $\mathscr{H}$ implies convergence in the L^2-sense, i.e. $\mathbf{m}_k \to \mathbf{m} \in L^2(\omega)^2$. Therefore, at least a subsequence converges pointwise almost everywhere towards $\mathbf{m}$. This proves $|\mathbf{m}| \le 1$ almost everywhere, i.e., $\mathbf{m} \in \mathscr{A}$. ☐

Finally we can now state the well-posedness of the thin-film problem in micromagnetics, and we can even prove certain uniqueness properties of minimizing magnetizations $\mathbf{m}^* \in \mathscr{A}$.

Theorem 3. *There is a minimizer* $\mathbf{m}^* \in \mathscr{A}$ *of* (19). *Moreover,* $\mathscr{P}\mathbf{m}^*$ *and* $q\mathbf{m}_2^*$ *are uniquely determined, i.e. for any minimizers* $\mathbf{m}^*, \widetilde{\mathbf{m}}^* \in \mathscr{A}$ *of* (19) *holds* $\mathscr{P}\mathbf{m}^* = \mathscr{P}\widetilde{\mathbf{m}}^*$ *as well as* $q\mathbf{m}_2^* = q\widetilde{\mathbf{m}}_2^*$.

Proof. Let $(\mathbf{m}_k)_{k\in\mathbb{N}} \subseteq \mathscr{A}$ be a sequence with

$$\lim_{k\to\infty} e(\mathbf{m}_k) = \inf_{\mathbf{m}\in\mathscr{A}} e(\mathbf{m}) =: M.$$

Plugging in $\mathbf{m} = 0 \in \mathscr{A}$, we obtain $M \le e(0) = 0$. This, however, implies for arbitrary $\varepsilon > 0$ and k sufficiently large

$$e(\mathbf{m}_k) = \frac{1}{2}\|\mathscr{P}\mathbf{m}_k\|^2_{L^2(\mathbb{R}^3)^3} + \frac{q}{2}\|\mathbf{m}_{k,2}\|^2_{L^2(\omega)} - \langle \mathbf{f}, \mathbf{m}\rangle_{L^2(\omega)} \le \frac{\varepsilon}{2}.$$

Using Cauchy's inequality and dropping the anisotropy energy, one obtains

$$\|\mathscr{P}\mathbf{m}_k\|^2_{L^2(\mathbb{R}^3)^3} \le 2\|\mathbf{f}\|_{L^2(\omega)^2}\|\mathbf{m}_k\|_{L^2(\omega)^2} + \varepsilon.$$

From the definition of $\mathscr{A}$, we derive $\|\mathbf{m}_k\|_{L^2(\omega)^2} \le |\omega|^{1/2}$. The equivalence $\|\mathscr{P}\mathbf{m}\|_{L^2(\mathbb{R}^3)^3} \sim \|\nabla\cdot\mathbf{m}\|_{\widetilde{H}^{-1/2}}$ therefore implies that $(\mathbf{m}_k)_{k\in\mathbb{N}}$ is bounded with respect to the $\mathscr{H}$-norm, namely

$$\|\mathbf{m}_k\|^2_{\mathscr{H}} \lesssim \|\mathbf{m}_k\|^2_{L^2(\omega)^2} + \|\mathscr{P}\mathbf{m}_k\|^2_{L^2(\mathbb{R}^3)^3} \le |\omega| + 2\|\mathbf{f}\|_{L^2(\omega)^2}|\omega|^{1/2} + \varepsilon.$$

Since $(\mathbf{m}_k)_{k\in\mathbb{N}}$ is a bounded sequence in the Hilbert space $\mathscr{H}$, we may assume that is has a weak limit $\mathbf{m} \in \mathscr{H}$. The set $\mathscr{A}$ is convex and closed. According to Mazur's lemma, it is thus also closed with respect to the weak topology in $\mathscr{H}$. This implies $\mathbf{m} \in \mathscr{A}$. Moreover, a convex and continuous functional is also weakly lower semicontinous [5], i.e.

$$e(\mathbf{m}) \le \liminf_{k\to\infty} e(\mathbf{m}_k) = M.$$

Altogether, $\mathbf{m} \in \mathscr{A}$ is a minimizer of $e(\cdot)$ in $\mathscr{A}$. Finally, the strict convexity of $e(\mathbf{m})$ with respect to $\mathscr{P}\mathbf{m}$ and $\mathbf{m}_2$, for $q > 0$, proves that both are uniquely determined for a minimizer $\mathbf{m}^* \in \mathscr{A}$ of $e(\mathbf{m})$. ☐

Remark. Note, that the proof of Theorem 3 only used some properties of the functional $e(\cdot)$ as well as the closedness and convexity of $\mathscr{A}$. Any finite dimensional subspace $X_h \le \mathscr{H}$ of the full energy space is obviously closed and convex. In particular, the discrete set of admissible functions $\mathscr{A}_h := \mathscr{A} \cap X_h$ is closed and convex as well. Therefore, Theorem 3 also applies for the discrete minimization problem, where we aim to minimize $e(\mathbf{m}_h)$ over $\mathscr{A}_h$. In particular, there is a discrete minimizer $\mathbf{m}_h^* \in \mathscr{A}_h$, and the stray field $\mathscr{P}\mathbf{m}_h^*$ and the second component $\mathbf{m}_{h,2}^*$ of two discrete minimizers $\mathbf{m}_h^*, \widetilde{\mathbf{m}}_h^* \in \mathscr{A}_h$ coincide. ☐

5 Discretization

We use lowest-order Raviart-Thomas finite elements that were introduced first in [15], which is also proposed in the recent thesis [8]: Let $\mathscr{T} = \{T_1, \dots, T_n\}$ be a regular triangulation of ω in the sense of Ciarlet, i.e.

- $\overline{\omega} = \bigcup_{j=1}^n T_j$, i.e. $\mathscr{T}$ covers ω,
- each T_j is a closed an non-degenerate triangle, i.e. $|T_j| > 0$,
- the intersection $T_i \cap T_j$, for $i \ne j$, is either empty, a common point, or a common edge.

Let $\mathscr{E}$ denote the set of all edges of the triangulation $\mathscr{T}$ and $\mathscr{E}_\omega$ the set of interior edges. Each edge $E \in \mathscr{E}_\omega$ belongs to precisely two elements T_+ and T_-. Let P_+ and P_- denote the vertices of T_+ and T_-, respectively, opposite from E, i.e. $T_\pm = \mathrm{conv}\{E \cup \{P_\pm\}\}$. We define the function

$$\psi_E := \begin{cases} \pm\frac{|E|}{2|T_\pm|}(x - P_\pm), & \text{for } x \in T_\pm, \\ 0, & \text{elsewhere.} \end{cases}$$

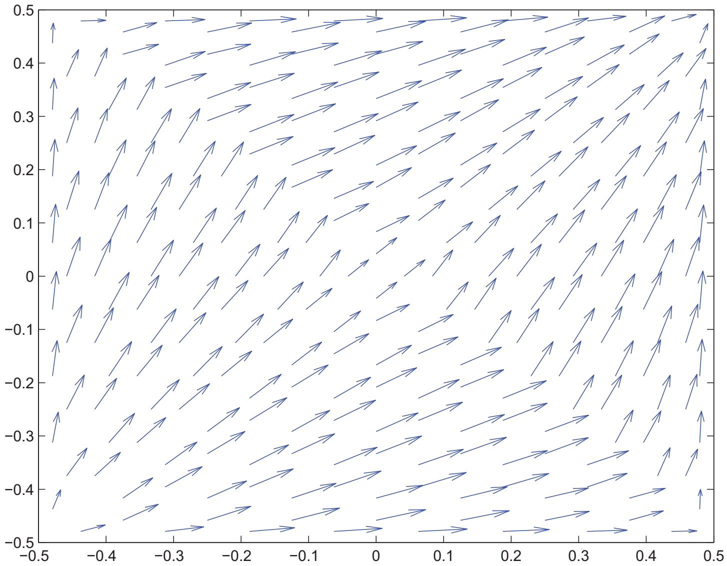

Figure 1: Magnetization of a soft ferromagnetic sample $\omega = (-0.5, 0.5)^2$ with $q = 0$ and $\mathbf{f} = (0.2, 0.2)$.

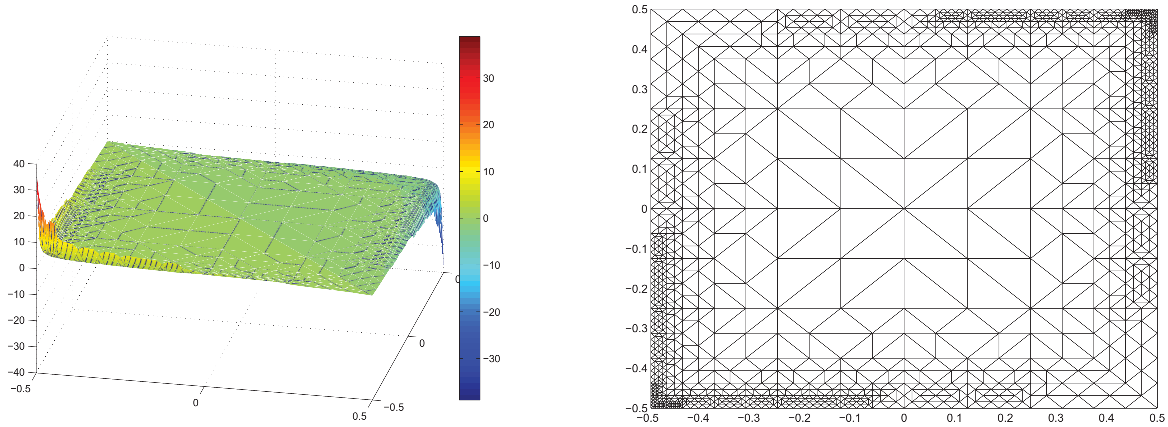

Figure 2: Discrete divergence of the magnetization $\mathbf{m}_h$ from Figure 1 (left), and corresponding generated mesh (right).

The set $\{\psi_E \,|\, E \in \mathscr{E}_\omega\}$ forms a basis of the Raviart-Thomas space $RT_0(\mathscr{T})$. There holds $RT_0(\mathscr{T}) \subseteq H_0^1(\nabla\cdot;\omega) \subseteq \mathscr{H}$, cf. [10]. Details on the the implementation of Raviart-Thomas finite elements in `Matlab` can be found, e.g., in [1].

We again stress that the results from Section 4 also hold for the discrete minimization problem with $\mathscr{A}_h := \mathscr{A} \cap RT_0(\mathscr{T})$. Moreover, the discrete energy coincides with the continuous energy $e_h(\cdot) = e(\cdot)|_{RT_0(\mathscr{T})}$.

When $\mathbf{f}$ is constant, the model problem even further simplifies. For smooth $\mathbf{m} \in \mathscr{D}(\omega)^2$, integration by parts and $\mathbf{m}\cdot\mathbf{n} = 0$ on $\gamma = \partial\omega$ yield

$$\int_\omega \mathbf{f}\cdot\mathbf{m}\,dx = -\int_\omega (\mathbf{f}\cdot x)(\nabla\cdot\mathbf{m})\,dx.$$

By use of density arguments, we thus conclude that the first derivative of the energy reads

$$De(\mathbf{m})(\cdot) = \langle \nabla\cdot\,, V(\nabla\cdot\mathbf{m})\rangle_{\widetilde{H}^{-1/2}(\omega)\times H^{-1/2}(\omega)} - \langle \nabla\cdot\,, \mathbf{f}\cdot x\rangle_{\widetilde{H}^{-1/2}(\omega)\times H^{-1/2}(\omega)}.$$

Recall that the unconstrained problem is convex so that the minimization problem can equivalently be stated in terms of the Euler-Lagrange equation [5]. Moreover, the Euler-Lagrange equation $De(\mathbf{m}) = 0$ is equivalent to Symm's integral equation

$$V(\nabla\cdot\mathbf{m}) = \mathbf{f}\cdot x \in H^{1/2}(\omega). \tag{20}$$

We compute the magnetization of a soft ferromagnetic film of quadratic shape, i.e. $\omega = [-0.5, 0.5]^2$ and $q = 0$. The applied field is chosen to be constant $\mathbf{f} = (0.2, 0.2)$. Computations in [8] show that we may solve the unconstrained linear system in the present case, i.e. the constraint $|\mathbf{m}| \leq 1$ is not active. Figure 1 shows a computed discrete magnetization $\mathbf{m}_h$. We observe that the magnetization tends to align with the applied field, at least in the interior of the simulation domain.

We use the $h-h/2$-error estimator from [9] in order to generate adaptive meshes resolving the singularities of the divergence $\nabla\cdot\mathbf{m}$. Figure 2 shows the computed solution of (20) for $\mathbf{f}=(0.2,0.2)$ (left) and the corresponding adaptively generated mesh (right). We observe strong singularities at the lower left and the upper right corner.

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