# AN ODE FOR THE RENAISSANCE

**Felix Breitenecker, Katharina Breitenecker, Florian Judex, Nikolas Popper, Siegfried Wassertheurer**

ARGESIM – ARGE Simulation News, Vienna University of Technology, Wiedner Hauptstrasse 8-10, 1040 Vienna, Austria

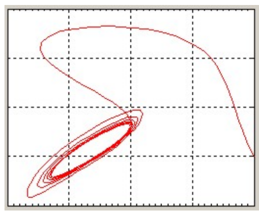
*Felix.Breitenecker@tuwien.ac.at (Felix Breitencker)*

## Abstract

- Is it possible to model in some detail the love dynamics between two person by ODEs ?
- Is it possible to describe a famous historic love story between two persons by ODEs ?
- Suppose, one person expresses its love by odes, is it possible to express it also by ODEs ?
- Is it possible to describe the poet's inspiration for writing odes by ODEs ?

All these questions can be answered positively, making use of three coupled nonlinear differential equations. Two states mimic the love for each other, coupled like a predator-prey model; a third state evaluates the poetic inspiration, following the poet's love emotions. Based on works by S. H. Strogatz, S. Rinaldi, and F. J. Jones (mathematics, literary science), one concludes:


$$\frac{dL}{dt} = -\alpha_1 L + R_L P + \beta_1 A_P$$
$$\frac{dP}{dt} = -\alpha_2 P + R_P L + \beta_2 A_L$$



- Laura, a beautiful, but married lady, inspired Petrarch for poems, which express ecstatic love as well as deep despair. Petrarch's grade of emotion in love for Laura can be established by evaluation of poems written between 1328 and 1350, showing Petrarch's emotional 'oscillating' behaviour between ecstatic love and deep despair.
- A nonlinear coupled ODE model can simulate love dynamics of two persons, taking into account mutual attraction, rejection, and neglect. Poetic inspiration can be modelled as function of love emotion.
- This model can be identified for Petrarch and Laura, getting an love dynamics for Petrarch, which coincides with Petrarch's emotional cycle derived from 'graded' poems.
- Analysis of the model shows a broad variety of phenomena: limit cycles, stable steady states, bifurcations, ….
- Experiments with this model and with an extended model for modern times, supported by a GUI, provide interesting case studies for different kind of love dynamics - attraction, rejection and neglect. Stable equilibria or limit cycles – which mimicry better real life love dynamics ?

**Keywords: Love Dynamics, Limit Cycles, Dynamical Systems**

### Presenting Author's biography

**Felix Breitenecker** studied 'Applied Mathematics' and acts as professor for Mathematical Modelling and Simulation at Vienna University of Technology He covers a broad research area, from mathematical modelling to simulator development, from DES via numerical mathematics to symbolic computation, from bio-medical and mechanical simulation to process simulation. He is active in various simulation societies: president and past president of EUROSIM since 1992, board member and president of the German Simulation Society ASIM, member of INFORMS, SCS, etc.He has published about 250 scientific publications, and he is author of two 3 books and editor of 22 books. Since 1995 he is Editor in Chief of the journal editing the journal Simulation News Europe.

*ISBN 978-3-901608-32-2*  

## 1 Introduction

Dynamic phenomena in physics, biology, economics, and all other sciences have been extensively studied with differential equations, since Newton introduced the differential calculus.

But the dynamics of love, perhaps the most important phenomenon concerning our lives, has been tackled only very rarely by this calculus. In literature two contributions can be found:

- *Love Affairs and Differential Equations* by S.H. Strogatz ([1]), - harmonic oscillators making reference to Romeo and Juliet, and
- *Laura and Petrach: an Intriguing Case of Cyclical Love Dynamics* by S. Rinaldi ([2]) – presenting a nonlinear ODE with cyclic solutions.

This contributions gives some results of works continuing the approach and analysis of Rinaldi ([2]), and of works with an extended model.

Section 2 presents the classification of Petrarch's poems by F. Jones, introducing Petrarch's emotional cycle. Sections 3 summarises the ODE model approach for the Laura-Petrarch model by S. Rinaldi. Section 4 introduces an equivalent transfer function model, which gives insight into the principle model structure and prepares a basis for a model extension.

Section 5 shows identification results for the Laura–Petrarch model based on a MATLAB implementation with a GUI (MATLAB graphical user interface). Section 6 presents analytical analysis for the Laura–Petrarch model as well as numerical analysis and experiments with this model.

Section 7 extends Laura–Petrarch model to a general model for the dynamic love cycle – the Woman-Man model, perhaps also applicable for modern times and modern love affairs, Section 8 concludes with experiments with the Woman-Man model, showing interesting case studies with changing appeal, etc.

## 2 Classification of Petrarch's poems

Francis Petrarch (1304-1374), is the author of the *Canzoniere*, a collection of 366 poems (sonnets, songs, sestinas, ballads, and madrigals). In Avignon, at the age of 23, he met Laura, a beautiful but married lady. He immediately fell in love with her and, although his love was not reciprocated, he addressed more than 200 poems to her over the next 21 years. The poems express bouts of ardour and despair, snubs and reconciliations, making Petrarch the most lovesick poet of all time.

Unfortunately, only a few lyrics of the *Canzoniere* are dated. The knowledge of the correct chronological order of the poems is a prerequisite for studying the lyrical, psychological, and stylistical development of Petrarch and his work.

Fig.1 Portraits of Laura and Petrarch, from Biblioteca Medicea Laurenziana, ms. Plut. cc. VIIIv- IX, Florence, Italy (courtesy of Ministero per i Bene Culturali e Ambientali), from [2]


Fig.2 Portraits of Laura and Petrarch, from Internet resources

For this reason, the identification of the chronological order of the poems of the *Canzoniere* has been for centuries a problem of major concern for scholars.

In 1995, Frederic Jones presented an interesting approach and solution to the chronological ordering problem of Petrarch's poems in his book *The Structure of Petrarch's Canzoniere* ([3]).

Jones concentrated on Petrarch's poems written at lifetime of Laura (the first, sonnet X, was written in 1330 and the last, sonnet CCXII, in 1347). First, he analysed 23 poems with fairly secure date. After a careful linguistic and lyrical analysis, he assigned grades for the poems, ranging from -1 to +1, establishing *Petrarch's emotional cycle*: the maximum grade (+1) stands for ecstatic love, while very negative grades correspond to deep despair.

The following examples (in quotations, the English version is taken from an English translation of the *Canzoniere* by Frederic Jones) illustrate some of these grades:

- Sonnet LXXVI, great love, grade +0.6:

  *Amor con sue promesse lunsingando,*
  *mi ricondusse alla prigione antica*

  [*Love's promises so softly flattering me*
  *have led me back to my old prison's thrall.*]

- Sonnet LXXIX, great despair, grade -0.6:

  *Cosi mancando vo di giorno in giorno,*
  *si chiusamente, ch'i' sol me ne accorgo*
  *et quella che guardando il cor mi strugge.*

  [*Therefore my strength is ebbing day by day,*
  *which I alone can secretly survey,*
  *and she whose very glance will*
  *scourge my heart.*]

- Sonnet CLXXVI, melancholy, grade -0.4:

  *Parme d'udirla, udendo i rami et l'ore*
  *et le frondi, et gli augei lagnarsi, et l'acque*
  *mormorando fuggir per l'erba verde.*

  [*Her I seem to hear, hearing bough and wind's caress, as birds and leaves lament, as murmuring flees the streamlet coursing through the grasses green.*]

Displaying the grades over time (Figure 3), F. Jones detected an oscillating behaviour, which he called *Petrarch's emotional cycle* $E(t)$, with a period of about four years.

In a second step, F. Jones analysed and 'graded' all the other poems with unknown date and checked, in which part of the cycle they could fit. Taking into account additional historical information, he could date these poems by locating them in Petrarch's emotional cycle $E(t)$, see Figure 3.

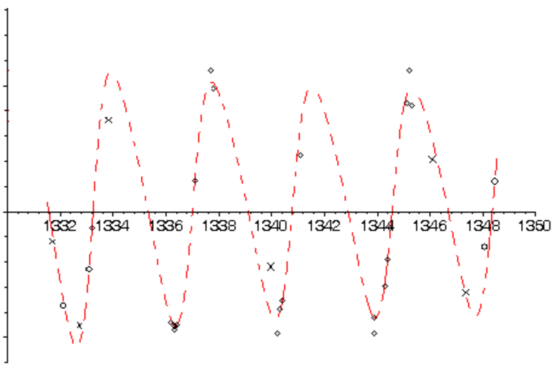



Fig.3 *Petrarch's emotional cycle* $E(t)$ – dashed line, with some 'graded' poems (crosses for securely dated poems, and circles for poems dated based on the emotional cycle)

## 3 Mathematical Modelling Approach

The big challenge for S. Rinaldi was to set up an ODE model for the cyclic love dynamics and emotional dynamics of Petrarch and Laura, which would fit the experimentally founded emotional cycle of Petrarch.

Basis for modelling is a predator-prey – type dynamics between the variable $L(t)$ and $P(t)$. There $L(t)$ represents Laura's love for the poet at time $t$; positive and high values of $L$ mean warm friendship, while negative values should be associated with coldness and antagonism.

$P(t)$ describes Petrarch's love for Laura, whereby high values of $P$ indicate ecstatic love, while negative values stand for despair. The general *Laura-Petrarch model* is given by the following equation:

$$\frac{dL(t)}{dt} = -\alpha_L L(t) + R_L(P(t)) + \beta_L A_P$$
$$\frac{dP(t)}{dt} = -\alpha_P P(t) + R_P(L(t)) + \beta_P A_L$$

There $R_L(P)$ and $R_P(L)$ are reaction functions to be specified later, and $A_P$ and $A_L$ is the appeal (physical, as well as social and intellectual) of Petrarch and Laura to each other.

Next, it is necessary to model in more detail the complex personality Petrarch. A second variable has to be used to describe his poetic inspiration $I_P(t)$, expressing his productivity for poems (in general modelling the inspiration for work related to the love), and influencing the appeal $A_L$:

$$\frac{dL(t)}{dt} = -\alpha_L L(t) + R_L(P(t)) + \beta_L A_P$$
$$\frac{dP(t)}{dt} = -\alpha_P P(t) + R_P(L(t)) + \beta_P \frac{A_L}{1+\delta_P I_P(t)}$$
$$\frac{dI_P(t)}{dt} = -\alpha_{IP} I_P(t) + \beta_{IP} P(t)$$

The rate of change of the love of Laura $L(t)$ is the sum of three terms. The first describes the forgetting process characterising each individual. The second term $R_L(P)$ is the reaction of Laura to the love of Petrarch, and the third is her response to his appeal.

The rate of change of the love of Petrarch $P(t)$ is of similar structure, with an extension: the response of Petrarch to the appeal of Laura depends also upon his inspiration $I_P(t)$. This takes into account the well-established fact that high moral tensions, like those associated with artistic inspiration, attenuate the role of the most basic instincts.

And there is no doubt that the tensions between Petrarch and Laura are of a passionate nature, as can be read in literature.

- In Sonett XXII, Petrarch writes:

*Con lei foss'io da che si parte il sole,*
*et non ci vedess' altri che le stelle,*
*sol una nocte, et mai non fosse l'alba.*

[*Would I were with her when first sets the sun,*
*and no one else could see us but the stars,*
*one night alone, and it were never dawn.*]

- And in his *Posteritati*, Petrarch confesses:

*Libidem me prorsus expertem dicere*
*posse optarem quidem, sed si dicat mentiar.*

[*I would truly like to say absolutely that*
*I was without libidinousness, but if I said*
*so I would be lying*].

The equation for the inspiration $I_P(t)$ says that the love of Petrarch sustains his inspiration which, otherwise, would exponentially decay.

The reaction functions $R_L(P)$ and $R_P(L)$ are partly nonlinear. A linear approach would simply say, that individuals love to be loved and hate to be hated. The linearity of $R_P(L)$ is more or less obvious, since in his poems the poet has very intense reactions to the most relevant signs of antagonism from Laura:

$$R_P(L(t)) = \beta_P \cdot L(t)$$

A linear reaction function is not appropriate for Laura. Only around $R_L(P) = 0$ it can be assumed to be linear, thus interpreting the natural inclination of a beautiful high-society lady to stimulate harmless flirtations. But Laura never goes too far beyond gestures of pure courtesy: she smiles and glances.

However, when Petrarch becomes more demanding and puts pressure on her, even indirectly when his poems are sung in public, she reacts very promptly and rebuffs him, as described explicitly in a number of poems.

- In Sonnet XXI, Petrach claims:

*Mille fiate, o dolce mia guerrera,*
*per aver co' begli occhi vostri pace*
*v'aggio proferto il cor; ma voi non piace*
*mirar si basso colla mente altera.*

[*A thousand times, o my sweet enemy,*
*to come to terms with your enchanting eyes*
*I've offered you my heart, yet you despise*
*aiming so low with mind both proud and free.*]

Consequently, the reaction function $R_L(P)$ should, for $P > 0$, first increase, and then decrease. But the behaviour of Laura' reaction is also nonlinear for negative values of $P$.

In fact, when $P < 0$ (when the poet despairs), Laura feels very sorry for him. Following her genuine Catholic ethic she arrives at the point of overcoming her antagonism by strong feelings of pity, thus reversing her reaction to the passion of the poet. This behavioural characteristic of Laura is repeatedly described in the *Canzoniere*.

- In Sonnet LXIII Petrarch writes:

*Volgendo gli occhi al mio novo colore*
*che fa di morte rimembrar la gente,*
*pieta vi mosse; onde, benignamente*
*salutando, teneste in vita il core.*

[*Casting your eyes upon my pallor new,*
*which thoughts of death recalls to all mankind,*
*pity in you I've stirred; whence, by your kind*
*greetings, my heart to life's kept true.*]

A good choice for Laura's reaction function $R_L(P)$ would be a cubic function of the following type, displayed in Figure 4:

$$R_L(P) = \beta_L P \cdot \left(1 - \left(\frac{P}{\gamma_L}\right)^2\right)$$

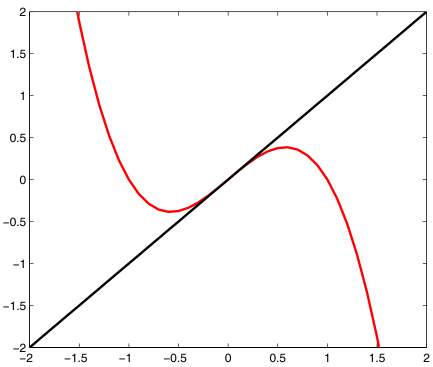



Fig.4 Nonlinear reaction function $R_L(P)$ of Laura - red line, and linear reaction function $R_P(L)$ of Petrarch – straight black line

With these approaches, the full *Laura-Petrarch model* is given by the following equations:

$$\frac{dL(t)}{dt} = -\alpha_L L(t) + \beta_L P\left(1 - \left(\frac{P}{\gamma_L}\right)^2\right) + \beta_L A_P$$

$$\frac{dP(t)}{dt} = -\alpha_P P(t) + \beta_P L(t) + \beta_P \frac{A_L}{1 + \delta_P I_P(t)}$$

$$\frac{dI_P(t)}{dt} = -\alpha_{IP} I_P(t) + \beta_{IP} P(t)$$

## 4 Model approach by transfer functions

It is worth trying a model approach by transfer functions with nonlinear elements: first it gives insight into the model structure, and second, it prepares a natural basis for extending the model.

At a first glance, supposing only linear relations, a model for Laura's and Petrarch's love dynamics $L(t)$ and $P(t)$ consist of first order transfer functions for the dynamic behaviour, with gain $\beta_L$ and time constant $\alpha_L$ for Laura, and with gain $\beta_P$ and time constant $\alpha_P$ for Petrarch.

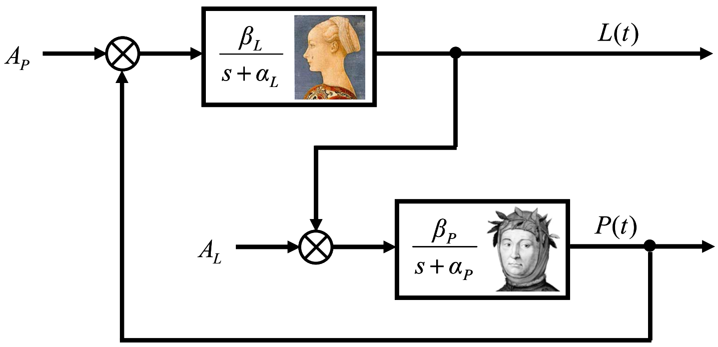



Fig.5 Linear transfer function model for Laura's and Petrarch's love dynamics $L(t)$ and $P(t)$

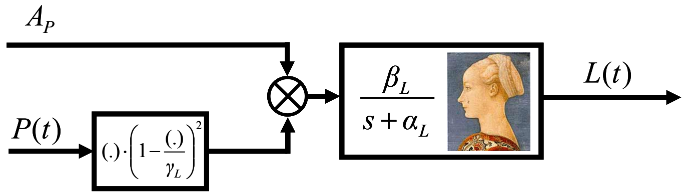



Fig.6 Transfer function model for Laura's love $L(t)$ with nonlinear gain for Petrarch's love $P(t)$ as reaction input

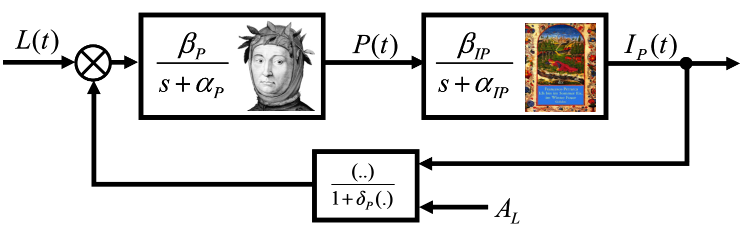



Fig.7 Transfer function model for Petrarch's love $P(t)$, with additional transfer function for poetic inspiration $I_P(t)$, setting up the nonlinear gain for Laura's appeal $A_L$

The love of both is driven by the appeal $A_P$ and $A_L$ to each other, being set point and input to the respective transfer function.

The second driving power for the love is clearly the reaction of Laura to Petrarch's love, and the reaction of Petrarch to Laura's emotions. Taking a linear approach, the reaction functions are simple the grade of love of the partner, being the second input to the respective transfer function. Figure 5 summarises this approach as block diagram.

The nonlinear reaction function $R_L(P)$ of Laura can be interpreted as additional additive nonlinear gain $R_L(P)$ (modified cubic gain with parameter $\gamma_L$) as input to Laura's transfer function, see Figure 6.

For Petrarch's nonlinear dynamics first an additional transfer function for his poetic inspiration $I_P(t)$ is introduced, with gain $\beta_{IP}$ and time constant $\alpha_{IP}$ . This variable acts as input for a nonlinear gain for Laura's appeal $A_L$ with parameter $\delta_P$; both additional blocks are summarised in Figure 7.

The nonlinearities are of different quality. Choosing in the nonlinear cubic-like gain for Laura's reaction $R_L(P)$ a big value for the parameter $\gamma_L$ , the nonlinear gain becomes almost linear (the nominator is bounded, usually less than 1). The nonlinear gain for Laura's appeal $A_L$ becomes linear, if the parameter $\delta_P$ is set to zero, letting the influence of Petrarch's poetic inspiration vanish.

Now the nonlinear transfer function models for Laura and Petrarch can be combined to a nonlinear transfer function model for the love dynamics of Laura and Petrarch $L(t)$ and $P(t)$, and the poetic inspiration dynamics of Petrarch $I_P(t)$, presented in Figure 8.

The structure of the transfer function model suggest a natural extension: also Laura writes poems, so that Petrarch's appeal is influenced by her poetic inspiration, and Petrarch shows more sensibility in his reaction to Laura. The model would become 'symmetric'; some aspects of an extended model will be discussed in Section 7 and Section 8.

## 5 Identification of the Laura–Petrarch model

The big challenge is to identify the model parameters in the nonlinear Laura-Petrarch model, with two appeal parameters, with three gains, with three time constants, and with two parameters for the nonlinearity – in sum ten parameters.

A brute-force identification starting with arbitrary values for these parameters is not successful, especially as the appeals may also be negative.

ISBN 978-3-901608-32-2

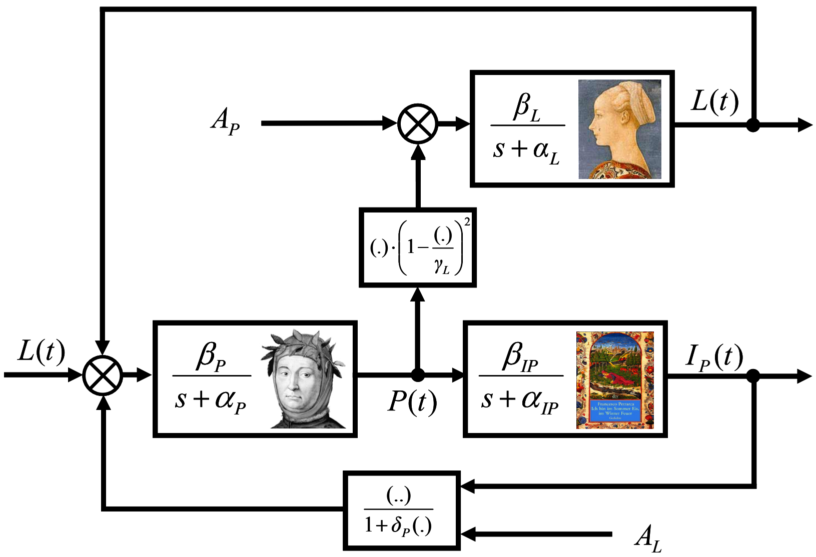



Fig.8 Nonlinear transfer function model for the love dynamics of Laura and Petrarch, $L(t)$ and $P(t)$, and for the poetic inspiration dynamics of Petrarch $I_P(t)$; model equivalent to ODE model

*Madonna e morta, et a*
*seco il mio core;*
*e volendol seguire,*
*interromper conven quest'anni rei,*
*perche mai veder lei*
*di qua non spero, et l'aspettar*
*m'e noia.*

[*It's time indeed to die,*
*and I have lingered more than I*
*desire.*
*My lady's dead, and with her my*
*heart lies;*
*and, keen with her to fly,*
*I now would from this wicked world*
*retire,*
*since I can no more aspire*
*on earth to see her, and delay will*
*me destroy*.]

Consequently between the time constants $\alpha_{IP}$ and $\alpha_p$ the relation $\alpha_{IP} < \alpha_p$ must hold.

Consequently first the size of the parameters and relations between them should be qualitatively analysed, following S. Rinaldi ([2]).

The time constants $\alpha_L$, $\alpha_p$, and $\alpha_{IP}$ describe the forgetting processes. For Laura and Petrarch obviously $\alpha_L > \alpha_p$ holds, because Laura never appears to be strongly involved, while the poet definitely has a tenacious attachment, documented by poems.

- In sonnet XXXV Petrarch claims:

  *Solo et pensoso i piu deserti campi*
  *vo mesurando a passi tardi e lenti,*
  …….
  *Ma pur si aspre vie ne' si selvage*
  *cercar non so ch' Amor non venga sempre*
  *ragionando con meco, et io col'lui.*

  [*Alone and lost in thought, each lonely strand*
  *I measure out with slow and laggard step,*
  ………
  *Yet I cannot find such harsh and savage trails*
  *where love does not pursue me as I go,*
  *with me communing, as with him do I.*]

The inspiration of the poet wanes very slowly, because Petrarch continues to write (over one hundred poems) for more than ten years after the death of Laura. The main theme of these lyrics is not his passion for Laura, which has long since faded, but the memory for her and the invocation of death.

- In sonett CCLXVIII, written about two years after Laura's demise, Petrarch remembers:

  *Tempo e ben di morire,*
  *et o tardato piu ch'i non vorrei.*

As Petrarch's inspiration holds about ten years, whereas Laura forgets Petrarch in about four months, and Petrarch's passion fades in one year, suitable relations and values are

$$\alpha_L \sim 3\cdot\alpha_P,\ \alpha_P \sim 10\cdot\alpha_{IP},\ \alpha_P \sim 1$$

The gains or reaction parameters $\beta_L$, $\beta_p$, and $\beta_{IP}$ also can be estimated qualitatively, with respect to the time constants:

$$\beta_L \sim \alpha_P,\ \beta_P \sim 5\cdot\alpha_P,\ \beta_{IP} \sim 10\cdot\alpha_P$$

Here the assumption is that Laura's reaction equals the forgetting time of Petrarch, and Petrarch reacts five time stronger.

For simplicity, the parameters $\gamma_L$ and $\delta_P$ are normalised to one, since it is always possible to scale $P(t)$ and $I_P(t)$ suitably.

The choice of the appeal parameters $A_L$ and $A_P$ is crucial, because these parameters determine the qualitative behaviour of the love dynamics – cyclic nonlinear behaviour, or damped oscillation toward an equilibrium. In case of Laura and Petrarch, cyclic love dynamics are expected in order to meet the experimentally founded emotional cycle $E(t)$ of Petrarch.

Clearly, Petrarch loves Laura, so $A_L > 0$ must hold. By contrast, Petrarch is a cold scholar interested in history and letters. He is appointed a *cappellanus continuus commensalis* by Cardinal Giovanni Colonna, and this ecclesiastic appointment brings him frequently to Avignon, where Laura lives.

Consequently Petrarch's appeal $A_P$ is assumed to be negative. Appropriate choices for the appeals $A_L$ and $A_P$ are:

$$A_L \sim 2, \;\; A_P \sim -1$$

The negativity of the appeal of Petrarch for Laura is somehow recognized by the poet himself:

- In sonnet XLV, while Petrarch is talking about Laura's mirror, he says

  *Il mio adversario in cui veder solete*
  *gli occhi vostri ch'Amore*
  *e'l ciel honora, ...*

  [*My rival in whose depths*
  *you're wont to see*
  *your own dear eyes which*
  *Love and heaven apprize, ...*]

The above estimated ten parameter values, together with zero initial values for the love dynamics and for the poetic inspiration, are a good choice for identification.

The ODE model has been implemented in MATLAB (for comparison, the transfer function model has been implemented in Simulink); for identification, a least squares method can be used:

$$\sum (P(t_k) - E_k)^2 \quad \rightarrow \quad \min$$

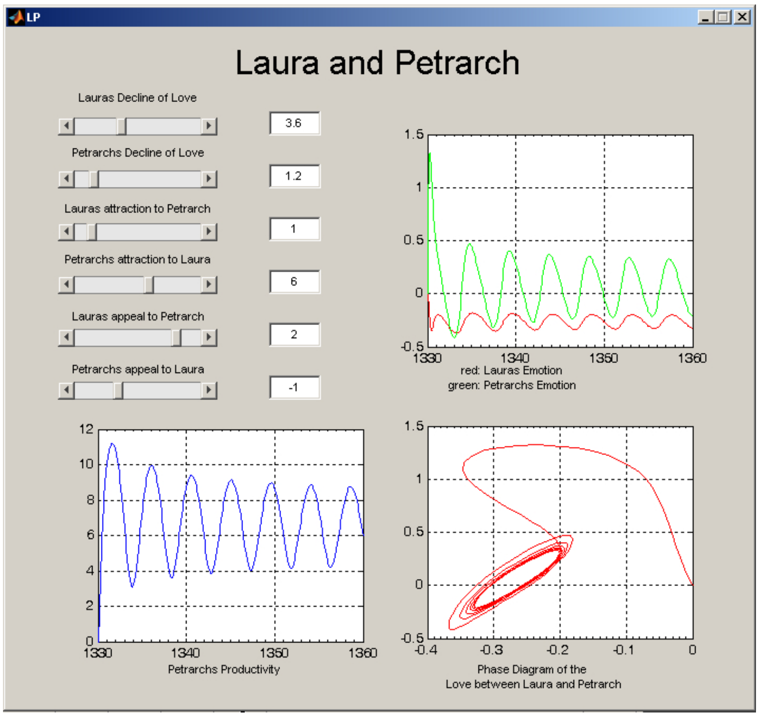



Fig. 10 MATLAB GUI for experimenting with the Laura-Petrarch model:
i) above left: sliders for gain $\beta_L$, time constant $\alpha_L$, gain $\beta_P$, time constant $\alpha_P$, and appeals $A_L$ and $A_P$;
ii) upper right: love dynamics for Laura - $L(t)$, green - and Petrarch - $P(t)$, red – over time period 1130 - 1360;
iii) lower left: poetic inspiration of Petrarch - $I_P(t)$, blue - over time period 1130 – 1360;
iv) lower right: phase portrait $P(L)$ of love dynamics of Petrarch and Laura – P over L

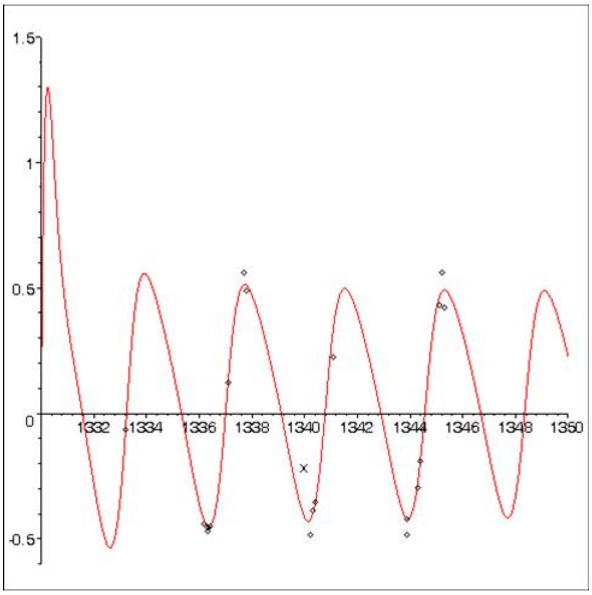


Fig.9 Result of model identification: love dynamics $P(t)$ for Petrarch coinciding with data from Petrarch's emotional cycle $E(t_k)$, with data $E_k$ (crosses and circles)

There, it makes sense to use relations between parameters, so that the number of parameters to be identified is reduced. As the quality of data is relatively poor, also different sets of parameter values may be seen as good approximation. Figure 9 shows an identification result for $P(t)$, with data $E_k$ ('graded' poems).

Of interest are of course also the love dynamics $L(t)$ for Laura and Petrarch's poetic inspiration $I_P(t)$. All variables are presented in a MATLAB GUI, which drives experiments with the Laura-Petrarch model. The GUI (Figure 10) offers parameter input (sliders) and displays time courses for $P(t)$ and $L(t)$ (together), the time course for $I_P(t)$, and a phase portrait $P(L)$.

Figure 10 shows all results for the identified parameters. The results of the numerical solution are qualitatively in full agreement with the *Canzoniere* and with the analysis of Frederic Jones. After a first high peak, Petrarch's love $P(t)$ tends toward a regular cycle characterised by alternate positive and negative peaks. Also, Laura's love $L(t)$ and Petrarch's poetic inspiration $I_P(t)$ tend towards a cyclic pattern.

At the beginning, Petrarch's inspiration $I_P(t)$ rises much more slowly than his love and then remains positive during the entire period. This might explain why Petrarch writes his first poem more than three years after he has met Laura, but then continues to produce lyrics without any significant interruption.

By contrast, Laura's love is always negative. This is in perfect agreement with the *Canzoniere*, where Laura is repeatedly described as adverse.

- In sonnet XXI, Petrarch calls Laura

  *dolce mia guerrera*

  [*my sweet enemy*].

- But in sonnet XLIV Petrarch says:

  *ne lagrima pero discese anchora*
  *da' be' vostr'occhi, ma disdegno et ira.*

  [*and still no tears your lovely eyes assail,*
  *nothing as yet, but anger and disdain*.]

The fit between $P(t)$ for Petrarch’s love and $E(t_k)$ for Petrarch’s emotional cycle is actually very good. It is of similar quality which is usually obtained when calibrating models of electrical and mechanical systems. Moreover, the fit could be further improved by slightly modifying the parameter values and by loose some parameter relations.

But improvement might be skipped, citing and agreeing with Rinaldi: *I do not want to give the impression that I believe that Petrarch had been producing his lyrics like a rigid, deterministic machine.*

Nevertheless, one can conclude that the Laura-Petrach model strongly supports Frederic Jones's conjecture on Petrarch’s emotional cycle.

## 6 Experiments and Analysis with the Laura–Petrarch model

Experiments with the parameters show, that the cyclic love dynamics may change to a damped oscillation converging to equilibrium. It is difficult to find out which parameter quality causes a cyclic behaviour, and which the damped oscillations.

Starting with the classic Laura–Petrarch parameters, for instance an increase of only one parameter $\alpha_L$ by a factor of 2.5 changes the qualitative behaviour essentially (Figure 11) – this parameter change means, that Laura forgets Petrarch in about half time then before.

In this case all variables for $P(t)$, $L(t)$, and $I_P(t)$, show strongly damped oscillation, reaching an equilibrium almost in ten years. The phase portrait $P(L)$ underlines this convergence in relatively fast time to an equilibrium of about $P =$ 0.05 and $L$ = -0.0105. This equilibrium has still a negative value for Laura’s love emotion, but it is very small – almost as small as the positive value for Petrarch’s emotion (see Figure 11).

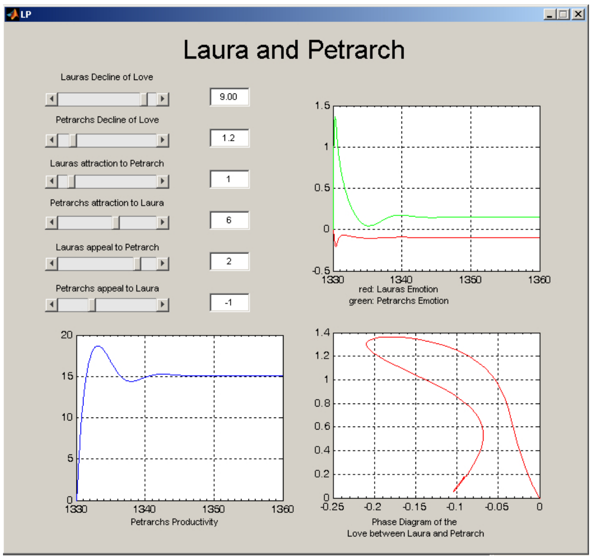



Fig.11 Experiment with Laura–Petrarch parameters and changed parameter $\alpha_L$ by a factor of 2.5: strongly damped variables converging to a an equilibrium with very small positive and very small negative value for $P$ and $L$, resp.

In principle, the existence of an equilibrium is tied to the existence of steady state solutions for the ODE. Setting all derivatives to zero, and substituting $I_P$ by the corresponding steady state for $P$, i.e.

$$I_P(t \to \infty) = \frac{\beta_{IP}}{\alpha_{IP}} P(t \to \infty),$$

results in two coupled nonlinear algebraic equations for $P$ and $L$ at steady state:

$$\begin{aligned} 0 &= -\alpha_L L + \beta_L P - \beta_L P^3 + \beta_L A_P \\ 0 &= -\alpha_P(\alpha_{IP} + \beta_{IP} P)P + \\ &\quad \beta_P(\alpha_{IP} + \beta_{IP} P)L + \alpha_{IP}\beta_P A_L \end{aligned}$$

Rinaldi investigates in detail by means of analytical methods the case $A_L < 0$ and $A_P > 0$. First he tested the robustness of the Laura – Petrarch cyclic love dynamics with respect to perturbations of the parameters. For this, the package LOCBIF, a professional software package for the analysis of the bifurcations of continuous-time dynamical systems, has been used. By varying only one parameter at a time, Rinaldi detected a supercritical Hopf bifurcation, by which the cycle eventually disappears.

Rinaldi continued with a detailed analysis of the limit cycle using substitution of the variable $L(t)$, transformation of parameters, and singular perturbation for $P(t)$ and $I_P(t)$. Results derive that indeed a supercritical Hopf bifurcation causes a change of the qualitative behaviour. One detailed result is a stability chart (Figure 12) for new parameters $\varepsilon$ and $\mu$:

$$\varepsilon = \alpha_{IP},\ \mu = \frac{\beta_{IP}}{\alpha_{IP}},\ \ \beta_P \cdot \beta_L > \alpha_P \cdot \alpha_L$$

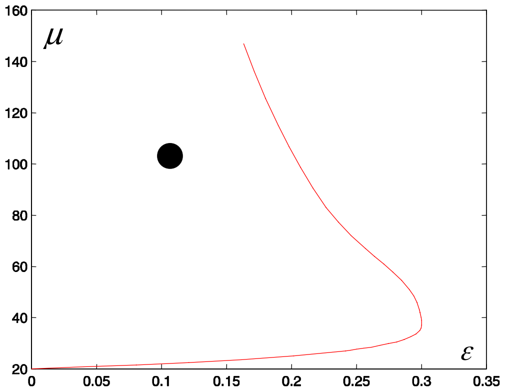



Fig.12 Stability chart $\mu(\varepsilon)$ showing the border (red line; Hopf bifurcation) between cyclic solutions (left) and solutions converging to equilibrium (right); the circle indicates the classical cyclic Laura–Petrarch solution

In Figure 12, the border between cyclic solutions and solutions converging to equilibrium can be seen, represented as graph $\mu(\varepsilon)$.

In general, cyclic solutions exist only in case of non-symmetric reactions, or as in simple case for appeal parameters with different signs. More details about these analytical investigations can be found in Rinaldi's work ([2]).

Usually, if people fall in love, both are attractive for each other, that means that $A_L$ and $A_P$ must be positive. In this case we meet only damped oscillations converging to a stable steady state:

$$A_L > 0, A_P > 0 \rightarrow \exists\,(P > 0, L > 0)$$

An interesting experiment is the case of an attractive Petrarch. Supposing e.g. that Petrarch is a young beautiful men, almost like Apollo, he may have the appeal $A_L \sim 6$ to Laura, three times the appeal of Laura to him (all other parameters unchanged).

Figure 13 shows the MATLAB GUI with the results of this experiment. All variables $P(t)$, $L(t)$, and $I_P(t)$, are very strongly damped and converge to steady states with relative high positive values. The poetic inspiration $I_P(t)$ shows almost (negative) exponential behaviour. Nevertheless $L(t)$ becomes negative for a short time period in the first three years. The phase diagram shows two crossings and it seems, that the love dynamics first cycles two times, before it decides to converge to a stable positive feeling of both lovers.

The MATLAB GUI allows experiments with a broad variety of parameter sets. Of interest are for instance also cases, which cycle more then hundred times, before they decide to converge to an equilibrium.

For 'extreme' parameter values the numerical solution may cause problems, or at least may take a long time; as ODE solver a stiff solver has been chosen, having the best relative performance.

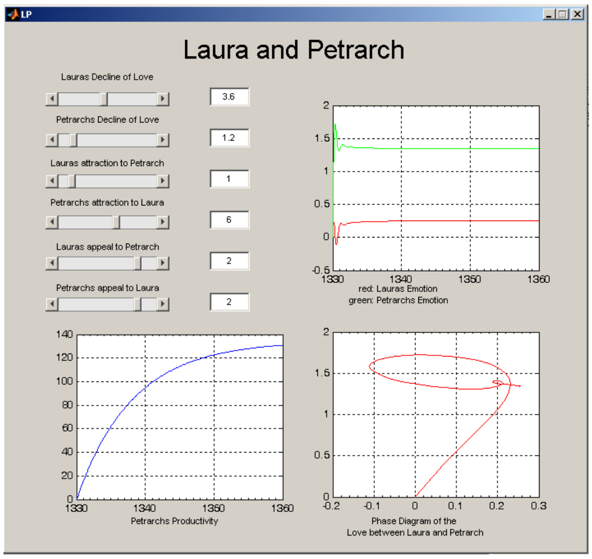



Fig.13 Experiment with Laura–Petrarch parameters and positive appeal $A_L \sim 6$ of Petrarch to Laura: very strongly damped variables converging to a an equilibrium with positive steady states for $P$ and $L$, resp.

## 7 From Laura-Petrarch model to Woman-Man - model

The structure of the transfer function model (Figure 8) suggests a genuine and natural extension: symmetric inputs, outputs, reaction functions and nonlinear gains as well for 'Petrarch' and for 'Laura', which should now generally represent a man and a women who fall in love. Figure 14 presents an extended nonlinear transfer function model with full symmetry.

In times of gender equality woman as well as men may play an active part in a love affair. Consequently also women express their love by poems or other media, and they confess their love to public. By this, the transfer function model for the woman's inspiration can be introduced easily.

For Laura and Petrarch this would mean, that also Laura writes poems, that Petrarch's appeal is influenced by Laura's poetic inspiration, and that Petrarch shows more sensibility in his reaction to Laura.

The *Woman–Man model* (Figure 14) describes the love dynamics $W(t)$ for a woman, and $M(t)$ for a man both falling in love to each other; the love inspire both the communicate their love to public, in letters, in videos, with CDs and DVDs, etc. – represented by the inspiration variables $I_W(t)$ and $I_M(t)$, which influence the appeal $A_M$ and $A_W$ to each other.

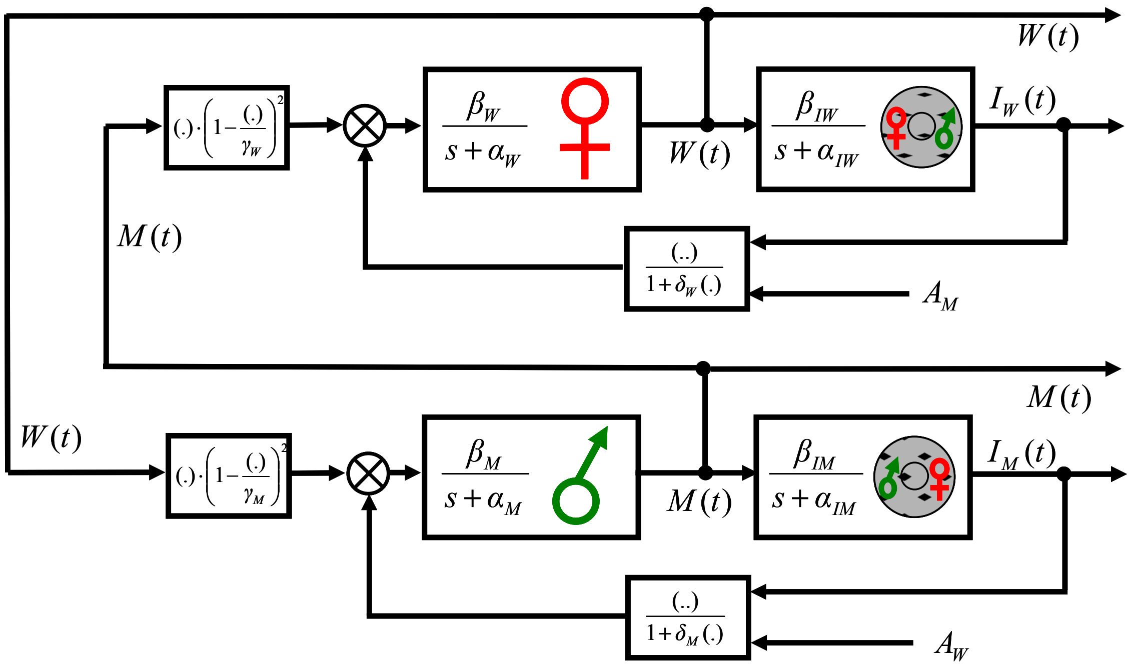



Fig.14 Transfer function – based *Woman–Man model* describing the love dynamics $W(t)$ for a woman, $M(t)$ for a man, and the inspiration variables $I_W(t)$ and $I_M(t)$ communicating the love to public

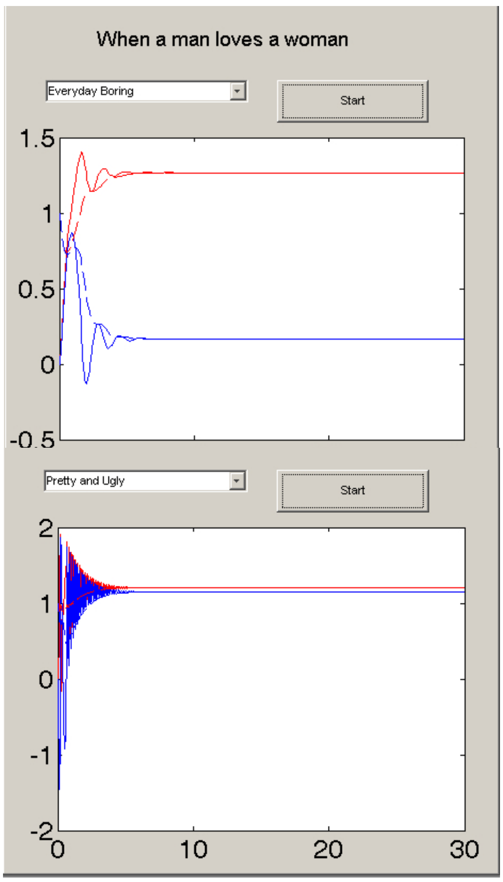



Fig.15 Experiment with Woman–Man model: case studies 'Everyday Boring' and 'Pretty and Ugly' (red/blue – woman's/men's love emotion; red/blue dashed – woman's/men's love inspiration)

These basic variables are modelled by linear transfer functions of first order, connected by nonlinear gain elements. These nonlinear gains are fully symmetric, representing the nonlinear reaction functions $R_W(M)$ and $R_M(W)$ and the nonlinear gains for the appeal parameters.

The number of parameters has increased now: four gains, four time constants, two appeal parameters, and four parameters in the nonlinear functions – in sum 14 parameters. An analysis of Man-Woman model is almost impossible, but numerical experiments may give interesting insight into love dynamics.

The equivalent ODE model also shows the symmetric structure:

$$\frac{dW(t)}{dt} = -\alpha_L W(t) + \beta_W M\left(1-\left(\frac{M}{\gamma_W}\right)^2\right) + \beta_W \frac{A_M}{1+\delta_W I_W(t)}$$

$$\frac{dI_W(t)}{dt} = -\alpha_{IW} I_W(t) + \beta_{IW} W(t)$$

$$\frac{dM(t)}{dt} = -\alpha_M M(t) + \beta_M W\left(1-\left(\frac{W}{\gamma_M}\right)^2\right) + \beta_M \frac{A_W}{1+\delta_M I_M(t)}$$

$$\frac{dI_M(t)}{dt} = -\alpha_{IM} I_M(t) + \beta_{IM} M(t)$$

As the Laura–Petrarch model, the Woman–Man model has been implemented in MATLAB (ODE model) and in Simulink (transfer functions model). Experiments can be controlled by a simplified MATLAB GUI, which suggest case studies (Figure 15).

Figure 15 shows results of two case studies in a GUI, with predefined parameters characterising typical parameter configurations ('Everyday Boring' and 'Pretty and Ugly').

## 8 Woman-Man model with dynamic appeal parameters

Does the Woman – Man model reflect reality ? The model is able to mimicry different situations, but with one assumption: the appeal parameters $A_M$ and $A_W$ are constant. This assumption may not meet reality, the appeal for each other may change and may be controlled.

A dynamic appeal can be easily modelled by time-dependent appeal variables $A_M(t)$ and $A_W(t)$, resulting in a small change in the ODE model:

$$\frac{dW(t)}{dt} = -\alpha_L W(t) + \beta_W M\left(1-\left(\frac{M}{\gamma_W}\right)^2\right) + \beta_W \frac{A_M(t)}{1+\delta_W I_W(t)}$$

$$\frac{dI_W(t)}{dt} = -\alpha_{IW} I_W(t) + \beta_{IW} W(t)$$

$$\frac{dM(t)}{dt} = -\alpha_M M(t) + \beta_M W\left(1-\left(\frac{W}{\gamma_M}\right)^2\right) + \beta_M \frac{A_W(t)}{1+\delta_M I_M(t)}$$

$$\frac{dI_M(t)}{dt} = -\alpha_{IM} I_M(t) + \beta_{IM} M(t)$$

Case studies may become now very complicated, because not only 14 parameters have to be chosen appropriately, but also the function $A_M(t)$ and $A_W(t)$ have to be provided meaningful. An extended version of the MATLAB GUI presented in Figure 15 allows additionally providing predefined appeal functions.

Figure 16 and Figure 17 show results for perhaps interesting cases: the appeal decreases exponentially – should happen, and second, the appeal $A_M(t)$ is an increasing step function - could model a plastic surgery.

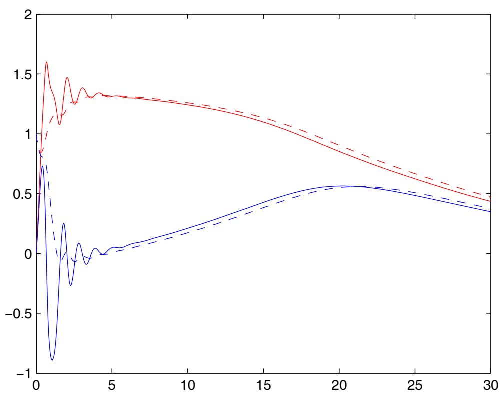


Fig.16 Experiment with Woman–Man model: exponentially decreasing appeal functions – all variables follow after five years the decreasing setup values given by the appeal functions (red/blue – woman's/men's love emotion; red/blue dashed – woman's/men's love inspiration)

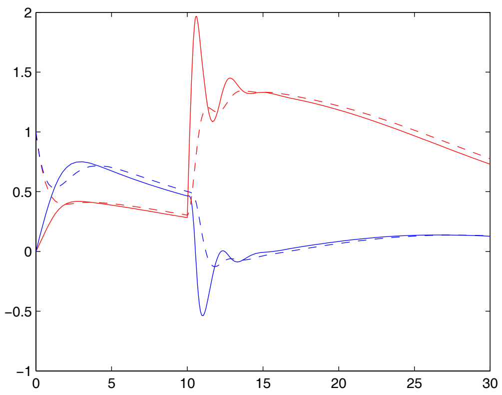


Fig.17 Experiment with Woman–Man model: positive step in appeal at ten years – before and after the step all variables converge to a steady state, different before and after the step (time courses as in Figure 16)

## 9 Conclusion

In principle, the contribution could answer the questions, raised in the abstract, positively:

- Is it possible to model in some detail the love dynamics between two person by ODEs ?
- Is it possible to describe a famous historic love story between two persons by ODEs ?
- Suppose, one person expresses its love by odes, is it possible to express it also by ODEs ?
- Is it possible to describe the poet's inspiration for writing odes by ODEs ?

The questions are more or less wordplays. Nevertheless they can be answered: the first question as well by the Laura–Petrarch model and by the Woman–Man model; the second by the Laura–Petrarch model; the third also by the Laura–Petrarch model, and the fourth by the poetic inspiration variable in the Laura–Petrarch model.

Of course, this contribution presents serious investigations. But is it possible to investigate the dynamics of love, perhaps the most important phenomenon concerning our lives, seriously by methods of mathematics and engineering? One could also conclude, it might be better not to tackle the secrets of love, because described and controlled by formula, it is not love anymore longer. In this view, the contribution might be seen as reference to Petrarch and the most beautiful love poems I ever read.

## 10 References


[1] S. H. Strogatz. Love affairs and differential equations. Math. Magazine, 61 (1988), p. 35.

[2] F. J. Jones. The Structure of Petrarch's Canzoniere. Brewer, Cambridge, UK, 1995.

[3] S. Rinaldi. Laura and Petrarch: an intriguing case of cyclical love dynamics. SIAM J. App.Math. Vol. 58 (1998), No. 4, pp. 1205-1221.