THE CONTROL OF LINEAR SYSTEMS UNDER FEEDBACK DELAYS[1]

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Abstract. The problems of measurement feedback control are at the heart of control theory. In this paper we consider problems of feedback control under delayed measurement output feedback and set-membership noise. The suggested solutions are based on a combination of Hamiltonian techniques with methods of set-valued analysis. They rely on ellipsoidal approximations of information and solvability sets which describe the solution strategies. These approaches allow to solve problems of realistically high dimensions. We present numerical examples for oscillating systems of high dimensions, including the dependence of solution on delay time.

1 Introduction

The problems of measurement feedback control are at the heart of control theory [5, 12, 7, 1]. These problems were mostly treated in a stochastic setting. In this paper we consider problems of feedback control under delayed measurement output feedback and set-membership noise. The delays considered here may be due to errors in communication channels as well as to the processing time in the observers and controllers.

The suggested solutions are based on a combination of Hamiltonian techniques with methods of set-valued analysis. They rely on ellipsoidal approximations of information and solvability sets which describe the solution strategies [10]. These approaches allow to solve problems of realistically high dimensions. Their practical implementation may be based on the Ellipsoidal Toolbox [11].

In Section 3 we consider the situation with feedback noise but no delay. This problem is dealt with by reduction to one in the metric space of information sets. The key point is that the problem further reduces to a standard one in finite-dimensional space.

In Section 4 we consider bounded noise in the delayed output measurement. A control strategy for the related problem of control is indicated. (Everything is exact.)

The final section 5 presents results of numerical modelling for oscillating systems (of dimension up to 20). We show the realized control inputs, and how the quality of solution depends on delay time.

In the Appendix A we summarize the main points of the solution techniques.

2 The Problem with Feedback Delay

Consider system

$$\dot{x}(t) = A(t)x(t) + B(t)u, \quad t \in [t_0, t_1], \tag{1}$$

where $x \in \mathbb{R}^n$ is the state variable, $u \in \mathbb{R}^m$ is the control input, $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times m}$ are continuous matrix functions. The time interval $[t_0, t_1]$ is fixed in advance.

The control input u is restricted by the hard bound $u[t] \in \mathscr{P}(t)$ where $\mathscr{P}(t)$ is a continuous set-valued mapping with non-empty convex compact values in $\mathbb{R}^m$, and $0 \in \mathscr{P}(t)$.

The feedback control belongs to the class $\mathscr{U}_1$ of functions $u = u[t] = U(t, x(t-h))$ defined by the following requirements: U is a set-valued function of t, x with convex compact values in $\mathscr{P}(t)$; within the interval $[t_0, t_0+h)$ the control U is only time-dependent; U also ensures the existence and extendibility of solutions to the closed-loop system. The controller is allowed to be with memory.

Problem 1. For a given terminal target set $\mathscr{M}$, find a delayed feedback control $U \in \mathscr{U}_1$ steering trajectories of the system (1) to $\mathscr{M}$ at terminal time t_1, i.e. $x(t_1) \in \mathscr{M}$.

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We look for the solution through reduction to an ordinary linear-convex problem without feedback delay (see Appendix A).

We assume that on time interval $[t_0, t_0+h)$ the control $u(t)$ is independent of the state. We thus take it to be given, setting it as $u[t] = 0$ for $t \in [t_0, t_0+h)$.

At time $t = t_0 + h$ the first state information is $x(t_0)$, which allows to reconstruct the on-line state[2]: $x(t_0+h) = X(t_0+h, t_0)x(t_0)$ (here we recall that $u(t) \equiv 0$ on $[t_0, t_0+h)$).

The control then relies on integrating equation

$$\dot{z}(t) = A(t)z(t) + B(t)u[t], \quad z(t_0+h) = X(t_0+h, t_0)x(t_0), \quad t \in [t_0+h, t_1]. \tag{2}$$

One may observe that $z(t) \equiv x(t)$ is such that a control of type $u(t,z)$ will be without feedback delay in contrast with $u(t, x(t-h))$.

We have reduced Problem 1 to the linear-convex problem for system (2) without delays. This problem may be treated by standard techniques (see Appendix A).

Clearly, it is possible to steer the on-line trajectory to the target set $\mathscr{M}$ if and only if $z(t)$ belongs to the solvability set of the linear-convex system, $\mathscr{W}[t]$.

For this problem we can define the *state* of the system in two ways. One is the triple $\{t, x(t-h), u[t-h, t]\}$ (where $u[t-h,t]$ denotes the history of control $u(\cdot)$ within $[t-h, t]$). This state is infinite-dimensional, since it memorizes the control history component. The other is finite dimensional — the pair $\{t, z(t)\}$. To construct the control we use the latter.

Remark 1. The constructed control depends only on one measurement of the state, namely, $x(t_0)$. Though theoretically correct, this may lead to a gradual deviation of $z(t)$ form $x(t)$ due to numerical errors. One may use more available measurements to reset $z(t)$ at any time τ as

$$z(\tau) = X(\tau, \tau-h)x(\tau-h) + \int_{\tau-h}^{\tau} X(\tau, \vartheta)B(\vartheta)u[\vartheta]d\vartheta.$$

However, this procedure requires to memorize the control history on at least an interval of length h. To avoid this storage we introduce an additional equation

$$\dot{z}_\tau(t) = A(t)z_\tau(t) + B(t)u[t], \quad z_\tau(\tau-h) = 0, \quad t \in [\tau-h, \tau]$$

for each instant τ when correction is necessary, and then to reset z as

$$z(\tau) = X(\tau, \tau-h)x(\tau-h) + z_\tau(\tau).$$

3 The Problem with Noisy Feedback and No Delay

In this section we introduce a measurement noise, assuming first the there is no delay. We have

$$y(t) = x(t) + \xi(t), \quad \xi(t) \in \mathscr{Q}(t), \tag{3}$$

where $\mathscr{Q}(t)$ is similar in nature to $\mathscr{P}(t)$ but now taken in $\mathbb{R}^n$. We further transform the coordinates to a new system assuming $z = X(\vartheta, t)x$. Performing this transformation for (2), (3), we come to

$$\dot{z} = \mathscr{B}(t)u, \quad \mathscr{B}(t) = X(\vartheta, t)B(t), \quad y(t) = H(t)x + \xi(t), \quad H(t) = X(t, \vartheta).$$

Returning to previous notations x, B instead of z, $\mathscr{B}$, we have

$$\text{(i)} \quad \dot{x} = B(t)u, \qquad \text{(ii)} \quad y(t) = H(t)x + \xi(t), \quad t \leq \vartheta. \tag{4}$$

For this system we are to solve a particular case of the measurement feedback control problem [8]. The particular properties of this simplified version allow to present a fairly simpler solution scheme than in the more general case. Here is a preliminary loose version of the problem.

Problem 2. For a given target set $\mathscr{M}$, find a measurement feedback control $U \in \mathscr{U}_1$ which steers system (4) to $\mathscr{M}$ at terminal time ϑ, so that $x(\vartheta) \in \mathscr{M}$, despite the disturbance $\xi(\cdot)$.

The first step here is to define *the state* of the system with feedback noise. According to [8] here we distinguish two problems — the one of *guaranteed state estimation* which gives us this state and the problem of *feedback control in the space of states*.

[2] $X(t, \tau)$ is the transition matrix of the homogeneous system, a solution to the matrix equation $\partial X(t,\tau)/\partial t = A(t)X(t,\tau)$, $X(\tau,\tau) = I$.

Suppose the measurement process begins at $t_0 < \vartheta$. Then the on-line state or *position* of the system is defined as the pair $\{\tau, \mathscr{X}[\tau]\}$ where $\mathscr{X}[\tau]$ is the *information set* consistent with the system model, the available measurement $y(s)$, $s \in [t_0, \tau]$, $\tau < \vartheta$, and the constraint $\mathscr{Q}$ on the unknown but bounded noise ξ.

Set $\mathscr{X}[t]$ is the solution to the problem of guaranteed state estimation [7, 6, 13]. With control and measurement realizations $u^*(t)$, $y^*(t)$, $t \in [t_0, \tau]$ given, the support function $\rho(l \mid \mathscr{X}[\tau]) = \max\{\langle \ell, x\rangle \mid x \in \mathscr{X}[\tau]\}$ may be calculated through techniques of convex analysis (see [14, 7, 8]). Note that the measurement equation produces at $t = t_0$ the inclusion $x(t_0) = x^0 \in H^{-1}(t_0)(y(t_0) - \mathscr{Q}(t_0)) = \mathscr{X}^0$. We have

$$\rho(\ell \mid \mathscr{X}[\tau]) = \tag{5}$$

$$= \inf\left\{ \int_{t_0}^{\tau} \psi(t)B(t)u^*(t)dt + \int_{t_0}^{\tau} (\lambda'(t)y(t) + \rho(-\lambda(t) \mid \mathscr{Q}(t)))dt + \rho(p \mid \mathscr{X}^0) \mid p, \lambda(\cdot) : \psi(\tau) = \ell \right\},$$

where the vector row $\dot\psi = \lambda(t)H(t)$, $\psi(t_0) = p$.

When the disturbance and hence the measurements are smooth enough, $\mathscr{X}(t)$ is the solution to the next *funnel equation* [9]

$$\lim_{\sigma \to 0+0} \sigma^{-1} h(\mathscr{X}(t+\sigma), (\mathscr{X}(t) + \sigma B(t)u^*[t]) \cap (y^*(t) - \mathscr{Q}(t))) = 0. \tag{6}$$

We thus observe that the state space position is $\{\tau, \mathscr{X}\}$, where $\mathscr{X}$ is a convex compact. We further proceed with *the measurement feedback control problem* in the *metric space* of convex compact sets. Our further aim will be to indicate that the last problem reduces to one in finite-dimensional space. We may now give a more precise formulation for Problem 2.

Problem 3. Given position $\{\tau, \mathscr{X}\}$, $\tau \in [t_0, \vartheta)$, indicate a feedback control strategy $U(t, \mathscr{X})$ which ensures the inclusions

$$\mathscr{X}(\vartheta, \tau, \mathscr{X}) = \mathscr{X}[\vartheta] \subseteq \mathscr{M},$$

$$H(t)\mathscr{X}[t] \subseteq (y(t) - \mathscr{Q}(t)), \quad t \in [\tau, \vartheta],$$

whatever be the measurement $y(t)$ (that is, despite the unknown measurement noise $\xi(t) \in \mathscr{Q}(t)$).

Here $\mathscr{X}[t] = \mathscr{X}(\vartheta, \tau, \mathscr{X})$ is the trajectory tube for system (4,i) emanating at time $t = \tau$ from $\mathscr{X}$ under the control $U(t, \mathscr{X})$.

Note that now the target set $\mathscr{M}$ has to be formally treated as the convex compact union $\mathscr{M} = \cup\{\mathscr{M}'\}$ of all its convex compact subsets $\mathscr{M}'$ and the terminal inclusion as $\mathscr{X}[\vartheta] \subseteq \mathscr{M}' \subseteq \mathscr{M}$ for some $\mathscr{M}'$.

We shall now look for the backward reach set $\mathscr{W}[\tau]$ for our problem. Recall that on the interval $t \in [\tau, \vartheta]$ the measurement realizations under given control u^* are of the next form:

$$y^*(t) = H(t)m - \int_t^{\vartheta} B(s)u^*(s)ds + \xi^*(t), \quad m \in \mathscr{M}' \tag{7}$$

for some ξ^*, m, $\mathscr{M}'$.

We now have the *subproblem W:* given $\mathscr{M}' \subseteq \mathscr{M}$ find set $\mathscr{W}'[\tau] = \mathscr{W}(\tau, \vartheta, \mathscr{M}')$ of all those vectors x for which there exists a control $u^*(t)$ which ensures that related $x^*(\vartheta, \tau, x) = x^*[\vartheta] = m \in \mathscr{M}'$ and that also inclusion (7) is true for same $m \in \mathscr{M}'$, no matter what be $\xi^*(t) \in \mathscr{Q}(t)$. Then, clearly

$$\mathscr{W}[\tau] = \mathscr{W}(\tau, \vartheta, \mathscr{M}) = \cup\{\mathscr{W}(\tau, \vartheta, \mathscr{M}') \mid \mathscr{M}' \subseteq \mathscr{M}\} = \cup\{\mathscr{W}'[\tau]\}.$$

The standard solution to this problem is that $\mathscr{W}[\tau] = \{x\}$ for all x that solve the inequality (note that here $\min\sup = \sup\min$)

$$\min_{u(\cdot), m} \sup_{\xi(\cdot), p, \lambda(\cdot)} \left\{ \langle s(\tau), x\rangle + \int_\tau^{\vartheta} \langle s(t), B(t)u(t)dt\rangle - \int_\tau^{\vartheta} \rho(\lambda(t) \mid \xi^*(t) - \mathscr{Q}(t))dt \le 0 \right\}$$

under constraints $u \in \mathscr{P}(t)$, $m \in \mathscr{M}'$, $\xi \in \mathscr{Q}(t)$, $p \in \mathbb{R}^n$, $\lambda(\cdot) \in \mathscr{L}_2(\cdot)$, where

$$ds = -\lambda(t)H(t)dt, \quad s(\vartheta) = p.$$

The last relation yields

Theorem 1. *The support function for the backward reachability set $\mathscr{W}[\tau] = \mathscr{W}(\tau, \vartheta, \mathscr{M})$ satisfies the relations*

$$\rho(\ell \mid \mathscr{W}[\tau]) = \rho(\ell \mid \cup\{\mathscr{W}'[\tau]\}) = \int_\tau^{\vartheta} \rho(-\ell \mid B(t)\mathscr{P}(t))dt + \rho(\ell \mid \mathscr{M}), \tag{8}$$

The last theorem indicates that the support function for the backward reachability set $\mathscr{W}[\tau]$ from target set $\mathscr{M}$, though originally taken in the space of convex compact sets, may be calculated in finite-dimensional space where it is the same as for the problem with noise-free complete measurements in space $\mathbb{R}^n$. It may also be calculated through the value function

$$V(\tau,x) = \min_u \{d(x(\vartheta),\mathscr{M}) \mid x(\tau) = x\}$$

for system $\dot{x} = B(t)u$, as the level set $\mathscr{W}[\tau] = \{x \mid V(\tau,x) \le 0\}$ for the solution to the HJB equation

$$V_t(t,x) + \min_u\{\langle V_x(t,x), B(t)u\rangle\} = 0, \quad V(\vartheta,x) = d(x,\mathscr{M}), \tag{9}$$

Now define

$$\mathscr{V}(t,\mathscr{X}) = h_+(\mathscr{X},\mathscr{W}[t]) = \min\{\varepsilon \mid \mathscr{X} \subseteq \mathscr{W}[t] + \varepsilon\mathbf{B}(0)\}$$

where $\mathbf{B}(0) = \{x \mid \langle x,x\rangle \le 1\}$. This is the Hausdorff semi-distance between $\mathscr{W}$ and $\mathscr{M}$. Then actually

$$\mathscr{V}(\tau,\mathscr{X}[\tau]) = \max\{d(x,\mathscr{W}[\tau]) \mid x \in \mathscr{X}[\tau]\} = \max\{\rho(\ell \mid \mathscr{X}[\tau]) - \rho(\ell \mid \mathscr{W}[\tau]) \mid \langle \ell,\ell\rangle \le 1\}, \tag{10}$$

where $d(x,\mathscr{M})$ is the euclidean distance.

Theorem 2. *The solution strategy $U^0(t,x)$ for Problem 3 is given as*

$$U^0(t,x) = \operatorname*{Argmin}_u\{dh_+(\mathscr{X},\mathscr{W})/dt \mid_{(4,i)}\} \le 0.$$

Let ℓ^0 be the unique maximizer in (10). Then with $u = u^0$ we have

$$dh_+(\mathscr{X}[\tau],\mathscr{W}[\tau])/d\tau \mid_{(4,i)} = \partial\rho(\ell^0|\mathscr{X}[\tau])/\partial\tau - \partial\rho(\ell^0|\mathscr{W}[\tau])/\partial\tau =$$

$$= \left\langle \ell^0, B(\tau)u^0\right\rangle + \max\{\left\langle -\ell^0, B(t)u\right\rangle \mid u \in \mathscr{P}(t)\} = 0$$

where the partial derivatives are directly calculated from formulas (5), (9). All such u^0 give us the set-valued solution $U^0(\tau,\mathscr{X}) = \{u^0\}$, which also satisfies an existence theorem for the differential inclusion $\dot{x} \in B(t)U^0(t,x)$.

A consequence of this reasoning is the proposition.

Theorem 3. *With $h_+(\mathscr{X}[\tau],\mathscr{W}[\tau]) = 0$ and $U = U^0(t,\mathscr{X})$ we come to $\mathscr{X}[\vartheta] \subseteq \mathscr{M}$, which also means $x[\vartheta] \in \mathscr{M}$ despite the measurement noise.*

The given scheme does not require a generalized solution to equation (9) and is achieved by the technique of directional differentiation. A similar result is obtained by applying control $U^*(\tau,x)$ which solves the optimization problem in the HJB equation (9) while the solution to the HJB equation allows a weakly generalized interpretation.

We now just have to include the delay into the obtained solution. This may be done through the simple formulas of Section 2.

4 The Case of Noisy Feedback with Delay

Here we combine the techniques of two previous sections ti solve the problem with delayed measurement feedback subjected to unknown but bounded noisy errors.

The controller gets a noisy measurement

$$y(t-h) = x(t-h) + \xi(t-h), \quad \xi(t-h) \in \mathscr{Q}(t-h). \tag{11}$$

The control is now allowed to depend on the histories of observations $y[t_0,t-h]$ together with its previous values $u[t_0,t)$. The corresponding class of controls is denoted as $\mathscr{U}_2$.

Problem 4. For a given terminal target set $\mathscr{M}$, find a delayed measurement feedback control $U \in \mathscr{U}_2$ steering trajectories of the system (1) to $\mathscr{M}$ at terminal time t_1, i.e. $x(t_1) \in \mathscr{M}$, despite the possible realizations of disturbance $\xi(\cdot)$.

We define the *state* of the system as the triple $\{t,\mathscr{X}(t-h),u[t-h,t]\}$. Here $\mathscr{X}(t-h)$ is the information set introduced in the previous section.

Introduce a linear operator:

$$\mathbf{T}_u u[\tau,t] = \int_\tau^t X(t,\vartheta)B(\vartheta)u(\vartheta)d\vartheta.$$

Now the set of all vectors $x(t)$ consistent with previous measurements and control history is

$$\hat{\mathscr{X}}[t] = X(t,t-h)\mathscr{X}(t-h) + \mathbf{T}_u u[t-h,t].$$

By reasoning similar to the previous section, we get the following estimate for the value function of this problem:

$$\mathscr{V}(t,\mathscr{X}(t-h),u[t-h,t]) \le d\left(X(t_1,t)\mathbf{T}_u u[t-h,t], \mathscr{M} \dot{-} X(t_1,t-h)\mathscr{X}(t-h) - \mathbf{T}_u\mathscr{P}[t,t_1]\right) =$$

$$= \mathscr{V}^+(t,\mathscr{X}(t-h),u[t-h,t]).$$

Here $\dot{-}$ denotes the geometric set difference: $A \dot{-} B = \{x \in \mathbb{R}^n \mid x + B \subseteq A\}$.

The function $\mathscr{V}^+(t,\mathscr{X}(t-h),u[t-h,t])$ is equal to the value function for the linear-convex control problem (see Appendix A) for the system (1) with terminal target set $\mathscr{M} \dot{-} X(t_1,t-h)\mathscr{X}(t-h)$ and initial state $x(t) = \mathbf{T}_u u[t-h,t]$. Therefore we set

$$U(t,\mathscr{X}(t-h),u[t-h,t]) = U_{LC}(t,\mathbf{T}_u u[t-h,t];t_1,\mathscr{M} \dot{-} X(t_1,t)\mathscr{X}(t-h)),$$

where U_{LC} is the feedback control for the linear-convex problem with the specified target set.

4.1 The Ellipsoidal Approximation

Here we assume that the sets $\mathscr{P}(t)$, $\mathscr{Q}(t)$ and $\mathscr{M}$ are ellipsoids:

$$\mathscr{P}(t) = \mathscr{E}(p(t),P(t)), \quad \mathscr{Q}(t) = \mathscr{E}(q(t),Q(t)), \quad \mathscr{M} = \mathscr{E}(m,M).$$

To find the external ellipsoidal approximation $\mathscr{Y}_+(t) = \mathscr{E}(\eta(t),Y(t))$ of the information set, we pass to a discrete-time analogue of (6) and then apply the formula for external approximation of intersection of two ellipsoids (see [11, 15]):

$$Y(t+\Delta t) = \alpha X^{-1}$$

$$X = \pi W_1 + (1-\pi)W_2$$

$$W_1 = (X(t+\Delta t,t)Y(t)X^T(t+\Delta t,t))^{-1}, \quad W_2 = Q^{-1}(t),$$

$$q_1 = X(t+\Delta t,t)\eta(t), \quad q_2 = y(t) - q(t),$$

$$\alpha = 1 - \pi(1-\pi)\left\langle q_2 - q_1, W_2 X^{-1} W_1 (q_2 - q_1)\right\rangle,$$

$$\eta(t+\Delta t) = X^{-1}(\pi W_1 q_1 + (1-\pi)W_2 q_2),$$

where parameter π is found numerically from the equation

$$\alpha \det^2 X \operatorname{tr}(X^{-1}(W_1 - W_2)) - \eta \det^2 X \big(2\langle \eta(t+\Delta t), W_1 q_1 - W_2 q_2\rangle +$$

$$\langle \eta(t+\Delta t), (W_2 - W_1)\eta(t+\Delta t)\rangle - \langle q_1, W_1 q_1\rangle + \langle q_2, W_2 q_2\rangle\big) = 0.$$

We then find the internal ellipsoidal approximation of set $\mathscr{M} \dot{-} X(t_1,t-h)\mathscr{Y}^+(t-h)$, $\mathscr{M}_- = \mathscr{E}(m',M')$:

$$m' = m - X(t_1,t-h)\eta(t-h),$$

$$M' = \left(1 - \left(\frac{\langle \ell, M\ell\rangle}{\langle \ell, \hat{Y}\ell\rangle}\right)^{\frac{1}{2}}\right) M + \left(1 - \left(\frac{\langle \ell, \hat{Y}\ell\rangle}{\langle \ell, M\ell\rangle}\right)^{\frac{1}{2}}\right)\hat{Y},$$

where ℓ is a good direction and $\hat{Y} = X(t_1,t-h)Y(t-h)X^T(t_1,t-h)$.

Finally we use an ellipsoid $\mathscr{E}(m',M')$ as the target set in the linear-convex problem to calculate the internal ellipsoidal approximation of the solvability set (see subsection A.1) and the ellipsoidal control synthesis (subsection A.2).

Remark 2. The described scheme requires to recalculate the solvability set for the linear-convex system at each time step. However, this can be reduced to recalculation at only some selected time steps, for example, when the information set shrinks significantly due to new information. Between these selected time steps, one uses the solvability tube calculated at last selected time step.

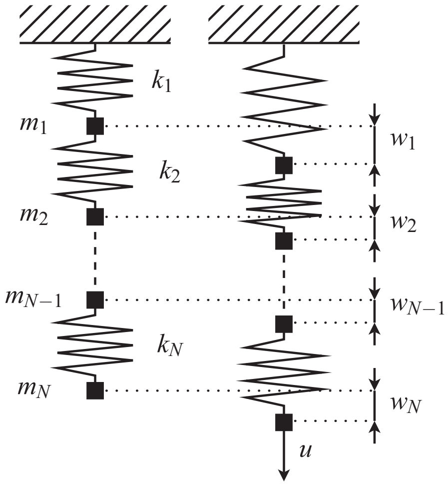

Figure 1: The chain of springs to be controlled in the equilibrium state (left) and in an arbitrary state (right)

5 Examples

Here we present the results of numerical simulations of designed control laws with the system described below. The presentation will be accompanied by computer animation.

The problem is to design a delayed feedback strategy to stop the oscillations of a suspended chain of a finite number of loaded springs by applying a bounded control force to a prescribed node of the chain (Fig. 1).

Apart from the springs, the chain also includes given loads attached in between the springs. We assume that the masses of springs are negligibly small compared to those of the loads. The upper end of the chain is rigidly attached to a fixed suspension. Then the oscillations of the chain could be described by the following system of second-order ODEs:

$$\begin{cases} m_1\ddot{w}_1 &= k_2(w_2 - w_1) - k_1 w_1, \\ m_i\ddot{w}_i &= k_{i+1}(w_{i+1} - w_i) - k_i(w_i - w_{i-1}), \\ m_N\ddot{w}_N &= -k_N(w_N - w_{N-1}) + u, \end{cases}$$

when $t > t_0$. Here N is the number of springs which are numbered from top to bottom. The loads are numbered similarly, so that the i-th load is attached to the lower end of the i-th spring. w_i is the displacement of the i-th load from the equilibrium, m_i is the mass of the i-th load, k_i is the stiffness coefficient of the i-th spring.

The initial state of the chain at time t_0 is given by the displacements w_i^0 and the velocities of the loads $\dot{w}_i^0$.

The equations of the springs system may be interpreted as a spatial discretization of a one-dimensional wave equation for a string with fixed left end and a control force applied to the free right end:

$$\rho(\xi)w_{tt}(t,\xi) = \left[E(\xi)w_\xi(t,\xi)\right]_\xi, \quad t > t_0, \quad 0 < \xi < L;$$

$$w(t,0) = 0, \quad w_\xi(t,L) = E^{-1}(L)u, \quad t \ge t_0; \quad w(t_0,\xi) = w^0(\xi), \quad w_t(t_0,\xi) = \dot{w}^0(\xi), \quad 0 \le \xi \le L.$$

(Here $w(t,\xi)$ is the displacement of point ξ at time t; each point ξ of the string is characterized by Young modulus $E(\xi)$ and mass density $\rho(\xi)$.)

The goal of the control is to steer the system to the equilibrium in given finite time.

In our numerical experiments we used the following values of parameters: $m_i = 1$, $k_i = 1$, $t_0 = 0$, $t_1 = 6\pi N$, $w_i^0 = \dot{w}_i^0 = 5$, $r(t) = 0$, $R(t) = \mathrm{diag}(10^{-4}I, 10^4 I)$ (i.e. displacements are measured with relatively small error $\pm 0{,}01$ and velocities are measured with large error ± 100), $h = 2N$, $\Delta t = 0{,}1$ for ellipsoidal filter, $p(t) = 0$, $P(t) = 1$, $m = 0$, $M = I$.

We used the worst-case measurement ($\xi(t) \equiv 0$), leading to the largest possible information set.

In Figure 2 we show the dependence of diameter of the ellipsoidal information set on model time $t - t_0$. Note that this plot is the same for any size of chain N.

In Figures 3 and 5 we present simulation results for $N = 2$ and $N = 10$ respectively. For both cases we indicate the realized control $u[t]$ and the total energy of the system as a function of time.

In Figure 4 we vary the feedback delay, h, in order to see how this affects the solution. For each value of h we plot the total energy of the damped system at the end of simulation (at time t_1). We see that up to a certain value of h (approximately 20) the increase in feedback delay does not influence the solution and the final energy is roughly zero. Beyond that level of h, the system cannot be dumped completely.

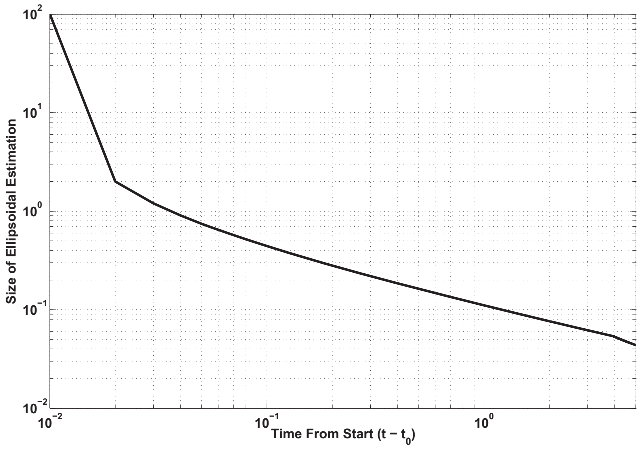

Figure 2: Diameter of ellipsoidal information set versus time from start $(t - t_0)$

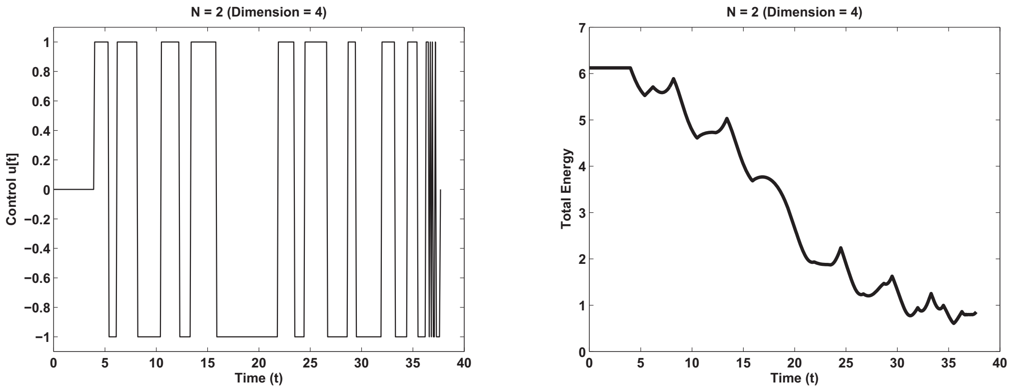

Figure 3: Simulation results for $N = 2$

A Appendix: The Solution to the Basic Linear-Convex Problem

In this section we summarize main facts about the solution of the linear-convex problem (see [3, 4]).

The problem under consideration is to steer a linear system (1) from initial state $(t, x(t))$ to the terminal target set $\mathscr{M}$ at fixed time t_1, by choosing a control u in feedback form. One has also to indicate the set of initial states from which the problem is solvable, i.e. the solvability set $\mathscr{W}[t]$.

The control $u = U(t,x)$ belongs to the class of set-valued mappings, measurable in t, upper semicontinuous in x, taking non-empty convex compact values in $\mathscr{P}(t)$. The closed-loop system is thus a differential inclusion. Existence and extendability of solutions to the latter is guaranteed by the mentioned properties of $U(t,x)$ [2].

The value function is equal to $V(t,x) = d\left(X(t_1,t)x, X(t_1,t)\mathscr{W}[t]\right)$.

The optimal feedback control is the minimizer in the HJB equation:

$$U(t,x) = \underset{u \in \mathscr{P}(t)}{\mathrm{Argmin}} \langle V_x, B(t)u \rangle \,. \tag{12}$$

A.1 The Ellipsoidal Approximation

The internal ellipsoidal approximation of the solvability set $\mathscr{W}_-[t] = \mathscr{E}\left(w(t), W(t)\right)$

$$\dot{w}(t) = A(t)w(t) + B(t)p(t), \quad w(t_1) = 0;$$

$$\dot{W}(t) = A(t)W(t) + W(t)A^T(t) +$$

$$+ W^{\frac{1}{2}}(t)S(t)(B(t)P(t)B^T(t))^{\frac{1}{2}} + (B(t)P(t)B^T(t))^{\frac{1}{2}}S^T(t)W^{\frac{1}{2}}(t),$$

$$W(t_1) = \varepsilon^2 I;$$

$$S(t)P^{\frac{1}{2}}(t)B^T(t)s(t) = \lambda(t)W^{\frac{1}{2}}s(t), \quad S^T(t)S(t) = I;$$

$$\dot{s}(t) = -A^T(t)s(t), \quad s(t_1) = \ell.$$

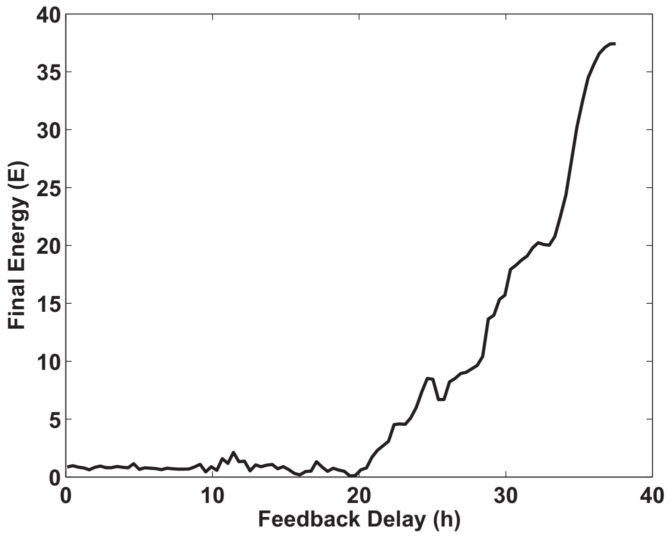

Figure 4: "Quality" of solution (expressed via chain energy at final time) versus feedback delay

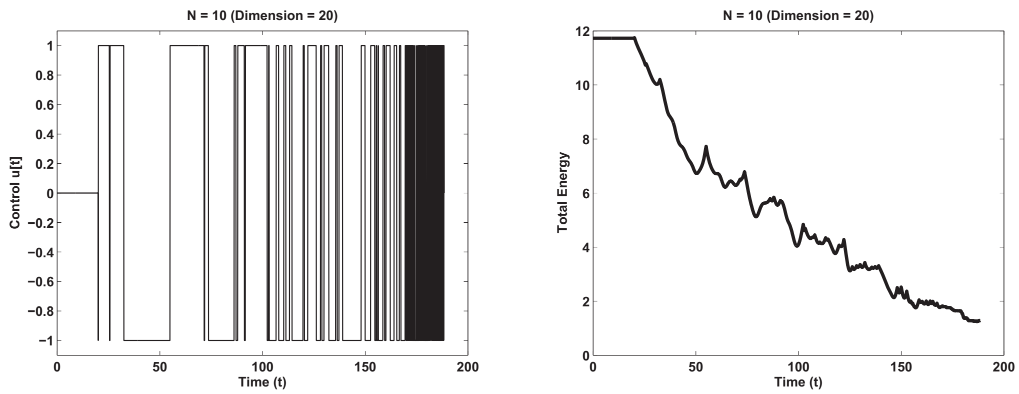

Figure 5: Simulation results for $N = 10$

The feedback control is found by substituting the upper estimate $V_+(t,x) = d(X(t_1,t)x, X(t_1,t)\mathscr{W}_-[t])$ into (12). Namely,

$$U(t,x) = \operatorname*{Argmin}_{u\in\mathscr{P}(t)} \langle \ell_0, B(t)u\rangle\,,$$

$$\ell^0 = 2\lambda(W+\lambda F)^{-1}(x-w), \quad F = X^T(t,t_1)X(t,t_1),$$

where λ is the unique non-negative root of

$$\big\langle (W+\lambda F)^{-1}(x-w), W(W+\lambda F)^{-1}(x-w)\big\rangle = 1,$$

or $\ell^0 = 0$ if there are no positive roots.

A.2 The Ellipsoidal Control Synthesis

One gets a simpler expression for feedback control using the following estimate of the value function:

$$\hat{V}(t,x) = \big(\big\langle x-w(t), W^{-1}(t)(x-w(t))\big\rangle - 1\big) \vee 0.$$

Then

$$U(t,x) = \operatorname*{Argmin}_{p\in\mathscr{P}(t)} \big\langle W^{-1}(t)(x-w(t)), B(t)u\big\rangle =$$

$$\begin{cases} \mathscr{P}(t), & B^T(t)W^{-1}(t)(x-w(t)) = 0; \\ p(t) - \dfrac{P(t)B^T(t)W^{-1}(t)(x-w(t))}{\langle B^T(t)W^{-1}(t)(x-w(t)), P^{-1}(t)B^T(t)W^{-1}(t)(x-w(t))\rangle}, & \text{otherwise.} \end{cases}$$

and $U(t,x) = \mathscr{P}(t)$ inside $\mathscr{W}_-[t]$.

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