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HANKEL-NORM-BASED LUMPING OF INTERCONNECTED LINEAR SYSTEMS

H. Sandberg[1]
[1] Royal Institute of Technology (KTH), Sweden

Corresponding author: H. Sandberg, Automatic Control Laboratory, School of Electrical Engineering, Royal Institute of Technology (KTH), Osquldas väg 10, SE-100 44 Stockholm, Sweden; hsan@kth.se

Abstract. In this paper, we consider the problem of how and when to lump subsystems together in large linear interconnected systems with external inputs and outputs. The motivation for this work is that we often want to reduce the model order of large interconnected systems, but before we do that we should identify what interconnection structures that are worth preserving in the reduction. For this purpose a constrained Hankel-norm is introduced in this paper, and a so-called lumping index is derived from it. A subsystem that is not a very independent subsystem in the interconnected system has a large lumping index, and we argue in the paper that it then is a good candidate for lumping. As an example, a large mechanical spring-mass system is considered.

1 Introduction

In this paper, a quantitative criterion for lumping of linear interconnected systems is introduced. Let us first illustrate the problem with an example. Consider an interconnection of linear systems, such as the one depicted in Figure 1. This system consists of six subsystems, $G_1,\ldots,G_6$, and it is being excited by the external signals w_1,w_2,w_3, and the signals z_1,z_2,z_3 are measurements on the system. We want to come up with a rationale for when to lump some of these subsystems together, in order to simplify its representation. Lumps are indicated with bar notation in the figure. There are good reasons for lumping subsystems together, if possible. The interconnection structure generally becomes less complicated for a lumped system, and it can be easier to understand and analyze the overall system behavior. Also, if we want to apply structure-preserving model reduction on the interconnected system, see for example [7, 3, 4], then the fewer structure constraints there are, the more the model order can be reduced.

Lumping is also frequently used for model reduction in chemical reaction systems, see for example [2]. The method we suggest for lumping in this paper is tailored to use together with the model reduction methods in [7, 4]. The method is based on a constrained version of the Hankel-norm. The Hankel-norm is an often used norm in model reduction, see [9]. It has also been used for other purposes, see for example [8], where it is used for input-output pairing in controller design.

A motivation for this paper is that many models that are of interest to the control community have a network structure, see [1]. Examples include models of the power grid, biological systems, formations of vehicles, but also control systems where controllers, actuators, and sensors are distributed over a computer network. In all of these examples there can be many subsystems that are interconnected in one way or another, and the order of the entire system can be very large. It is often desirable to obtain a model with simpler structure, to simplify analysis of the system and controller synthesis.

The structure of the paper is as follows. In Section 2, the model framework is introduced. A known method for structured model reduction is described, and lumping within the used model framework is presented. In Section 3, a constrained Hankel-norm is introduced and a so-called "lumping index" is introduced. Finally, in Section 4 the use of the norm and the index is illustrated on two numerical examples.

Notation. Most notation in the paper is standard notation from the robust control literature, see [9], for example. For realizations of rational transfer function matrices $G(s)$ we use the notation $G(s)=C(sI-A)^{-1}B+D=:\left[\begin{array}{c|c}A & B\\ \hline C & D\end{array}\right]$, or alternatively $G(s)=C(sI-A)^{-1}B+D=:[A,B,C,D]$. The set RH_∞ is the set of real and rational transfer function matrices in the Hardy space H_∞, see [9]. Let $\|G\|_\infty$ denote the H_∞-norm of $G(s)$:

$$\|G\|_\infty := \sup_{s\in\mathbb{C}_+} \|G(s)\|,$$

where $\|G(s)\|$ is the induced Euclidean norm of $G(s)$ (the largest singular value), and $\mathbb{C}_+$ is the open right complex half plane. By $\|u\|_{2,[a,b]}$ we mean the L_2-norm of u over the interval $[a,b]$. With $P>0$ ($P<0$) we mean that P is a positive (negative) definite matrix, with $|x|_P$ the weighted Euclidean norm $\sqrt{x^TPx}$, and with $\mathrm{diag}\{P_1,P_2\}$ the block-diagonal matrix $\begin{bmatrix}P_1 & 0\\ 0 & P_2\end{bmatrix}$.

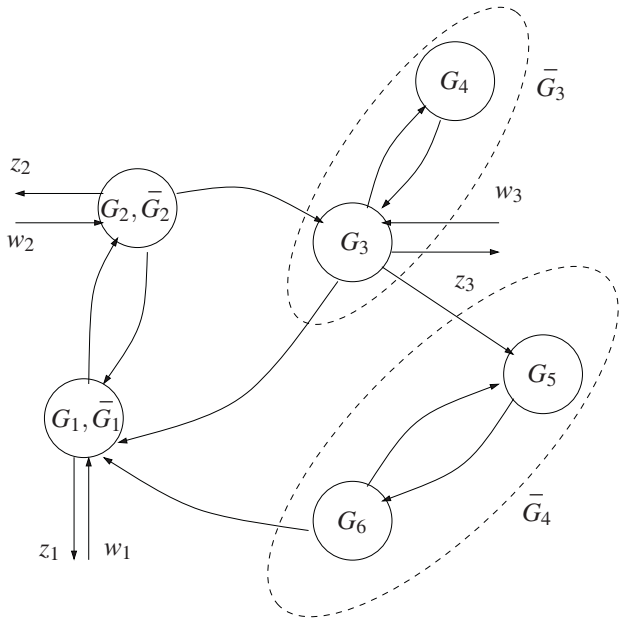

Figure 1: An example of an interconnected linear system with subsystems G_i, $i = 1,\ldots,6$. We want to know when to lump subsystems together based on how the system is interconnected, excited, and measured. Lumps are indicated with bar notation, such as $\bar{G}_3, \bar{G}_4$. Lumps can contain only one subsystem, such as $G_1, \bar{G}_2$.

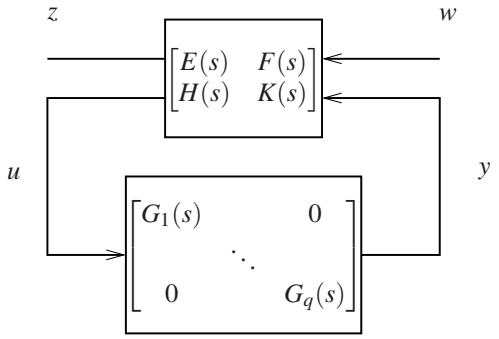

Figure 2: A general linear interconnected system modelled by a linear fractional transformation $\mathscr{F}_l(N,G)$.

2 Model framework, model reduction, and lumping

The same model framework as in [4, 5] is used here, and we repeat some definitions and results without proof. We model interconnected linear systems in the frequency domain using the linear fractional transform $\mathscr{F}_l(N,G)$, where the interconnection topology and dynamics is modelled by N, and the subsystem dynamics in G, see Figure 2 and the following equations:

$$\mathscr{F}_l(N,G)(s) = E(s) + F(s)(I - G(s)K(s))^{-1}G(s)H(s) \tag{1}$$

$$= \left[\begin{array}{cc|c} A_N + B_{N,2}I_{GK}D_GC_{N,2} & B_{N,2}I_{GK}C_G & B_{N,1} + B_{N,2}I_{GK}D_GD_H \\ B_GI_{KG}C_{N,2} & A_G + B_GI_{KG}D_KC_G & B_GI_{KG}D_H \\ \hline C_{N,1} + D_FD_GI_{KG}C_{N,2} & D_FI_{GK}C_G & D_E + D_FD_GI_{KG}D_H \end{array}\right] =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \tag{2}$$

$$I_{GK} := (I - D_GD_K)^{-1}, \quad I_{KG} := (I - D_KD_G)^{-1}.$$

The realization (2) is called a *structured realization* of $\mathscr{F}_l(N,G)$. The q subsystems are stored in the block-diagonal transfer function matrix

$$G(s) = \operatorname{diag}\{G_1(s),\ldots,G_q(s)\} =: \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right] \tag{3}$$

where

$$A_G = \operatorname{diag}\{A_1,\ldots,A_q\}, \quad B_G = \operatorname{diag}\{B_1,\ldots,B_q\},$$
$$C_G = \operatorname{diag}\{C_1,\ldots,C_q\}, \quad D_G = \operatorname{diag}\{D_1,\ldots,D_q\},$$

and

$$A_k \in \mathbb{R}^{n_k \times n_k}, \qquad B_k \in \mathbb{R}^{n_k \times m_k},$$
$$C_k \in \mathbb{R}^{p_k \times n_k}, \qquad D_k \in \mathbb{R}^{p_k \times m_k}, \quad k = 1,\ldots,q.$$

The interconnection topology and dynamics is modelled by

$$N(s) = \begin{bmatrix} E(s) & F(s) \\ H(s) & K(s) \end{bmatrix} =: \left[\begin{array}{c|cc} A_N & B_{N,1} & B_{N,2} \\ \hline C_{N,1} & D_E & D_F \\ C_{N,2} & D_H & D_K \end{array}\right],$$

where

$$\begin{aligned} A_N &\in \mathbb{R}^{n_N \times n_N}, & B_{N,1} &\in \mathbb{R}^{n_N \times m_N}, \\ C_{N,1} &\in \mathbb{R}^{p_N \times n_N}, & D_E &\in \mathbb{R}^{p_N \times m_N}. \end{aligned}$$

The element K of N models how the subsystems $G_1, \ldots, G_q$ are connected to each other, and E, F, H model the external excitation and measurements of the interconnected system. Throughout the paper it is assumed that $\mathscr{F}_l(N,G)$ is a well-posed and stable feedback interconnection, i.e., $\|\mathscr{F}_l(N,G)\|_\infty < \infty$. In [5], it is shown how a mechanical systems fits to this framework.

To quantify how controllable the interconnected system is from the input w, and how observable it is from the output z, *controllability and observability Gramians* are often computed [9]. The controllability Gramian P and the observability Gramian Q satisfy the Lyapunov equations

$$\begin{aligned} AP + PA^T + BB^T = 0, \quad P > 0, \\ A^T Q + QA + C^T C = 0, \quad Q > 0, \end{aligned} \tag{4}$$

for an asymptotically stable system $\mathscr{F}_l(N,G)$, where $[A,B,C,D]$ is a minimal structured realization as defined in (2). Let us impose the following structure on the Gramians,

$$\begin{aligned} Q &= \begin{bmatrix} Q_N & Q_{NG} \\ Q_{NG}^T & Q_G \end{bmatrix}, \quad Q_G = \begin{bmatrix} Q_1 & \cdots & Q_{1q} \\ \vdots & \ddots & \vdots \\ Q_{1q}^T & \cdots & Q_q \end{bmatrix}, \\ P &= \begin{bmatrix} P_N & P_{NG} \\ P_{NG}^T & P_G \end{bmatrix}, \qquad P_G = \begin{bmatrix} P_1 & \cdots & P_{1q} \\ \vdots & \ddots & \vdots \\ P_{1q}^T & \cdots & P_q \end{bmatrix}, \end{aligned} \tag{5}$$

such that $P_k, Q_k \in \mathbb{R}^{n_k \times n_k}$, conformably to the structured realization (2).

2.1 Balanced truncation of interconnected linear systems

In [6, 7, 4, 5], balanced truncation of interconnected linear systems are studied. The model reduction problem is to find a new set of subsystems $\hat{G}$ with the same block-diagonal structure as G in (3), but of smaller McMillan degree, and such that $\|\mathscr{F}_l(N,G) - \mathscr{F}_l(N,\hat{G})\|_\infty$ is small. In the above papers, extensions to balanced truncation are proposed to solve this problem, and a summary is given next.

We say the structured realization of $\mathscr{F}_l(N,G)$ and the corresponding Gramians are *subsystem balanced* if the internal coordinates are such that the block-diagonal elements of the Gramians take the form

$$\begin{aligned} Q_k = P_k = \Sigma_k = \operatorname{diag}\{\sigma_{k,1} \ldots, \sigma_{k,n_k}\}, \\ \sigma_{k,1} \geq \ldots \geq \sigma_{k,n_k} > 0, \quad k = 1 \ldots q. \end{aligned} \tag{6}$$

We call $\sigma_{k,i}$ *structured Hankel singular values* of the interconnected system. They are invariant under structured block-diagonal coordinate transformations, see Proposition 1, and can be computed as

$$\sigma_{k,i} = \sqrt{\lambda_i(P_k Q_k)}, \tag{7}$$

where P_k, Q_k come from any structured realization. The following results are motivated and shown in [6, 7, 4, 5].

Proposition 1. *If there exist Gramians P and Q (5) for a structured realization of the interconnected system $\mathscr{F}_l(N,G)$, then there exist a block-diagonal ("structured") coordinate transformation $\bar{x} = Tx$,*

$$T = diag\{T_N, T_1, \ldots, T_q\},$$

$T_N \in \mathbb{R}^{n_N \times n_N}$, $T_k \in \mathbb{R}^{n_k \times n_k}$, $k = 1, \ldots, q$ that makes the realization and the Gramians subsystem balanced (6):

$$T_k^T P_k T_k = T_k^{-T} Q_k T_k^{-1} = \Sigma_k.$$

The model reduction procedure that will be used in Section 4 is as follows.

Procedure 1. *Assume the interconnected system $\mathscr{F}_l(N,G)$ is* subsystem balanced *(6). Let the realizations of the subsystems G_k and $\hat{G}_k$, $k = 1 \dots q$, be given by*

$$G_k(s) = \left[\begin{array}{cc|c} A_{k,11} & A_{k,12} & B_{k,1} \\ A_{k,21} & A_{k,22} & B_{k,2} \\ \hline C_{k,1} & C_{k,2} & D_k \end{array}\right], \quad \hat{G}_k(s) = \left[\begin{array}{c|c} A_{k,11} & B_{k,1} \\ \hline C_{k,1} & D_k \end{array}\right],$$

where $A_{k,11} \in \mathbb{R}^{r_k \times r_k}$, $B_{k,1} \in \mathbb{R}^{r_k \times m_k}$, and $C_{k,1} \in \mathbb{R}^{p_k \times r_k}$, and the reduced-order system be $\hat{G} = diag\{\hat{G}_1, \dots, \hat{G}_q\}$.

Small structured Hankel singular values indicate that there are states in the corresponding subsystems that are not so important for preserving the input-output map $w \mapsto z$. There are a priori error bounds based on truncated structured Hankel singular values under certain circumstances and interconnection topologies, see [4, 5].

The larger blocks T_k that we allow for in the structured coordinate transformations, the better approximations we can expect to obtain using the above procedure. This is exactly what lumping achieves: When two subsystems are lumped into one, two small blocks in the structured coordinate transformation T are replaced by one large block.

2.2 Subsystem lumping

By lumping we mean that subsystems are grouped together. In Figure 1, the subsystems G_3, G_4 are lumped together into G_3, for example. Lumping is an operation that does not reduce the state dimension of the model; it simply changes the partitioning of the state space, and leaves the input-output map $w \mapsto z$ invariant. Lumping is a natural step to perform before model reduction as described in Section 2.1.

In the above model framework, lumping is described as follows. If the original system $\mathscr{F}_l(N,G)$ has subsystems

$$G(s) = \mathrm{diag}\{G_1(s), \dots, G_q(s)\},$$

then admissible lumping gives rise to new systems $\mathscr{F}_l(\bar{N},\bar{G})$, where

$$\bar{G}(s) = \mathrm{diag}\{\bar{G}_1(s), \dots, \bar{G}_{\bar{q}}(s)\},$$

such that $\bar{q} \leq q$, and

$$\mathscr{F}_l(\bar{N},\bar{G}) = \mathscr{F}_l(N,G). \tag{8}$$

The new interconnection structure $\bar{N}$ is induced by the the choice of $\bar{G}$, and (8). The system $\mathscr{F}_l(\bar{N},\bar{G})$ is called a lumped system.

Lumping is most easily understood by means of examples.

Example 1. *The most extreme form of lumping is to drop all internal interconnection structure and to put $\bar{G} = \mathscr{F}_l(N,G)$. The lumped system is then a complete black-box model, and $\bar{N} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$.*

Example 2. *Consider the system in Figure 1 and assume all signals are scalar, that transfer functions are single-input–single-output, and that inputs are added together and all outputs of a subsystem are identical. Before lumping we have*

$$G(s) = diag\{G_1(s), \dots, G_6(s)\}, \quad E(s) = 0,$$

$$F(s) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad H(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad K(s) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

After lumping, as indicated in Figure 1, the same input-output dynamics is realized with

$$\bar{G}(s) = diag\{\bar{G}_1(s), \dots, \bar{G}_4(s)\}, \quad \bar{E}(s) = 0,$$

$$\bar{F}(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \bar{H}(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{K}(s) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

where $\bar{N} = \begin{bmatrix} \bar{E} & \bar{F} \\ \bar{H} & \bar{K} \end{bmatrix}$. For the lumped system, the states of the subsystems G_3, G_4 and G_5, G_6 are allowed to be mixed in the structured coordinate transformations.

How many subsystems to lump together is a trade off. If many subsystems are lumped together, there is a larger potential for model reduction (reducing number of states), but important structural information of the model might be lost. On the other hand, if no lumping is performed, one might use a model with too detailed interconnection structure that is not motivated by the chosen input-output map $w \mapsto z$.

Next, we introduce an easily computable index that should be a help to choose when and what subsystems G_k to lump.

3 A constrained Hankel-norm and a lumping index

The Hankel-norm of a linear system $G : w \mapsto z$ with a transfer function matrix $G(s) \in RH_\infty$ is defined as the maximum amplification of energy from past inputs to future outputs, see [9]. By definition, the Hankel-norm is given by

$$\|G\|_H := \sup_{w \in L_2(-\infty,0] \neq 0} \frac{\|Gw\|_{2,[0,\infty)}}{\|w\|_{2,(-\infty,0]}}, \tag{9}$$

where Gw is the time-domain output of the linear system G when the input w is applied. If $[A,B,C,D]$ is a minimal realization of $G(s)$, the Hankel-norm can be computed as [9]

$$\|G\|_H = \sqrt{\max_x \frac{x^T Q x}{x^T P^{-1} x}} = \sqrt{\lambda_{\max}(PQ)},$$

where P, Q satisfy the Lyapunov controllability and observability equations (4), and $\lambda_{\max}$ is the largest eigenvalue. The Hankel-norm is often used in model reduction and is invariant under coordinate transformations. The Hankel-norm is large if G has states that are easy to control from w and are very visible in the output z. This intuition will be used next to quantify whether a subsystem G_k should be lumped or not.

Let us now consider a structured realization of $\mathscr{F}_l(N,G)$, and let us compute the amount of output energy the states in the subsystem G_k result in. Assume that all other states are zero at $t = 0$, and that the input $w = 0$ for $t \geq 0$. Then it holds that

$$\|z\|^2_{2,[0,\infty)} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_{k,0} \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T Q \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_{k,0} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = x_{k,0}^T Q_k x_{k,0}, \tag{10}$$

where $x_{k,0}$ is the initial state of G_k, see [7]. Here Q_k is the k-th block of the observability Gramian (5).

Let us next consider the problem of controlling the interconnected system from rest at $t = -\infty$ to an arbitrary state of subsystem G_k, i.e., $x_k(0) = x_{k,0}$ such that all other states are zero, i.e., $x_{-k}(0) = 0$. The energy of the minimum such control w^* is, see [7],

$$\|w^*\|^2_{2,(-\infty,0]} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_{k,0} \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T P^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_{k,0} \\ 0 \\ \vdots \\ 0 \end{bmatrix} =: x_{k,0}^T (P^{-1})_k x_{k,0}, \tag{11}$$

where $(P^{-1})_k$ is the diagonal block of P^{-1} corresponding to the states of G_k.

We can now define a constrained Hankel-norm by

$$\|\mathscr{F}_l(N,G)\|_{H,k} := \sup_{w \in L_2(-\infty,0] \neq 0; x_{-k}(0)=0} \frac{\|\mathscr{F}_l(N,G)w\|_{2,[0,\infty)}}{\|w\|_{2,(-\infty,0]}}.$$

Hence, the interconnected system is at rest at $t = -\infty$, and is then maximally excited during $(-\infty, 0]$ using w such that all the states x_{-k} (all states except x_k) are zero at $t = 0$. Then the interconnected system is released from this state with zero input, and we measure the energy in the output. If the subsystem G_k is highly controllable and observable independently of all other subsystems in the interconnected system, then the number $\|\mathscr{F}_l(N,G)\|_{H,k}$ is large. The constrained Hankel-norm can be computed as shown in the next proposition.

Proposition 2. *Given a minimal structured realization $[A,B,C,D]$ of $\mathscr{F}_l(N,G)$, we can compute the constrained Hankel-norm as*

$$\|\mathscr{F}_l(N,G)\|_{H,k} = \sqrt{\lambda_{\max}([(P^{-1})_k]^{-1}Q_k)},$$

where $(P^{-1})_k, Q_k$ are defined in (10)–(11). Furthermore, the norm is invariant under structured coordinate transformations, as defined in Proposition 1.

Proof. Use the identities (10) and (11), and we get

$$\|\mathscr{F}_l(N,G)\|_{H,k}^2 = \max_{x_{k,0}} \frac{x_{k,0}^T Q_k x_{k,0}}{x_{k,0}^T (P^{-1})_k x_{k,0}}.$$

Since the realization is minimal, $(P^{-1})_k$ is positive definite, and thus invertible, and the result follows since this is a generalized Rayleigh quotient. □

Remark 1. *For the above formula for $\|\mathscr{F}_l(N,G)\|_{H,k}$ to work, the system must be controllable (P^{-1} exists), which is true for minimal realizations. How to best compute the constrained norm for a non-minimal realization is still an open issue.*

The constraint in the above Hankel-norm can be relaxed, and we can let more subsystems participate in the energy transfer from the past into the future. It is clear that the more systems that are allowed to participate, the larger the Hankel-norm becomes, as stated in the following proposition.

Proposition 3. *For an interconnected linear system $\mathscr{F}_l(N,G)$, it holds that*

$$\|\mathscr{F}_l(N,G)\|_{H,1} \leq \|\mathscr{F}_l(N,G)\|_{H,[1,2]} \leq \ldots \leq \|\mathscr{F}_l(N,G)\|_{H,[1,2,\ldots,q]} \leq \|\mathscr{F}_l(N,G)\|_{H,[N,1,2,\ldots,q]} = \|\mathscr{F}_l(N,G)\|_H,$$

where $\|\mathscr{F}_l(N,G)\|_{H,[1,\ldots,k]}$ means that the states of $G_1,\ldots,G_k$ at $t=0$ are free variables. Similar inequalities hold for $\|\mathscr{F}_l(N,G)\|_{H,2},\ldots,\|\mathscr{F}_l(N,G)\|_{H,q}$.

We are now ready to define the lumping index mentioned in the introduction.

Definition 1. *The* lumping index *γ_k of subsystem G_k in the interconnected system $\mathscr{F}_l(N,G)$ is defined by*

$$\gamma_k := \frac{1}{\|\mathscr{F}_l(N,G)\|_{H,k}} = \frac{1}{\sqrt{\lambda_{\max}([(P^{-1})_k]^{-1}Q_k)}},$$

where the last equality holds for a minimal realization of $\mathscr{F}_l(N,G)$.

One interpretation of the lumping index γ_k is that it indicates how independent the subsystem G_k is with respect to the inputs and outputs w,z, and the interconnected system. A subsystem with a large γ_k is not a very visible and controllable subsystem by itself. Note that the index depends heavily on the surrounding and the chosen inputs and outputs. Hence, a subsystem can have a small lumping index in one interconnected system, and large index in another interconnected system.

If subsystems are lumped together as defined in Section 2.2, it follows from Proposition 3 that the lumping index of the lump is smaller than the lumping indices of the subsystems in the lump. That is, if G_k belongs to the lump $\bar{G}_{k'}$, then

$$\bar{\gamma}_{k'} \leq \gamma_k.$$

We propose that one computes the lumping indices for all subsystems, $\gamma_1,\ldots,\gamma_q$, and compare them to each other. The lumping index is a relative measure and has no real absolute meaning. If there is a large difference in the magnitudes of the lumping indices in an interconnected system, then we propose that subsystems with large indices are lumped together. After a successful lumping, the lumping indices, $\bar{\gamma}_1,\ldots,\bar{\gamma}_{\bar{q}}$, should be roughly of the same magnitude. The rationale for this procedure is that the new subsystems $G_1,\ldots,G_{\bar{q}}$, are roughly equally controllable and observable subsystems, and thus of similar importance in the interconnected system.

This procedure does not tell how subsystems should be lumped together, only that a subsystem with a large index is a *candidate* for lumping. We suggest that neighboring subsystems with large indices are either lumped together, or that all subsystems with a large index are lumped into one large "rest-of-the-world" or "environment" lump.

In the following section, we illustrate how the lumping index can be used in two numerical examples.

ε	$\|\mathscr{F}_l(N,G)\|_H$	$\|\mathscr{F}_l(N,G)\|_{H,1}$	$\|\mathscr{F}_l(N,G)\|_{H,2}$
0.01	$6.45\cdot 10^{-2}$	$1.40\cdot 10^{-3}$	$1.74\cdot 10^{-7}$
0.1	$6.95\cdot 10^{-2}$	$1.38\cdot 10^{-3}$	$1.74\cdot 10^{-5}$
1	$1.06\cdot 10^{-1}$	$1.16\cdot 10^{-3}$	$1.29\cdot 10^{-3}$

Table 1: The constrained and unconstrained Hankel-norms for Example 3.

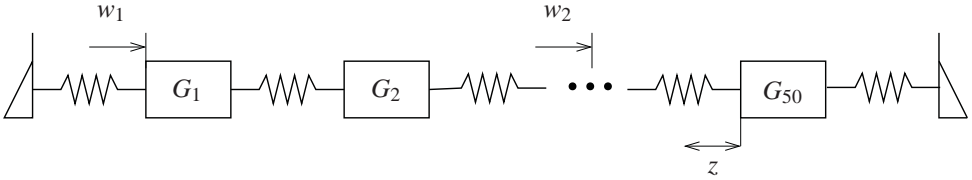

Figure 3: The interconnected spring-mass system in Example 4.

4 Numerical Examples

In the first example, we compute the constrained Hankel-norm for a simple example, to get some intuition.

Example 3. *Consider the subsystems*

$$G_1(s) = \frac{s+1}{(s+2)(s+10)}, \quad G_2(s) = \frac{s+1}{s+2+\varepsilon},$$

that are connected in parallel, such that $\mathscr{F}_l(N,G) = G_1 + G_2$*. We compute the constrained Hankel-norms for various* ε*, see Table 1. As* $\varepsilon \to 0$ *there is a pole in the subsystem* G_2 *that is getting closer and closer to the pole in* $s = -2$ *in* G_1*. This means that the subsystem* G_2 *is getting harder to control and to observe independently. The constrained Hankel-norm of subsystem* G_1 *is much less sensitive to the changes in* ε *because it also has a pole in* $s = -10$*, and thus* G_1 *contains dynamics that is unique for this interconnected system.*

Note that just because $\|\mathscr{F}_l(N,G)\|_{H,2}$ *is small does not mean that* G_2 *is unimportant in the interconnected system* $\mathscr{F}_l(N,G)$*. It just means that* G_1 *is not a very independent subsystem in this particular interconnection structure.*

In the next numerical example, we lump and reduce a model of a spring-mass system.

Example 4. *In Figure 3, an interconnection of* 50 *rigid masses is illustrated. The masses are interconnected with linear springs of uniform stiffness. There is also uniform viscous friction at each mass in the system to make the system asymptotically stable. Each subsystem* G_k *has two states, position* x_k *and velocity* v_k*, and thus the interconnected system has order* 100*. There are two forces* w_1, w_2 *acting as inputs. The first force acts at* G_1 *and the second at mass* G_{25}*. The output* z *is the position of the mass* G_{50}*.*

In Figure 4, the lumping indices γ_k *for the* 50 *subsystems* G_k *are shown. It is seen that there is a large variance in magnitude throughout the interconnected system. The indices* $\gamma_1, \gamma_{25}, \gamma_{50}$ *are especially small since there are inputs or an output at those respective subsystems (these subsystems are especially controllable or observable). In between these subsystems, the indices are larger. In order to make the lumping indices more uniform, we create five lumps:* $\bar{G}_1, \ldots, \bar{G}_5$*, where* $\bar{G}_1 = G_1$*,* $\bar{G}_3 = G_{25}$*, and* $\bar{G}_5 = G_{50}$*. The lumps* $\bar{G}_2$ *and* $\bar{G}_4$ *contain the masses that lie in between. After lumping, the indices are more or less of equal order, as seen in Figure 4. Hence, in the lumped system* $\mathscr{F}_l(\bar{N},\bar{G})$*, all subsystems* $\bar{G}_k$ *are roughly equally independent.*

After lumping, it is natural to try to reduce the order of the dynamical system. The structured Hankel singular values for $\bar{G}_2$ *and* $\bar{G}_4$*, see Section 2.1, are shown in Figure 5. After inspection of the singular values, it is determined that* 8 *and* 10 *states can be removed, respectively. These are removed using Procedure 1. The approximation error becomes* $\|\mathscr{F}_l(N,G) - \mathscr{F}_l(\bar{N},\hat{\bar{G}})\|_\infty = 0.0137$*, where* $\|\mathscr{F}_l(N,G)\|_\infty = 0.1258$*, and the approximation is good enough for most purposes.*

In conclusion: We have reduced a chain of 100 *rigid masses into a chain of five lumped masses, where two of the lumped masses can be thought of as non-rigid masses. We argue that this lumped and reduced model is a simpler and more natural model of how the underlying physical system works, given the specified forces* w_1, w_2 *and the measurement* z*.*

5 Conclusions

In this paper, we have studied the problem of lumping of interconnected linear systems. A model framework was first presented along with a model reduction procedure. In order to decide when and what subsystems to lump, a constrained Hankel-norm was introduced, and a lumping index was derived from it. The index was later successfully tested on two simple examples. Even though the lumping index seems promising, its properties and usefulness must be further studied and evaluated. For example, how the index can be robustly computed for large nearly uncontrollable/unobservable systems remains an open issue.

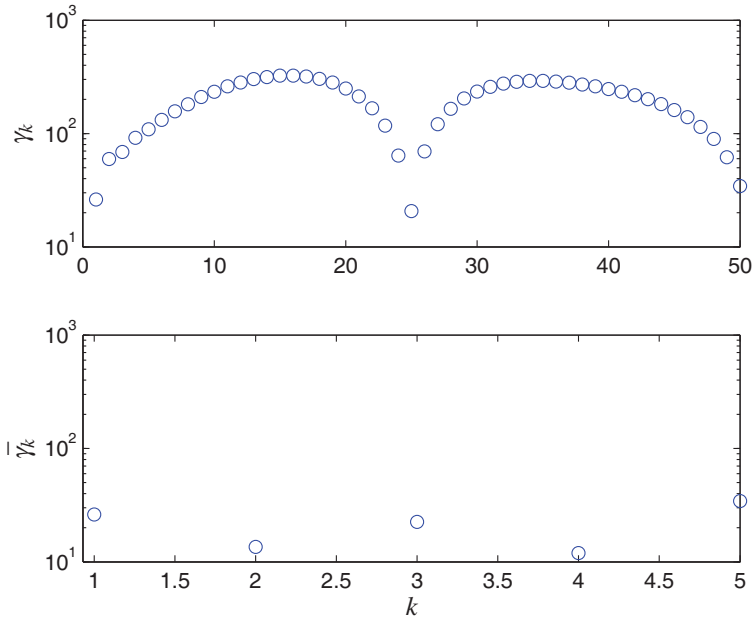

Figure 4: Lumping indices for the spring-mass system in Example 4, before (top) and after (bottom) lumping. After lumping, all the indices are roughly of the same order.

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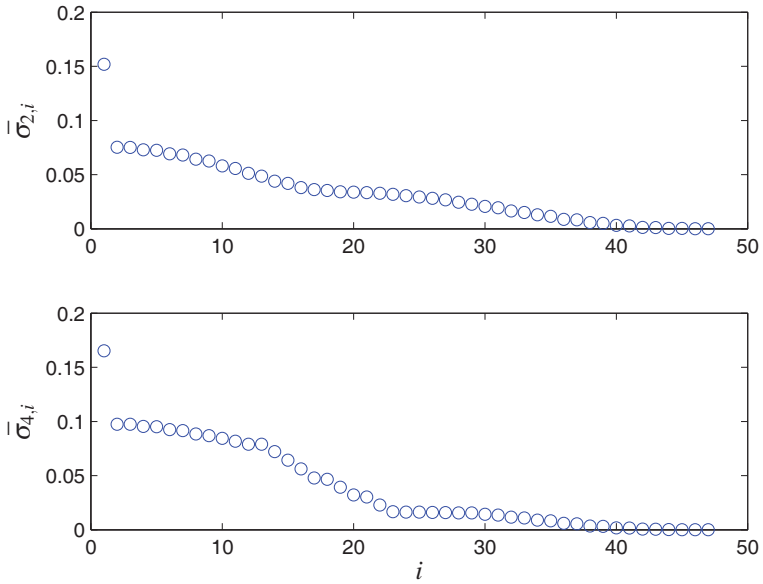

Figure 5: The structured Hankel singular values for the lumped subsystems $\bar{G}_2$ and $\bar{G}_4$ in Example 4.