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GEOMETRY OF COUPLED INFINITE-DIMENSIONAL SYSTEMS

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Abstract. In the contribution a possible geometric covariant description for the class of coupled lumped- and distributed-parameter systems is proposed, where especially boundary and coupling conditions as well as (lumped and distributed) system inputs and outputs are taken into account. By applying differential geometrical methods dynamic systems are associated with suitable intrinsic geometric objects, which reflect their dynamics. In this context the systems equations are supposed to describe (locally) a family of regular fibred submanifolds of some appropriately-constructed manifolds. Moreover, it is shown that the introduced geometric structures are adequate with respect to the first order Lagrange formalism. Several examples are arranged to illustrate the proposed theory.

1 Introduction

Modeling is an essential or even the most important step for the analysis and design of dynamic systems. Hence, it may turn out that certain components of physical systems cannot be described by finite-dimensional systems due to their inadequateness to incorporate some physical effects like transportation delays, spatially-distributed parameters, hysteresis nonlinearities, and the like; this often leads to infinite-dimensional systems, which are frequently termed distributed-parameter systems. In particular, the present work considers coupled lumped- and distributed-parameter systems, whose evolution along continuous-time is allowed to be governed by coupled ordinary and/or partial differential equations in general.

In many publications, see, e.g., [2, 3, 6, 8] and references therein, differential geometric methods have already emerged as a (standard and) useful tool for the geometric analysis of finite- and infinite-dimensional systems. Hence, the approaches inherently rely on a proper geometric description of the considered dynamic systems. With regard to the class of coupled infinite-dimensional systems there are often important related aspects neglected resp. can not be captured appropriately, like (distributed and lumped) system inputs, system outputs, boundary conditions, etc. Complementarily, the main issue of this contribution is to emphasize that by some adequate adaptions and extensions of those methods as well as the introduction of appropriate geometric structures a proper geometric description of the considered systems is obtained. Thereby, the dynamic systems are associated with some suitable geometric objects reflecting their dynamics in such a manner that the description is provided in a coordinate-free manner. In addition, the first order Lagrange formalism (with first-order Lagrangian density) is considered for coupled infinite-dimensional systems. It is illustrated that the proposed geometric structures fit well into this theory resp. for the derived systems.

The article is organized as follows. In Section II (coupled) infinite-dimensional dynamic systems of interest are discussed, and a suitable geometric description is provided. In Section III dynamic systems are considered, which are derived by means of the first order Lagrange formalism based upon the geometric structures, which are introduced in Section II. In addition, some examples are arranged to illustrate the proposed theory. Finally, the contribution finishes with some conclusions. The utilized notation and relevant mathematical preliminaries are introduced successively when necessary.

2 Geometry and Dynamic Systems

This contribution applies the concept of smooth manifolds and bundles, see, e.g., [1, 2, 8] for an introduction and much more details. Let $\mathscr{Z}$ and $\mathscr{B}$ be smooth manifolds, then a bundle is a triple $(\mathscr{Z}, \pi_{\mathscr{Z}}, \mathscr{B})$ with the total space $\mathscr{Z}$, the base space $\mathscr{B}$ and the projection (or fibration) $\pi_{\mathscr{Z}} : \mathscr{Z} \to \mathscr{B}$, where $\pi_{\mathscr{Z}}^{-1}(p)$ for any $p \in \mathscr{B}$ denotes the fiber over p. If there is no danger for confusion, a bundle is denoted $\pi_{\mathscr{Z}} : \mathscr{Z} \to \mathscr{B}$ or simply $\mathscr{Z} \to \mathscr{B}$ for short. The manifold $\mathscr{B}$ has the coordinates $\left(Z^i\right)$ and $\mathscr{Z}$ the adapted coordinates (Z^i, z^α). Z^i, $i = 1 \ldots n_Z$ are the independent coordinates and z^α, $\alpha = 1 \ldots n_z$ the dependent coordinates, where the terminology is self-explanatory if the concept of sections is introduced. A section γ of the bundle $\mathscr{Z} \to \mathscr{B}$ is a map $\gamma : \mathscr{B} \to \mathscr{Z}$ such that $\pi \circ \gamma = \mathrm{id}_{\mathscr{B}}$ with the identity map $\mathrm{id}_{\mathscr{B}}$ on $\mathscr{B}$. For brevity, the Einstein summation convention is utilized throughout the article and the ranges of the indices are not always stated explicitly if they are clear form the context. The term $C^\infty(\mathscr{M})$ denotes the set of smooth functions on a manifold $\mathscr{M}$.

In the main part of this section extensive use of two methods is made to construct new bundles from given ones. If $\mathscr{Z} \to \mathscr{B}$ and $\overline{\mathscr{Z}} \to \mathscr{B}$ are bundles over the same base space $\mathscr{B}$, then the fibred product bundle is given by the triple $(\mathscr{Z}\times_{\mathscr{B}}\bar{\mathscr{Z}},\pi_{\mathscr{Z}\times_{\mathscr{B}}\bar{\mathscr{Z}}},\mathscr{B})$, where the total manifold is defined by $\{(a_1,a_2)\in\mathscr{Z}\times\bar{\mathscr{Z}}:\pi_{\mathscr{Z}}(a_1)=\pi_{\bar{\mathscr{Z}}}(a_2)\}$ and the projection by $(\pi_{\mathscr{Z}\times_{\mathscr{B}}\bar{\mathscr{Z}}})(a_1,a_2):=\pi_{\mathscr{Z}}(a_1)=\pi_{\bar{\mathscr{Z}}}(a_2)$. If $\mathscr{N}$ is a manifold and $\rho:\mathscr{N}\to\mathscr{B}$ a map, then the pull-back bundle of $\mathscr{Z}\to\mathscr{B}$, defined by ρ, is the triple $(\rho^*(\mathscr{Z}),\rho^*(\mathscr{Z}),\mathscr{N})$, where the total space is defined by $\{(a_1,a_2)\in\mathscr{N}\times\mathscr{Z}:\rho(a_1)=\pi_{\mathscr{Z}}(a_2)\}$ and the projection by $\rho^*(\pi_{\mathscr{Z}})(a_1,a_2)=a_1$. If ρ is an embedding, the pull-back bundle is also called restricted bundle, see, e.g., [8].

For a section $\gamma:\mathscr{B}\to\mathscr{Z}$ the k^{th}-order partial derivatives are given by

$$\gamma_J^\alpha=\partial_J\gamma^\alpha=\frac{\partial^k}{\partial_1^{j_1}\dots\partial_{n_X}^{j_{n_X}}}\gamma^\alpha,\ \partial_i=\frac{\partial}{\partial Z^i}$$

with the ordered multi-index $J=j_1,\dots,j_{n_X}$, $k=\#J=\sum_{i=1}^{n_X}j_i$ and $\partial_J\gamma^\alpha=\gamma^\alpha$ for $\#J=0$. For brevity $j_i=\delta_{ij}$, $j\in\{1,\dots,n_X\}$ will be denoted as 1_j and $j_i+\delta_{ij}$ as $J+1_j$ with the Kronecker symbol δ. Then, the section γ can be extended to a map $j^n(\gamma)(Z)=(Z^i,\partial_J\gamma^\alpha(Z))$ with $0\leqslant\#J\leqslant n$, which is called the n^{th} jet of γ at Z. The set of n^{th} jets (or n^{th}-order prolongations) of sections $\mathscr{B}\to\mathscr{Z}$ can also be equipped with the structure of manifold $J^n(\mathscr{Z})$, which is called the n^{th} jet manifold and has the adapted coordinates (Z^i,z_J^α), $0\leqslant\#J\leqslant n$ where $x^\alpha=x_J^\alpha$ for $\#J=0$. By means of $J^n(\mathscr{Z})$ the bundles $\pi^n:J^n(\mathscr{Z})\to\mathscr{B};\ (Z^i,z_J^\alpha)\mapsto(Z^i)$ and $\pi_{n-1}^n:J^n(\mathscr{Z})\to\mathscr{Z};\ (Z^i,z_J^\alpha)\mapsto(Z^i,z^\alpha)$ among others can be constructed.

The previous differential geometrical constructs and objects permit to introduce a unifying formal framework for (coupled) distributed-parameter systems, where all governing equations, inputs and outputs are considered in a convenient manner. Hence, first the main underlying structures must be provided. In this work a dynamic system is assumed to be described by different sets of equations, where some of those may have different (spatial) carriers. These domains are denoted by

$$\tilde{\mathscr{D}}_i\ ,\ i=1,\dots,n_{dom}\ , \tag{1}$$

and are supposed to be $n_{\tilde{\mathscr{D}}_i}$-dimensional, compact manifolds ($\forall i\ n_{\tilde{\mathscr{D}}_i}\in\mathbb{N}_0$) with global volume form and the coherently orientable boundaries $\partial\tilde{\mathscr{D}}_i$ (if the dimension is $n_{\tilde{\mathscr{D}}_i}>0$), respectively. Later on the geometric containers for the system equations rely on the geometric structures

$$\Omega_\delta\to\mathscr{T}\ ,\ \delta=1,\dots,n_d\ , \tag{2}$$

where the total spaces are constructed as follows

$$\Omega_\delta=\mathscr{T}\times\mathscr{D}_\delta\ ,\ \mathscr{D}_\delta=\tilde{\mathscr{D}}_i\ ,\ i=1,\dots,n_{dom}\qquad\text{or}\qquad\Omega_\delta=\mathscr{T}\times\mathscr{D}_\delta\ ,\ \mathscr{D}_\delta=\{\}\ , \tag{3}$$

where the domains (1) are applied. Concretely, Ω_δ are the so-called time-space cylinders, which all share the same time manifold $\mathscr{T}=\mathbb{R}$. The (fibred product) manifolds Ω_δ are equipped with a set of coordinates (X^{i_δ}), where X^{1_δ} is equal to the time coordinate[1] and (X^{j_δ}), $j_\delta\to j=2,\dots,n_{X,\delta}$ are the spatial coordinates.

Remark 1 A special index notation is applied here to handle the variable sets for the different manifolds Ω_l. The index set is given by some $I\subset\mathbb{N}^2$, where the corresponding family is defined by $f:I\to\{X^{j_l}\};(j,l)\mapsto X^{j_l}$. In particular, the term X^{j_l} represents variables on Ω_l, where the notation $j_\delta\to j=1,\dots,n_{X,\delta}$ indicates that for fixed $\delta\in[1,\dots,n_d]$, the index j has range $[1,\dots,n_{X,\delta}]$ with $n_{X,\delta}\in\mathbb{N}$.

In general the dynamic systems are assumed to be of the form

$$\begin{aligned}0=&\quad f^{\mu_\delta}(X,x_J,u_K)\quad,\mu_\delta\to\mu=1,\dots,n_{e,\delta}\,,\\ y^{\zeta_\delta}=&\quad g^{\zeta_\delta}(X,x_J,u_K)\quad,\zeta_\delta\to\zeta=1,\dots,n_{y,\delta}\,,\ \delta=1,\dots,n_d\end{aligned} \tag{4}$$

with $\sum_\delta n_{e,\delta}$ system equations, $\sum_\delta n_{y,\delta}$ output equations, and the sets of variables

$$\begin{aligned}X=&\quad(X^{j_\delta})\quad,j_\delta\to j,\dots,n_{X,\delta}\,,\\ x_J=&\quad(x_{J_\delta}^{\alpha_\delta})\quad,\alpha_\delta\to\alpha,\dots,n_{x,\delta}\,,\\ u_K=&\quad(u_{K_\delta}^{\varsigma_\delta})\quad,\varsigma_\delta\to\varsigma,\dots,n_{u,\delta}\,,\end{aligned}$$

where the (sets of) multi-indices,

$$J=(J_\delta)\ ,\quad K=(K_\delta)\ ,\quad ,J_\delta,K_\delta\in\mathbb{N}_0^{n_{X,\delta}}\ ,\quad\#J_\delta\le n\,,\#K_\delta\le n$$

[1] Note that all time coordinates X^{1_δ} on Ω_δ coincide with each other.

are applied. In particular, we follow here the concept of Remark 1, i.e., the index δ of f^{μ_δ} and g^{ζ_δ} indicates the carriers Ω_δ of the equations. Moreover, for simplicity the functions f^{μ_δ}, g^{ζ_δ} are assumed to be smooth[2]. Although this assumption is not often necessary, hence, it simplifies matters since many distinctions of cases can be avoided.

Before a suitable geometric description for dynamic systems of the form (4) is actually provided, some further appropriate bundles are required, which build upon the structures (2). By means of the manifolds Ω_δ the bundles

$$\mathscr{E}_{\mathscr{X},\delta} \to \Omega_\delta\,, \quad \mathscr{E}_{\mathscr{U},\delta} \to \Omega_\delta\,, \tag{5}$$

with $\delta = 1,\ldots,n_d$ can be introduced, where $(X^{i_\delta}, x^{\alpha_\delta})$ and $(X^{i_\delta}, u^{\varsigma_\delta})$ are the (local) coordinates on $\mathscr{E}_{\mathscr{X},\delta}$ and $\mathscr{E}_{\mathscr{U},\delta}$, respectively. Thereby, X^{i_δ}, $i_\delta \to i = 1\ldots n_{X,\delta}$ denote the independent coordinates, and x^{α_δ}, $\alpha_\delta \to \alpha = 1,\ldots,n_{x,\delta}$ as well as u^{ς_δ}, $\varsigma_\delta \to \varsigma = 1\ldots n_{u,\delta}$ the dependent coordinates. Obviously, the previous geometric structures share the same base spaces Ω_δ and, thus, the fibred product bundles

$$\mathscr{E}_\delta = \mathscr{E}_{\mathscr{X},\delta} \times_{\Omega_\delta} \mathscr{E}_{\mathscr{U},\delta} \to \Omega_\delta \tag{6}$$

can be constructed with the help of (5). Since we are interested in a geometric definition of dynamic systems of the form (4) the jet bundles

$$J^n(\mathscr{E}_\delta) \to \Omega_\delta \tag{7}$$

are required, where n^{th} jet manifold $J^n(\mathscr{E}_\delta)$ possesses the coordinates $(X^{i_\delta}, x^{\alpha_\delta}_{J_\delta}, u^{\varsigma_\delta}_{K_\delta})$.

In order to state an suitable container for the system equations (4) the concept of a pull-back bundle is exploited. This procedure permits to incorporate boundary and coupling conditions in a convenient way. Let us first consider the bundle $J^n\left(\mathscr{E}_{\mathscr{X},\delta_1}\right) \to \Omega_{\delta_1}$ with $\delta_1 \in [1,\ldots,n_d]$ and $n \geq 0$. If

$$\iota_{\Omega_{\delta_2},\Omega_{\delta_1}} : \Omega_{\delta_2} \to \Omega_{\delta_1}\,, \quad \delta_1 \neq \delta_2\,, \delta_2 \in [1,\ldots,n_d] \tag{8}$$

is an embedding, then the following pull-back bundle

$$\iota^*_{\Omega_{\delta_2},\Omega_{\delta_1}}\left(J^n\left(\mathscr{E}_{\mathscr{X},\delta_1}\right)\right) \to \Omega_{\delta_2} \tag{9}$$

can be constructed with the (local) coordinates $(X^{\alpha_{\delta_2}}, X^{\alpha_{\delta_1}}, x^{\alpha_{\delta_1}}_{J_{\delta_1}})$ on $\iota^*_{\Omega_{\delta_2},\Omega_{\delta_1}}\left(J^n\left(\mathscr{E}_{\mathscr{X},\delta_1}\right)\right)$, as depicted in figure 1.

$$\begin{tikzcd}
J^n(\mathscr{E}_{\mathscr{X},\delta_1}) \arrow[d, "\pi^n_{\mathscr{E}_{\mathscr{X},\delta_1}}"'] & \iota^*_{\Omega_{\delta_2},\Omega_{\delta_1}}(J^n(\mathscr{E}_{\mathscr{X},\delta_1})) \arrow[d, "\iota^*_{\Omega_{\delta_2},\Omega_{\delta_1}}(\pi^n_{\mathscr{E}_{\mathscr{X},\delta_1}})"'] \\
\Omega_{\delta_1} & \Omega_{\delta_2} \arrow[l, "\iota_{\Omega_{\delta_2},\Omega_{\delta_1}}"']
\end{tikzcd}$$

Figure 1: Pull-back bundle

A section $\gamma_{\Omega_{\delta_2}} : \Omega_{\delta_2} \to \iota^*_{\Omega_{\delta_2},\Omega_{\delta_1}}\left(J^n\left(\mathscr{E}_{\mathscr{X},\delta_1}\right)\right)$ is related to a section $\gamma_{\Omega_{\delta_1}} : \Omega_{\delta_1} \to J^n\left(\mathscr{E}_{\mathscr{X},\delta_1}\right)$ by

$$\gamma_{\Omega_{\delta_2}} = \gamma_{\Omega_{\delta_1}} \circ \iota_{\Omega_{\delta_2},\Omega_{\delta_1}} = \iota^*_{\Omega_{\delta_2},\Omega_{\delta_1}}(\gamma_{\Omega_{\delta_1}})\,, \tag{10}$$

which actually justifies the notation $\iota^*_{\Omega_{\delta_2},\Omega_{\delta_1}}(\cdot)$. Apparently, a more concise notation for the coordinates on $\iota^*_{\Omega_{\delta_2},\Omega_{\delta_1}}\left(J^n\left(\mathscr{E}_{\mathscr{X},\delta_1}\right)\right)$ is

$$(X^{\alpha_{\delta_2}}, \iota^*_{\Omega_{\delta_2},\Omega_{\delta_1}}\left(X^{\alpha_{\delta_1}}\right), \iota^*_{\Omega_{\delta_2},\Omega_{\delta_1}}(x^{\alpha_{\delta_1}}_{J_{\delta_1}}))\,, \tag{11}$$

however, the embeddings are neglected in the work if they are clear from the context.

Example 1 Let us consider a string with normalized physical parameters defined on the domain $\mathscr{D}_1 = [0,1]$, which is described by

$$f^{1_1}(X, x_J) = x^{1_1}_{20} - x^{1_1}_{02} = 0\,,$$

with $f^{1_1}(X,x_J) \in C^\infty\left(J^2(\mathscr{E}_1)\right)$. Thereby, the bundle $\mathscr{E}_1 \to \Omega_1$ is utilized with the coordinates $(X^{1_1}, X^{1_2}, x^{1_1})$, (X^{1_1}, X^{1_2}) on $\mathscr{E}_1$ and $\Omega_1 = \mathscr{T} \times \mathscr{D}_1$, respectively. If the string is clamped at $\mathscr{D}_2 = [0]$, apparently, it is not suitable to set $f^{2_1} = \iota^*_{\Omega_{\delta_2},\Omega_{\delta_1}}(x^{1_1}_{00}) = 0$ with f^{2_1} being an element of $C^\infty(J^2(\mathscr{E}_1))$. Hence, a consistent alternative is

$$\iota^*_{\Omega_2,\Omega_1}(x^{1_1}_{00}) = f^{1_2}(X, x_J) \in C^\infty\left(\iota^*_{\Omega_2,\Omega_1}\left(J^2(\mathscr{E}_1)\right)\right)\ .$$

[2]Note that the solutions do not have to be smooth.

For the considered problems pull-back bundle structures are essential and even required to consider interactions between different subsystems. Following the previous concepts the special manifolds

$$\mathscr{M}^n_\delta = J^n(\mathscr{E}_\delta) \times \iota^{\times,*}_{\Omega_{\breve{\delta}}}(J^n(\mathscr{E})) \ , \tag{12}$$

as well as the bundle structures

$$\mathscr{M}^n_\delta \to \mathscr{M}^{n-1}_\delta \to \ldots \to \mathscr{M}^0_\delta \to \Omega_\delta \tag{13}$$

can be introduced with

$$\iota^{\times,*}_{\Omega_{\breve{\delta}}}(\cdot) = \bigtimes_{\breve{\delta}} \iota^*_{\Omega_\delta,\Omega_{\breve{\delta}}}\left((\cdot)_{\breve{\delta}}\right) \quad \forall \breve{\delta} \in [1,\ldots,n_d] \ ,$$

and

$$\iota_{\Omega_\delta,\Omega_{\breve{\delta}}} : \Omega_l \to \Omega_{\breve{\delta}} \text{ being an embedding.} \tag{14}$$

Analogously, in (12) we have

$$\mathscr{M}^n_\delta = J^n(\mathscr{E}_\delta) \times \iota^{\times,*}_{\Omega_{\breve{\delta}}}(J^n(\mathscr{E})) = J^n(\mathscr{E}_\delta) \times \bigtimes_{\breve{\delta}} \iota^*_{\Omega_\delta,\Omega_{\breve{\delta}}}\left(J^n\left(\mathscr{E}_{\breve{\delta}}\right)\right) \ .$$

With the manifolds $\mathscr{M}^n_\delta$ at our disposal we are able to give a geometric definition of a so-called jet system. Neither it requires (local) coordinates nor equations since a jet system is defined as an intrinsic geometric object, namely a family of submanifolds.

Definition 1 *A* jet system *of order n is a family of regular fibred submanifolds $S^n_\delta \subset \mathscr{M}^n_\delta$ of $\mathscr{M}^n_\delta$.*

Remark 2 In comparison to, e.g., [2, 3, 8], a jet system resp. a (system of) differential equation(s) is not defined as a regular fibred submanifold of a single jet bundle. This is due to the fact that we consider here the class of coupled lumped and distributed parameter systems, where, in particular, the aspects of boundary and coupling conditions, different validity areas, (lumped and distributed) system inputs and outputs, etc. are taken into account.

Definition (1) enables us now to provide an adequate container for the system equations (4), which are assumed to define a jet system, i.e., a jet system is described by systems of equations (locally).

Theorem 1 *A (coupled) dynamic system of the form (4) is a jet system if for any $\delta \in \{1,\ldots,n_e\}$ the set of system equations $f^{\mu_\delta}(X, x_J, u_K) = 0$ valid on Ω_δ describes a (locally) regular fibred submanifold $S^n_\delta \subset \mathscr{M}^n_\delta$ of $\mathscr{M}^n_\delta$.*

Beside the geometric container for the system equations (4) the output equations are supposed to satisfy

$$y^{\zeta_\delta} = g^{\zeta_\delta}(X, x_J, u_K) \in C^\infty\left(\mathscr{M}^n_\delta\right) \ .$$

From now on the notions of dynamic systems and jet systems are used synonymously, even though they represent two different mathematical objects. A jet system states the family of submanifolds S^n_δ, whereas a dynamic system is a local representation of S^n_δ. It is worth mentioning that obviously both a jet system and a dynamic system of the form (4) cannot possess a unique (local) representation.

Let us apply the theory to an example in order to illustrate the proposed methods.

Example 2 Suppose a flexible mechanical system as depicted in Figure (2). Basically it represents gantry crane with a heavy chain and a horizontally-movable carriage. For the modeling we assume that the displacements of the chain are small, the chain length L, the chain force $P(X^{2_1}) = g\left(\rho\left(L - X^{2_1}\right)\right)$ for $0 \le X^{2_1} \le L$, the density ρ and the carriage mass m_c; any rotary inertia of the chain is neglected. A possible domain structure is given by $\mathscr{D}_1 = [0,L]$, $\mathscr{D}_2 = [L]$, $\mathscr{D}_3 = [0]$, wherewith the bundles[3] $\mathscr{E}_1 \to \Omega_1$, $\mathscr{E}_2 \to \Omega_2$ can be introduced with the (local) coordinates $\left(X^{1_1}, X^{2_1}\right) \in \Omega_1$, $\left(X^{1_2}\right) \in \Omega_2$, $\left(X^{1_3}\right) \in \Omega_3$ and $\left(X^{1_1}, X^{2_1}, x^{1_1}\right) \in \mathscr{E}_1$, $\left(X^{1_2}, x^{1_2}, u^{1_2}, y^{1_2}, y^{2_2}\right) \in \mathscr{E}_2$. Then,

[3]For this example a bundle $\mathscr{E}_3 \to \Omega_3$ is obviously not required.

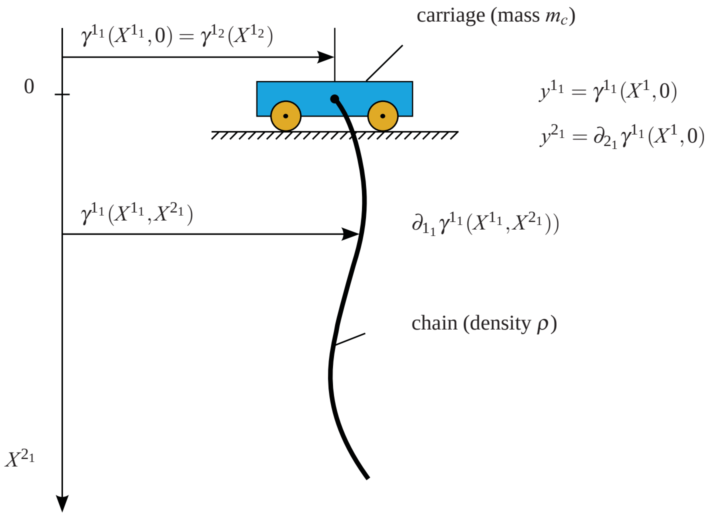

Figure 2: Schematic diagram of gantry crane system

aligned to [4] a mathematical model for the system in the form (4) reads as[4]

$$
\begin{aligned}
0 &= \quad f^{1_1}(X,x_J,u_K) = \rho x_{20}^{1_1} + g\rho x_{01}^{1_1} - g(\rho(L - X^{2_1}))x_{02}^{1_1}\,,\\
0 &= \quad f^{1_2}(X,x_J,u_K) = m_c x_{20}^{1_2} - g\rho L \iota^*_{\Omega_2,\Omega_1}(x_{01}^{1_1}) - u^{1_2}\,,\\
0 &= \quad f^{2_2}(X,x_J,u_K) = \iota^*_{\Omega_2,\Omega_1}(x_{00}^{1_1}) - x_0^{1_2}\,,\\
0 &= \quad f^{3_2}(X,x_J,u_K) = \iota^*_{\Omega_2,\Omega_1}(x_{10}^{1_1}) - x_1^{1_2}\,,\\
0 &= \quad f^{1_3}(X,x_J,u_K) = \iota^*_{\Omega_3,\Omega_1}(x_{01}^{1_1})\,,\\
y^{1_2} &= \quad g^{1_2}(X,x_J,u_K) = \iota^*_{\Omega_2,\Omega_1}(x_{00}^{1_1})\,,\\
y^{2_2} &= \quad g^{2_2}(X,x_J,u_K) = \iota^*_{\Omega_2,\Omega_1}(x_{01}^{1_1})\,,
\end{aligned}
$$

where a suitable geometric container for the system equations is given by the following family of submanifolds

$$
\begin{aligned}
S_1^2 &:= \left\{p \in \mathscr{M}_1^2 : f^{1_1}(p) = 0\right\} &&\subset \mathscr{M}_1^2 = J^2(\mathscr{E}_1)\,,\\
S_2^2 &:= \left\{p \in \mathscr{M}_2^2 : f^{1_2}(p) = f^{2_2}(p) = f^{3_2}(p) = 0\right\} &&\subset \mathscr{M}_2^2 = J^2(\mathscr{E}_2) \times \iota^*_{\Omega_2,\Omega_1}\left(J^2(\mathscr{E}_1)\right)\,,\\
S_3^2 &:= \left\{p \in \mathscr{M}_3^2 : f^{1_3}(p) = 0\right\} &&\subset \mathscr{M}_3^2 = J^2(\mathscr{E}_3) \times \iota^*_{\Omega_3,\Omega_1}\left(J^2(\mathscr{E}_1)\right)\,.
\end{aligned}
$$

$$
\begin{tikzcd}
\mathscr{M}_2^2 \arrow[d, phantom, "||"] & \mathscr{M}_1^2 \arrow[d, phantom, "||"] & \mathscr{M}_3^2 \arrow[d, phantom, "||"] \\
J^2(\mathscr{E}_2) \times \iota^*_{\Omega_2,\Omega_1}(J^2(\mathscr{E}_1)) \arrow[d, "\pi^2_{\mathscr{E}_2} \times \iota^*_{\Omega_2,\Omega_1}(\pi^2_{\mathscr{E}_1})"'] & J^2(\mathscr{E}_1) \arrow[d, "\pi^2_{\mathscr{E}_1}"'] & \iota^*_{\Omega_2,\Omega_1}(J^2(\mathscr{E}_1)) \arrow[d, "\iota^*_{\Omega_3,\Omega_1}(\pi^2_{\mathscr{E}_1})"] \\
\Omega_2 \arrow[r, "\iota_{\Omega_2,\Omega_1}"] & \Omega_1 & \Omega_3 \arrow[l, "\iota_{\Omega_3,\Omega_1}"']
\end{tikzcd}
$$

Figure 3: Geometric structures for beam system

Henceforth, for simplicity and legibility the following product bundles are constructed and applied throughout the work, namely

$$\mathscr{E}_{\mathscr{X}} \to \Omega\,,\ \mathscr{E}_{\mathscr{U}} \to \Omega\,,\ \mathscr{E} \to \Omega \tag{15}$$

with $\mathscr{E}_{\mathscr{X}} = \times_\delta\ \mathscr{E}_{\mathscr{X},\delta}$, $\mathscr{E}_{\mathscr{U}} = \times_\delta\ \mathscr{E}_{\mathscr{U},\delta}$, $\mathscr{E} = \times_\delta\ \mathscr{E}_\delta$ as well as $\Omega = \times_\delta\ \Omega_\delta$, $l = 1,\ldots,n_d$.

[4]The embeddings should actually be clear from the context and are only stated here for illustration.

Finally, we need of course the notion of a solution, which is represented by a section in the presented geometric framework. Due to the previous bundle constructions a solution of the system (4) is a section $(\gamma,\mu):\Omega\to\mathscr{E}$ with the trajectory $\gamma:\Omega\to\mathscr{E}_{\mathscr{X}}$ and the input $\mu:\Omega\to\mathscr{E}_{\mathscr{U}}$, which satisfies the equations

$$0 = f^{\mu_\delta}\left(X,x_J,u_K\right)\circ j^n\left(\gamma,\mu\right)\left(X\right). \tag{16}$$

The associated outputs follow as

$$y^{\zeta_\delta}\circ j^n\left(\mu,\gamma\right)\left(X\right) = g^{\zeta_\delta}\circ j^n\left(\gamma,\mu\right)\left(X\right). \tag{17}$$

Topics like solvability and well-posedness of solutions are not addressed within this contribution. Here, the equations are considered from a theoretical point of view, and some geometric definition and description are provided. Hence, closely related with the notion of a jet system is the requirement that there are no further, more restricting, equations, which are hidden behind the system equations. This leads us directly to the concept of *local solvability*.

Definition 2 *A dynamic system of the form (4) or a jet system is* locally solvable *if for any point $p_0\in S^n_\delta$, $\delta=1,\ldots,n_d$ there is at least one (local) solution $(\gamma,\mu):\Omega\to\mathscr{E}_{\mathscr{X}}$ of the system.*

Geometrically, the local solvability of a dynamic system requires that there are no further integrability conditions. In general those can be found by suitable prolongation and projection of the system equations, see, e.g., [9, 3], and may arise due to cross-derivatives and the prolongation of lower-order equations.

3 Lagrange Formalism

Next, the investigations are focused on free (coupled) dynamic systems, which are derived by means of the first order Lagrangian formalism, where problems are described by first-order Lagrangian densities

$$l_\lambda\,\mathrm{dX}_\lambda \qquad \text{with} \qquad l_\lambda\in C^\infty(J^1(\mathscr{E}_{\mathscr{X},\lambda}))\,,\quad \lambda=1,\ldots,n_l \tag{18}$$

and the volume form $\mathrm{dX}_\lambda=\bigwedge_{i=1}^{n_{X,\lambda}}\mathrm{d}X^{i_\lambda}$ on Ω_λ. The theory can be extended to higher order as well, hence ambiguities arise, see, e.g., [2], which are omitted here for simplicity. The final result is stated by sets of system equations including boundary and coupling conditions, which can also be captured under certain conditions by the geometric picture of dynamic systems as introduced in Section II.

The tangent and cotangent bundle with respect to a smooth manifold $\mathscr{B}$ are denoted by $\mathscr{T}(\mathscr{B})\to\mathscr{B}$ and $\mathscr{T}^*(\mathscr{B})\to\mathscr{B}$, respectively, which are equipped with the induced coordinates $(Z^i,\dot{Z}^i)$ and $(Z^i,\dot{Z}_i)$ with respect to the holonomic bases $\{\partial_i\}$ and $\{\mathrm{d}Z^i\}$. The exterior algebra is denoted by $\wedge(\mathscr{T}^*(\mathscr{B}))$ with the exterior derivative $\mathrm{d}:\wedge_k(\mathscr{T}^*(\mathscr{B}))\to\wedge_{k+1}(\mathscr{T}^*(\mathscr{B}))$, the interior product $\rfloor:\wedge_{k+1}(\mathscr{T}^*(\mathscr{B}))\to\wedge_k(\mathscr{T}^*(\mathscr{B}))$ written as $v\rfloor\omega$, $v:\mathscr{Z}\to\mathscr{T}(\mathscr{B})$ and $\omega:\mathscr{B}\to\wedge^{k+1}(\mathscr{T}^*(\mathscr{B}))$, and the exterior product $\wedge$, where $\wedge^k(\mathscr{T}^*(\mathscr{Z}))\to\mathscr{Z}$ is the exterior k-form bundle on $\mathscr{Z}$. The Lie derivative of a form $\omega:\mathscr{B}\to\wedge(\mathscr{T}^*(\mathscr{B}))$ along a vector field $f:\mathscr{B}\to\mathscr{T}(\mathscr{B})$ is given by $f(\omega)$.

Let us consider the same bundle $\pi_{\mathscr{Z}}:\mathscr{Z}\to\mathscr{B}$ as used at the beginning of Section II. Then, the vertical vector bundle $\mathscr{V}(\mathscr{Z})\to\mathscr{Z}$, which is a subbundle of $\mathscr{T}(\mathscr{Z})\to\mathscr{Z}$, consists of all fields $v:\mathscr{Z}\to\mathscr{T}(\mathscr{Z})$, which satisfy $\pi_*(v)=0$. A π-vertical vector field locally generates a fiber-preserving bundle automorphism

$$(f_{\mathscr{B}},f_{\mathscr{Z}})=(\mathrm{id}_{\mathscr{B}},\exp(\varepsilon v))):\mathscr{B}\times\mathscr{Z}\to\mathscr{B}\times\mathscr{Z}\,, \tag{19}$$

see, e.g., [8]. Furthermore, instead of prolonging the automorphism to $J^n(\mathscr{Z})$, its infinitesimal generator can be prolonged by the following general formula,

$$j^n v = d_J(v^\alpha)\partial_\alpha^J\,,\quad 0\leqslant \#J\leqslant n\,,\quad v=v^\alpha\partial_\alpha\,,\quad d_J=(d_1)^{j_1}\circ\cdots\circ(d_{n_X})^{j_{n_Z}} \tag{20}$$

where $d_i:J^n(\mathscr{Z})\to\pi^{n,*}_{n-1}(\mathscr{T}(J^{n-1}(\mathscr{Z})))$ and $d_J(v^\alpha)=v^\alpha$ for $\#J=0$. The operator d_i, which meets $(d_if)\circ j^{n+1}\gamma=\partial_i f(j^n\gamma)$ for all $f\in C^\infty(J^n(\mathscr{Z}))$ and $\gamma:\mathscr{B}\to\mathscr{Z}$, is called the total derivative with respect to the independent variable Z^i. It is defined by $d_i=\partial_i+z^\alpha_{J+1_i}\partial^J_\alpha$, $\partial^J_\alpha=\frac{\partial}{\partial z^\alpha_J}$ in adapted coordinates (Z^i,z^α_J).

Based on the introduced bundle structures of Section II the Lagrangian densities (18) are applied to define the Lagrangian functional $\mathscr{L}:\Gamma(\mathscr{E}_{\mathscr{X}})\to\mathbb{R}$, which reads as

$$\mathscr{L}(\gamma)=\sum_{\lambda=1}^{n_l}\int_{\Omega_\lambda}(j^1\gamma)^*(l_\lambda\,\mathrm{dX}_\lambda) \tag{21}$$

with $\Gamma(\mathscr{E}_{\mathscr{X}})$ being the set of sections $\gamma:\Omega\to\mathscr{E}_{\mathscr{X}}$. If admitting only fiber preserving variations a section $\gamma:\Omega\to\mathscr{E}_{\mathscr{X}}$ will be called extremal, iff

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathscr{L}(\phi_\varepsilon\circ\gamma)\bigg|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathscr{L}(\exp(\varepsilon v_{\mathscr{L}})\gamma)\bigg|_{\varepsilon=0} = \sum_{\lambda=1}^{n_l}\int_{\Omega_\lambda}(j^1\gamma)^*(j^1 v_{\mathscr{L}}(l_\lambda\,\mathrm{dX}_\lambda))=0 \tag{22}$$

is met for any valid variational field $v_{\mathscr{L}} : \mathscr{E}_{\mathscr{X}} \to \mathscr{V}(\mathscr{E}_{\mathscr{X}})$ with the flow $\phi : \mathbb{R} \times \mathscr{E}_{\mathscr{X}} \to \mathscr{E}_{\mathscr{X}}$ and the shortcut $\phi_\varepsilon = \phi(\varepsilon, \cdot) : \mathscr{E}_{\mathscr{X}} \to \mathscr{E}_{\mathscr{X}}$. Since only fiber preserving variations are permitted, the field $v_{\mathscr{L}}$ represents a vertical vector field and, therefore, the bundle automorphism (19) is utilized. The reader should note that due to the underlying geometric structures actually a set of vector fields

$$v_{\mathscr{L}} = (v_{\mathscr{L},\lambda}) \qquad \text{with} \qquad v_{\mathscr{L},\lambda} : \mathscr{E}_{\mathscr{X},\lambda} \to \mathscr{V}(\mathscr{E}_{\mathscr{X},\lambda})$$

and a set of flows

$$\phi_\varepsilon = (\phi_{\varepsilon,\lambda}) \qquad \text{with} \qquad \phi_{\varepsilon,\lambda} : \mathscr{E}_{\mathscr{X},\lambda} \to \mathscr{E}_{\mathscr{X},\lambda}$$

are considered. By using the relation $j^1 v_{\mathscr{L}}(\omega_\lambda) = j^1 v_{\mathscr{L}} \rfloor \mathrm{d}(\omega_\lambda) + \mathrm{d}(j^1 v_{\mathscr{L}} \rfloor \omega_\lambda)$ for $\omega_\lambda : \Omega_\lambda \to \pi^{n,*}(\wedge(\mathscr{T}^*(\Omega_\lambda)))$ and Stokes' Theorem, see, e.g., [1], the functional (22) can be rewritten in the form

$$\underbrace{\sum_{\lambda=1}^{n_l} \int_{\Omega_\lambda} (j^1\gamma)^*(j^1 v_{\mathscr{L}} \rfloor \mathrm{d}l_\lambda \wedge \mathrm{dX}_\lambda)}_{=\mathscr{I}_1(\gamma)=0} + \underbrace{\sum_{\lambda=1}^{n_l} \int_{\partial\Omega_\lambda} \iota^*_{\Omega_\lambda}(j^1\gamma)^*(j^1 v_{\mathscr{L}} \rfloor l_\lambda \mathrm{dX}_\lambda)}_{=\mathscr{I}_2(\gamma)=0} = 0 \tag{23}$$

with the inclusions $\iota_{\Omega_\lambda} : \partial\Omega_\lambda \to \Omega_\lambda$. The term $\mathscr{I}_2(\gamma)$ vanishes since $j^1 v_{\mathscr{L}} \rfloor l\mathrm{dX} = 0$ due to the use of a vertical vector field $v_{\mathscr{L}}$. Hence, it may happen that the remaining term $\mathscr{I}_1(\gamma)$ is not appropriate in the current form since $(j^1\gamma)^*(j^1 v_{\mathscr{L}} \rfloor \mathrm{d}l_\lambda \wedge \mathrm{dX}_\lambda)$ can depend on the (spatial) derivatives of components of the vector field $v_{\mathscr{L}}$ and, thus, can contribute to the so far vanishing boundary term $\mathscr{I}_2(\gamma)$. Fortunately, the variational problem can be replaced with the variational problem for so-called Lepegian equivalents $\rho_{L,\lambda}$, which satisfy $h_0(\rho_{L,\lambda}) = l_\lambda \mathrm{dX}_\lambda$ with the horizontal projection $h_0 : \wedge^1 J^1(\mathscr{E}_{\mathscr{X}}) \to \wedge^{1,0} J^1(\mathscr{E}_{\mathscr{X}})$, see, e.g., [2]. The attention is especially restricted here to the so-called Poincare-Cartan forms $c_\lambda = l_\lambda \mathrm{dX}_\lambda + \partial^{1_i}_{\alpha_\lambda} l_\lambda (\mathrm{d}x^{\alpha_\lambda} - x^{\alpha_\lambda}_{1_i} \mathrm{d}X^{i_\lambda})$. The conditions[5] on the domains Ω_λ and on the (spatial) boundaries $\mathscr{T} \times \partial\mathscr{D}_\lambda$ follow as

$$\sum_{\lambda=1}^{n_l} \int_{\Omega_\lambda} (j^2\gamma)^*(v_{\mathscr{L}} \rfloor (\delta_{\alpha_\lambda}(l_\lambda)\mathrm{d}x^{\alpha_\lambda} \wedge \mathrm{dX}_\lambda) + \sum_{\lambda=1}^{n_l} \int_{\mathscr{T}\times\partial\mathscr{D}_\lambda} \iota^*_{\Omega_\lambda}(j^1\gamma)^*(j^1 v_{\mathscr{L}} \rfloor (\partial^{1_i}_{\alpha_\lambda} l_\lambda \mathrm{d}x^{\alpha_\lambda} \wedge \partial_{i_\lambda} \rfloor \mathrm{dX}_\lambda)) = 0 \tag{24}$$

with the Euler-Lagrange operator $\delta_{\alpha_\lambda} : \partial_{\alpha_\lambda} + d_{i_\lambda} \partial^{1_i}_{\alpha_\lambda}$.

It is worth noting that more specific conditions may be implied by physics or the concrete choice. However, these conditions have to fulfill equation (24) in any case. Thus, it will be incumbent upon the modeler to set suitable and valid conditions. Finally, by a suitable choice of boundary and coupling conditions and by appropriate extraction of the conditions one yields[6]

$$f^{\tau_\lambda}(X,x) = 0 \qquad \text{or} \qquad x^{\alpha_\lambda} = \sigma^{\alpha_\lambda}(X^{i_\lambda}) \qquad \text{on} \qquad \Omega_\lambda\,, \tag{25}$$

where the right-hand side results by the explicit selection of functions $\sigma^{\alpha_\lambda}(X^{i_\lambda})$. Moreover, similarly, conditions of the form

$$f^{\tau_{\bar\theta}}(X,x) = 0 \qquad \text{or} \qquad \iota^*_{\Omega_{\bar\theta},\Omega_\lambda} = \bar\sigma^{\alpha_\lambda}(\iota^*_{\Omega_\lambda}(X^{i_\lambda})) \qquad \text{on} \qquad \Omega_\theta\,, \tag{26}$$

with $\Omega_\theta \subset \mathscr{T} \times \partial\mathscr{D}_\lambda$ for some $\lambda \in [1,\ldots,n_l]$, are gathered. Here, it is worth mentioning that the appearing embeddings exhibit a form

$$\iota_{\Omega_{\bar\theta},\Omega_\lambda} = \iota_{\partial\Omega_\lambda,\Omega_\lambda}\big|_{\Omega_\theta} = \iota_{\Omega_\lambda}\big|_{\Omega_\theta} : \Omega_\theta \to \Omega_\lambda$$

in comparison to the general case (14).

Apparently, the equations (25) and (26) can be written in the form (4) and if they obey Theorem 1, the equations are a (local) representation of a jet system. Thus, the first-order Lagrange formalism for (coupled) dynamic systems is conform with the geometric structures introduced in the preceding section.

Example 3 Let us consider the same flexible mechanical structure as in Example 2, hence, where the force F is neglected. Then, the Lagrangian functional is obtained by taking the difference of kinetic and potential energy,

$$\mathscr{L}(\gamma) = \int_{\Omega_1} (j^1\gamma)^*(\underbrace{\left(\frac{\rho}{2}(x^{1_1}_{10})^2 - \frac{P(X^{2_1})}{2}(x^{1_1}_{01})^2\right)\mathrm{d}X^{1_1} \wedge \mathrm{d}X^{2_1}}_{l_1\mathrm{dX}_1}) + \int_{\Omega_2} (j^1\gamma)^*(\underbrace{\frac{m_c}{2}(x^{1_1}_{10})^2 \mathrm{d}X^{1_2}}_{l_2\mathrm{dX}_2}).$$

In accordance to (24) we yield the condition

$$\int_{\Omega_1} (j^2\gamma)^*(\delta_{1_1}(l_1)\mathrm{dX}_1) + \int_{\mathscr{T}\times\partial\mathscr{D}_1} \iota^*_{\Omega_1}(j^1\gamma)^*(j^1 v_{\mathscr{L}} \rfloor (\underbrace{\partial^{1_2}_{1_1} l_1 \mathrm{d}x^{1_1} \wedge \partial_{2_1} \rfloor \mathrm{dX}_1}_{-g\rho L x^{1_1}_{01}\mathrm{d}x^{1_1}\wedge \mathrm{d}X^{2_1}})) + \int_{\Omega_2} (j^2\gamma)^*(\delta_{1_2}(l_2)\mathrm{dX}_2) = 0\,.$$

[5] Note that there is no variation permitted on the time boundaries $\partial\mathscr{T} \times \mathscr{D}_\lambda$.

[6] The well-known second-order Euler-Lagrange equations $\delta_{\alpha_\lambda}(l_\lambda) = \partial_{\alpha_\lambda} l_\lambda + d_{i_\lambda}(\partial^{1_i}_{\alpha_\lambda} l_\lambda) = 0$ may appear as part of the equations (25).

Then, by applying

$$\iota_{\Omega_2,\Omega_1} = \iota_{\partial\Omega_1,\Omega_1}\big|_{\Omega_2} = \iota_{\Omega_1}\big|_{\Omega_2} = \Omega_2 \to \Omega_1; (X^{1_2}) \mapsto (X^{1_1} = X^{1_2}, X^{2_1} = 0)$$

and the selection of valid and reasonable boundary and coupling conditions the mathematical model is gathered, which reads as

$$\begin{aligned}
0 = \quad & f^{1_1}(X,x_J) = \delta_{1_1}(l_1) = \rho x^{1_1}_{20} + g\rho x^{1_1}_{01} - g(\rho(L - X^{2_1}))x^{1_1}_{02}\,, \\
0 = \quad & f^{1_2}(X,x_J) = \delta_{1_2}(l_2) - g\rho L\iota^*_{\Omega_2,\Omega_1}(x^{1_1}_{01}) = m_c x^{1_2}_{20} - g\rho L\iota^*_{\Omega_2,\Omega_1}(x^{1_1}_{01})\,, \\
0 = \quad & f^{2_2}(X,x_J) = \iota^*_{\Omega_2,\Omega_1}(x^{1_1}_{00}) - x^{1_2}_0\,, \\
0 = \quad & f^{3_2}(X,x_J) = \iota^*_{\Omega_2,\Omega_1}(x^{1_1}_{10}) - x^{1_2}_1\,, \\
0 = \quad & f^{1_3}(X,x_J) = \iota^*_{\Omega_3,\Omega_1}(x^{1_1}_{01})\,.
\end{aligned}$$

Obviously, the previous equations are of the form (4) and, moreover, they represent a jet system according to Definition 1 and Theorem 1.

4 Conclusions

In this contribution a possible geometric description for the class of coupled lumped- and distributed-parameter systems was proposed, which involves and incorporates important aspects like boundary and coupling conditions, system inputs and outputs, etc. Thereby, intrinsic geometric objects are associated with dynamic systems, which reflect their corresponding dynamics. In addition, it was outlined that systems derived by means of the first-order Lagrange formalism are captured by this geometric picture under certain conditions. It is worth mentioning that such a covariant container for systems equations is very gainful for a control- and system-theoretical analysis to study certain qualitative properties of systems, see, e.g., [4, 5, 7].

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