COMPUTATION OF POSITIVE REALIZATIONS OF MIMO HYBRID LINEAR SYSTEMS WITH DELAYS USING THE STATE VARIABLE DIAGRAM METHOD

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Abstract

In this paper the realization problem for 2D positive multi-input and multi-output (MIMO) linear hybrid systems with delays in state vector and input is addressed. A method based on the state variable diagram for finding positive realizations of a given proper transfer matrix is proposed. The essence of proposed method for solving of the realization problem for positive 2D hybrid systems with delays in state vector and input will be presented on single-input single-output (SISO) system. The solution for MIMO systems will be obtained by generalization of method proposed for SISO systems. Sufficient conditions for the existence of a positive realization of a given proper transfer matrix are established. This conditions gives only the answer is there exists positive realization for given proper transfer matrix, they do not consider the stability of obtained realization. A procedure for computation of a positive realization is proposed for SISO systems and generalized for MIMO systems. The considerations are illustrated by two numerical examples. First example illustrate solving procedure for SISO system with one delay and the second example illustrate solving procedure for system with two inputs, two outputs and one delay.

Keywords: hybrid 2D system, multi-input multi-output, delay, positive realization, computation.

Presenting Author's biography

Łukasz Sajewski. Born 08.12.1981 in Poland. Completed his studies in Bialystok Technical University in faculty of Electrical Engineering in the field of Control Engineering and Microprocessor Techniques. In July 05 2006 was awarded the professional title of MSc with the grade Very Good. Same year he started PhD studies in Bialystok Technical University in faculty of Electrical Engineering in the field of Positive Systems.
His research interests cover the automatic control systems theory, specially positive 1D and 2D systems.

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1 Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of art in positive systems theory is given in the monographs [2, 5]. The realization problem for positive discrete-time and continues-time systems without and with delays was considered in [1, 2, 5-10]. The reachability, controllability and minimum energy control of positive linear discrete-time systems with delays have been considered in [3]. The relative controllability of stationary hybrid systems has been investigated in [15] and the observability of linear differential-algebraic systems with delays has been considered in [16]. A new class of positive 2D hybrid linear system has been introduced in [11], and the realization problem for this class of systems has been considered in [12].

The main purpose of this paper is to present a new method for computation of positive realizations of a given proper transfer matrix using the state variable diagram method. Sufficient conditions for the existence of a positive realization of a given proper transfer matrix will be established and a procedure for computation of positive realizations will be proposed.

2 Preliminaries and problem formulation

Consider a hybrid system with delays in state vector and input described by the equations [11]

$$\begin{bmatrix} \dot{x}_1(t,i) \\ x_2(t,i+1) \end{bmatrix} = \sum_{k=0}^{h} \begin{bmatrix} A_{11}^k & A_{12}^k \\ A_{21}^k & A_{22}^k \end{bmatrix} \begin{bmatrix} x_1(t-kd,i) \\ x_2(t,i-k) \end{bmatrix} + \sum_{k=0}^{h} \begin{bmatrix} B_1^k \\ B_2^k \end{bmatrix} u(t-kd,i-k)$$
$$y(t,i) = \sum_{k=0}^{h} \begin{bmatrix} C_1^k & C_2^k \end{bmatrix} \begin{bmatrix} x_1(t-kd,i) \\ x_2(t,i-k) \end{bmatrix} + \sum_{k=0}^{h} D^k u(t-kd,i-k) \tag{1}$$

for $t \in R_+ = [0, +\infty]$, $i \in Z_+$

where $\dot{x}_1(t,i) = \dfrac{\partial x_1(t,i)}{\partial t}$, $x_1(t,i) \in R^{n_1}$, $x_2(t,i) \in R^{n_2}$, $u(t,i) \in R^m$, $y(t,i) \in R^p$ and $d > 0$ is delay.

Let $R_+^{n\times m}$ be the set of $n\times m$ real matrices with nonnegative entries and $R_+^n = R_+^{n\times 1}$. Let M_n be the set of $n\times m$ Metzler matrices (real matrices with nonnegative off-diagonal entries).

Definition 1. The hybrid system with delays (1) is called (internally) positive if $x_1(t,i) \in R_+^{n_1}$, $x_2(t,i) \in R_+^{n_2}$, and $y(t,i) \in R_+^p$, $t \in R_+$, $i \in Z_+$ for arbitrary boundary conditions $x_1(-kd,0) \in R_+^{n_1}$, $x_2(0,-k) \in R_+^{n_2}$, $k = 0,1,...,h$ and inputs $u(t,-k) \in R_+^m$, $t \in [-hd, 0)$, $k = 0,1,...,h$.

Theorem 1. The hybrid system with delays (1) is internally positive if and only if

$$\begin{gathered} A_{11}^0 \in M_{n_1},\ A_{11}^1,...,A_{11}^h \in R_+^{n_1\times n_1},\ A_{12}^k \in R_+^{n_1\times n_2}, \\ A_{21}^k \in R_+^{n_2\times n_1},\ A_{22}^k \in R_+^{n_2\times n_2},\ B_1^k \in R_+^{n_1\times m}, \\ B_2^k \in R_+^{n_2\times m},\ C_1^k \in R_+^{p\times n_1},\ C_2^k \in R_+^{p\times n_2},\ D^k \in R_+^{p\times m} \end{gathered} \tag{2}$$

for $k = 0,1,...,h$.

Proof.
Necessity. Let $e_{n_1}^1$ be the ith ($i = 1,...,n_1$) column of the identity matrix I_{n_1}. From (1) for t = 0, i = 0 and $x_1(0,0) = e_{n_1}^i$, $x_1(-kd,0) = 0$, $k = 1,...,h$ $x_2(0,-j) = 0$, and inputs $u(-jd,-j)$, $j = 0,1,...,h$ we have $\dot{x}_1(0,0) = A_{11}^0 e_i$. The trajectory does not live the orthant $R_+^{n_1}$ only if $A_{11}^0 e_i \geq 0$, what implies $a_{ij} \geq 0$, $i \neq j$. Therefore, the matrix A_{11}^0 has to be the Metzler matrix. For the same reasons, for $x_2(0,0) = e_{n_2}^i$, $x_1(-kd,0) = 0$, $k = 0,1,...,h$ $x_2(0,-j) = 0$, $j = 1,...,h$ and inputs $u(-kd,-k) = 0$, $k = 0,1,...,h$ we have $\dot{x}_1(0,0) = A_{11}^0 e_{n_2}^i \geq 0$ that implies $A_{12}^0 \in R_+^{n_1\times n_2}$. In a similar way we may show that the hybrid system with delay (1) is internally positive only if the conditions (2) are satisfied.
Sufficiency. From (1) for i = 0 we have

$$\begin{aligned} \dot{x}_1(t,0) &= A_{11}^0 x_1(t,0) + A_{12}^0 x_2(t,0) + A_{11}^1 x_1(t-d,0) \\ &+ A_{12}^1 x_2(t,-1) + ... + A_{11}^h x_1(t-hd,0) \\ &+ A_{12}^h x_2(t,-h) + B_1^0 u(t,0) \\ &+ ... + B_1^h u(t-hd,-h \end{aligned} \tag{1a}$$

$$\begin{aligned} x_2(t,1) &= A_{21}^0 x_1(t,0) + A_{22}^0 x_2(t,0) + A_{21}^1 x_1(t-d,0) \\ &+ A_{22}^1 x_2(t,-1) + \ldots + A_{21}^h x_1(t-hd,0) \\ &+ A_{22}^h x_2(t,-h) + B_2^0 u(t,0) \\ &+ \ldots + B_2^h u(t-hd,-h) \end{aligned} \quad (1b)$$

For given nonnegative initial conditions $x_1(t-kd,0)$, $k=0,1,\ldots,h;$ $x_2(0,-j)=0$, $j=1,\ldots,h$ and the input, $u(t-kd,i-k)$, $k=0,1,\ldots,h$ we may find the solution of the equation (1a)

$$x_1(t,0) = e^{A_{11}^0 t} x_1(0,0) + \int_0^t e^{A_{11}^0 (t-\tau)} [A_{12}^0 x_2(\tau,0) + \ldots + B_1^h u(\tau-hd,-h)] d\tau$$

Which is nonnegative $t \geq 0$ if the conditions (2) are met. Knowing $x_1(t,0) \in R_+^{n_1}$ from the equation (1b) we obtain $x_2(t,1) \in R_+^{n_2}$ if the conditions (2) are met. Continuing the procedure for $i=1,\ldots,h$ we may show that if the conditions (2) are satisfied the $x_1(t,i) \in R_+^{n_1}$ and $x_2(t,i) \in R_+^{n_2}$ for $t \geq 0$ and $i \in Z_+$. □

Transfer function of the system (1) is given by equation

$$\begin{aligned} T(s,z) &= \left(C^0 + C^1 w + \ldots + C^h w^h\right) \\ &\times \left[\begin{bmatrix} I_{n_1} s & 0 \\ 0 & I_{n_2} z \end{bmatrix} - A^0 - A^1 w - \ldots - A^h w^h\right]^{-1} \\ &\times \left(B^0 + B^1 w + \ldots + B^h w^h\right) \\ &+ D^0 + D^1 w + \ldots + D^h w^h \quad \in R^{p\times m}(s,w) \end{aligned} \quad (3)$$

where $w = e^{-sd}$, $R^{p\times m}(s,z)$ is the set of $m \times n$ rational matrices in s and z.

Definition 2. The matrices (2) are called the positive realization of the transfer matrix $T(s,z)$ if they satisfy the equality (3). A realization is called minimal if the matrices A_{11}^k and A_{22}^k have minimal dimensions among all positive realizations of $T(s,z)$.

The realization problem can be stated as follow.

Given a proper rational matrix $T(s,z) \in R^{p\times m}(s,z)$, find its positive realization.

3 Problem solution for SISO systems

The essence of proposed method for solving of the realization problem for positive 2D hybrid systems with h delays will be presented on single-input single-output (SISO) system.
Consider a hybrid system with the transfer function

$$T(s,z) = \frac{b_{n,m}(w)s^n z^m + b_{n,m-1}(w)s^n z^{m-1} + \ldots + b_{11}(w)sz + b_{10}(w)s + b_{01}(w)z + b_{00}(w)}{s^n z^m - a_{n,m-1}(w)s^n z^{m-1} - \ldots - a_{11}(w)sz - a_{10}(w)s - a_{01}(w)z - a_{00}(w)} = \frac{\sum_{i=0}^{n}\sum_{j=0}^{m} b_{ij}(w)s^i z^j}{s^n z^m - \left(\sum_{\substack{i=0 \\ i+j\neq n+m}}^{n}\sum_{j=0}^{m} a_{ij}(w)s^i z^j\right)} \quad (4)$$

where $b_{ij}(w) = b_{ij}^h w^h + \ldots + b_{ij}^1 w + b_{ij}^0$, $a_{ij}(w) = a_{ij}^h w^h + \ldots + a_{ij}^1 w + a_{ij}^0$ for $i=0,1,\ldots,n$; $j=0,1,\ldots,m$, and coefficients $b_{ij}(w)$, $a_{ij}(w)$ for $i=0,1,\ldots,n-1$; $j=m$ are equal $b_{im}(w)=b_{im}^0$ $a_{im}(w)=a_{im}^0$.

Multiplying the numerator and denominator of transfer function (4) by $s^{-n}z^{-m}$ we obtain

$$T(s,z) = \frac{b_{n,m}(w) + b_{n,m-1}(w)z^{-1} + b_{n-1,m}(w)s^{-1} + \ldots + b_{00}(w)s^{-n}z^{-m}}{1 - a_{n,m-1}(w)z^{-1} - a_{n-1,m}(w)s^{-1} - \ldots - a_{00}(w)s^{-n}z^{-m}} = \frac{Y}{U} \quad (5)$$

and

$$\begin{aligned} E &= U + (a_{n,m-1}(w)z^{-1} + a_{n-1,m}(w)s^{-1} \\ &\quad + \ldots + a_{00}(w)s^{-n}z^{-m})E \\ Y &= (b_{n,m}(w) + b_{n,m-1}(w)z^{-1} + b_{n-1,m}(w)s^{-1} \\ &\quad + \ldots + b_{00}(w)s^{-n}z^{-m})E \end{aligned} \quad (6)$$

Using (6) we may draw the state variable diagram shown in Fig. 1.

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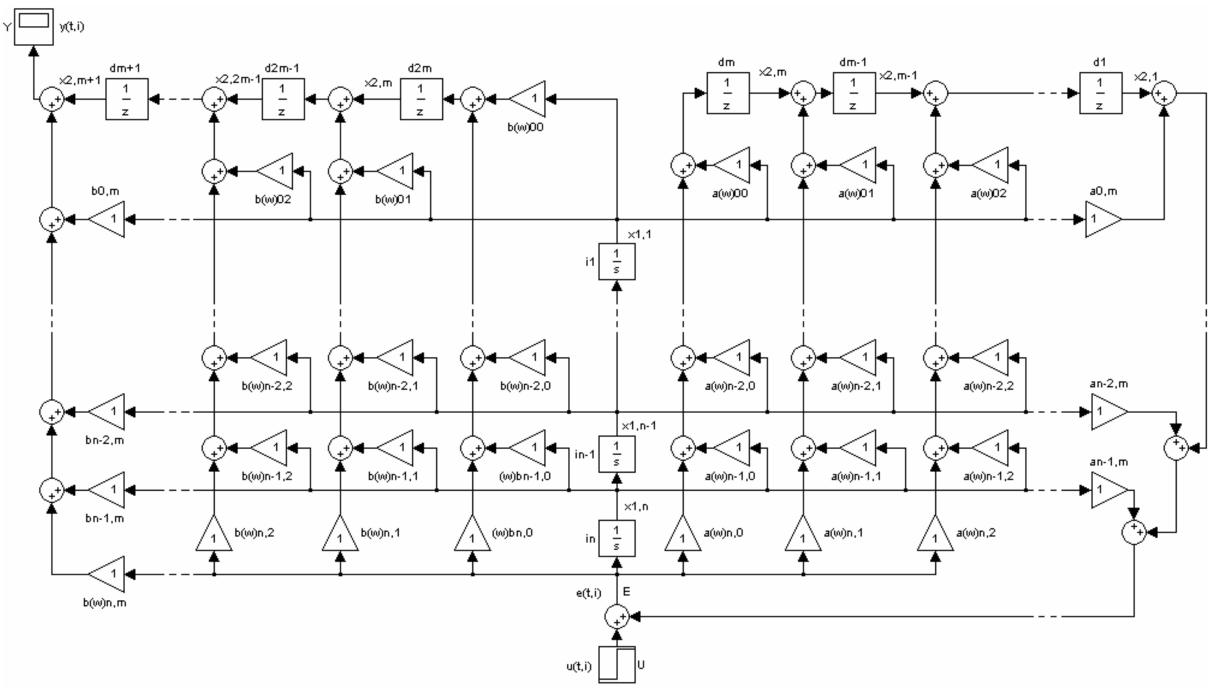

Fig. 1 MATLAB/SIMULINK state variable diagram for transfer function (4).

Remark. Blocks $b_{ij}(w), a_{ij}(w)$ for $i = 0,1,...,n-1$; $j = 0,1,...,m-1$ and $b_{nm}(w)$ have the form shown in Fig. 2

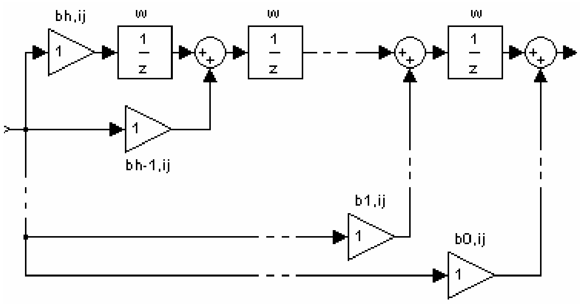

Fig. 2 MATLAB /SIMULINK state variable diagram for blocks $b_{ij}(w), a_{ij}(w)$

with exception of blocks $b_{ij}(w), a_{ij}(w)$ for $i = 0,1,...,n-1$; $j = m$ which have no delay unit w and they are equal to $b_{im}(w) = b_{im}^0$, $a_{im}(w) = a_{im}^0$.

As a state variable we choose the outputs of integrators ($x_{1,1}(t,i)$, $x_{1,2}(t,i)$, ... ,$x_{1,n}(t,i)$) and of delay elements ($x_{2,1}(t,i)$, $x_{2,2}(t,i)$, ..., $x_{2,2m}(t,i)$). Using state variable diagram (Fig. 1) and taking into account (Fig. 2) we can write the following differential and difference equations

$$
\begin{aligned}
\dot{x}_{1,1}(t,i) &= x_{1,2}(t,i) \\
\dot{x}_{1,2}(t,i) &= x_{1,3}(t,i) \\
&\vdots \\
\dot{x}_{1,n-1}(t,i) &= x_{1,n}(t,i) \\
\dot{x}_{1,n}(t,i) &= e(t,i)
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
x_{2,1}(t,i+1) &= a_{0,m-1}(w)x_{1,1}(t,i) + a_{1,m-1}(w)x_{1,2}(t,i) \\
&\quad + ... + a_{n-1,m-1}(w)x_{1,n}(t,i) + x_{2,2}(t,i) \\
&\quad + a_{n,m-1}(w)e(t,i) \\
x_{2,2}(t,i+1) &= a_{0,m-2}(w)x_{1,1}(t,i) + a_{1,m-2}(w)x_{1,2}(t,i) \\
&\quad + ... + a_{n-1,m-2}(w)x_{1,n}(t,i) + x_{2,3}(t,i) \\
&\quad + a_{n,m-2}(w)e(t,i) \\
&\vdots \\
x_{2,m-1}(t,i+1) &= a_{0,1}(w)x_{1,1}(t,i) + a_{1,1}(w)x_{1,2}(t,i) \\
&\quad + ... + a_{n-1,1}(w)x_{1,n}(t,i) + x_{2,m}(t,i) \\
&\quad + a_{n,1}(w)e(t,i) \\
x_{2,m}(t,i+1) &= a_{00}(w)x_{1,1}(t,i) + a_{10}(w)x_{1,2}(t,i) \\
&\quad + ... + a_{n-1,0}(w)x_{1,n}(t,i) + a_{n,0}(w)e(t,i) \\
x_{2,m+1}(t,i+1) &= b_{0,m-1}(w)x_{1,1}(t,i) + b_{1,m-1}(w)x_{1,2}(t,i) \\
&\quad + ... + b_{n-1,m-1}(w)x_{1,n}(t,i) + x_{2,m+2}(t,i) \\
&\quad + b_{n,m-1}(w)e(t,i) \\
x_{2,m+2}(t,i+1) &= b_{0,m-2}(w)x_{1,1}(t,i) + b_{1,m-2}(w)x_{1,2}(t,i) \\
&\quad + ... + b_{n-1,m-2}(w)x_{1,n}(t,i) + x_{2,m+3}(t,i) \\
&\quad + b_{n,m-2}(w)e(t,i) \\
&\vdots \\
x_{2,2m-1}(t,i+1) &= b_{0,1}(w)x_{1,1}(t,i) + b_{1,1}(w)x_{1,2}(t,i) \\
&\quad + ... + b_{n-1,1}(w)x_{1,n}(t,i) + x_{2,2m}(t,i) \\
&\quad + b_{n,1}(w)e(t,i) \\
x_{2,2m}(t,i+1) &= b_{00}(w)x_{1,1}(t,i) + b_{10}(w)x_{1,2}(t,i) \\
&\quad + ... + b_{n-1,0}(w)x_{1,n}(t,i) + b_{n,0}(w)e(t,i)
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
y(t,i) &= b_{0,m}x_{1,1}(t,i)+b_{1,m}x_{1,2}(t,i)+\ldots+b_{n-1,m}x_{1,n}(t,i)\\
&\quad +x_{2,m+1}(t,i)+b_{n,m}(w)e(t,i)
\end{aligned}
\tag{7}
$$

where

$$
\begin{aligned}
e(t,i) &= a_{0,m}x_{1,1}(t,i)+a_{1,m}x_{1,2}(t,i)\\
&\quad +\ldots+a_{n-1,m}x_{1,n}(t,i)+x_{2,1}(t,i)+u(t,i)
\end{aligned}
\tag{8}
$$

Substituting (8) into (7) we obtain

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1(t,i) \\ x_2(t,i+1) \end{bmatrix} &= \sum_{k=0}^{h} A^k \begin{bmatrix} x_1(t-kd,i) \\ x_2(t,i-k) \end{bmatrix} + \sum_{k=0}^{h} B^k u(t-kd,i-k)\\
y(t,i) &= \sum_{k=0}^{h} C^k \begin{bmatrix} x_1(t-kd,i) \\ x_2(t,i-k) \end{bmatrix} + \sum_{k=0}^{h} D^k u(t-kd,i-k)
\end{aligned}
\tag{9}
$$

where

$$
\begin{aligned}
&A^k = \begin{bmatrix} A_{11}^k & A_{12}^k \\ A_{21}^k & A_{22}^k \end{bmatrix},\quad B^k = \begin{bmatrix} B_1^k \\ B_2^k \end{bmatrix}, \quad \text{for } k = 0,1,\ldots,h\\
&C^k = \begin{bmatrix} C_1^k & C_2^k \end{bmatrix},\quad D^k = \begin{bmatrix} b_{nm}^k \end{bmatrix}
\end{aligned}
\tag{10}
$$

and

$$
A_{11}^0 = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ a_{0,m}^0 & a_{1,m}^0 & a_{2,m}^0 & \ldots & a_{n-1,m}^0 \end{bmatrix} \in R^{n\times n},
$$

$$
A_{11}^k = [0] \in R^{n\times n},\quad k = 1,\ldots,h
$$

$$
A_{12}^0 = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix} \in R^{n\times 2m},
$$

$$
A_{12}^k = [0] \in R^{n\times 2m},\quad k = 1,\ldots,h
$$

$$
A_{21}^k = \begin{bmatrix} \bar{a}_{0,m-1}^k & \bar{a}_{1,m-1}^k & \bar{a}_{2,m-1}^k & \ldots & \bar{a}_{n-1,m-1}^k \\ \bar{a}_{0,m-2}^k & \bar{a}_{1,m-2}^k & \bar{a}_{2,m-2}^k & \ldots & \bar{a}_{n-1,m-2}^k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{a}_{00}^k & \bar{a}_{10}^k & \bar{a}_{20}^k & \ldots & \bar{a}_{n-1,0}^k \\ \bar{b}_{0,m-1}^k & \bar{b}_{1,m-1}^k & \bar{b}_{2,m-1}^k & \ldots & \bar{b}_{n-1,m-1}^k \\ \bar{b}_{0,m-2}^k & \bar{b}_{1,m-2}^k & \bar{b}_{2,m-2}^k & \ldots & \bar{b}_{n-1,m-2}^k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{b}_{00}^k & \bar{b}_{10}^k & \bar{b}_{20}^k & \ldots & \bar{b}_{n-1,0}^k \end{bmatrix} \in R^{2m\times n}
\tag{11}
$$

$$
k = 0,1,\ldots,h
$$

$$
A_{22}^0 = \left[\begin{array}{cccccccccccccccc}
a_{n,m-1}^0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
a_{n,m-2}^0 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n,2}^0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
a_{n,1}^0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
a_{n,0}^0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
b_{n,m-1}^0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
b_{n,m-2}^0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
b_{n,2}^0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\
b_{n,1}^0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \\
b_{n,0}^0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0
\end{array}\right]
$$

$$
\in R^{2m\times 2m}
$$

$$
A_{22}^k = \begin{bmatrix} a_{n,m-1}^k & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,0}^k & 0 & \ldots & 0 \\ b_{n,m-1}^k & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ b_{n,0}^k & 0 & \ldots & 0 \end{bmatrix} \in R^{2m\times 2m},
$$

$$
k = 1,\ldots,h
$$

$$
B_1^0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in R^{n\times 1},\quad B_1^k = [0] \in R^{n\times 1},\quad k = 1,\ldots,h;
$$

$$
B_2^0 = \begin{bmatrix} a_{n,m-1}^k \\ a_{n,m-2}^k \\ \vdots \\ a_{n,0}^k \\ b_{n,m-1}^k \\ b_{n,m-2}^k \\ \vdots \\ b_{n,0}^k \end{bmatrix} \in R^{2m\times 1},\quad k = 0,1,\ldots,h
$$

$$
\begin{aligned}
&C_1^0 = \begin{bmatrix} \bar{b}_{0,m}^0 & \bar{b}_{1,m}^0 & \ldots & \bar{b}_{n-1,m}^0 \end{bmatrix} \in R^{1\times n},\\
&C_2^0 = \begin{bmatrix} C_{21}^0 & C_{22}^0 \end{bmatrix} \in R^{1\times 2m},\\
&C_{21}^0 = \begin{bmatrix} b_{nm}^0 & 0 & \ldots & 0 \end{bmatrix} \in R^{1\times m},\\
&C_{22}^0 = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix} \in R^{1\times m}\\
&C_1^k = \begin{bmatrix} \bar{c}_{0,m}^k & \bar{c}_{1,m}^k & \ldots & \bar{c}_{n-1,m}^k \end{bmatrix} \in R^{1\times n},\\
&C_2^k = \begin{bmatrix} C_{21}^k & C_{22}^k \end{bmatrix} \in R^{1\times 2m},\\
&C_{21}^k = \begin{bmatrix} b_{nm}^k & 0 & \ldots & 0 \end{bmatrix} \in R^{1\times m},\\
&C_{22}^k = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix} \in R^{1\times m},\quad k = 1,\ldots,h\\
&D^k = \begin{bmatrix} b_{nm}^k \end{bmatrix} \in R^{1\times 1},\quad k = 0,1,\ldots,h
\end{aligned}
\tag{11}
$$

and $\bar{a}_{ij}^k = a_{ij}^k + a_{im}^0 a_{nj}^k$, $\bar{b}_{ij}^k = b_{ij}^k + a_{im}^0 b_{nj}^k$, $\bar{c}_{ij}^k = a_{im}^0 b_{nj}^k$ for $k = 0,1,...,h$; $i = 0,1,...,n-1$; $j = 0,1,...,m-1$.

Therefore, the following theorem has been proved

Theorem 2. There exists a positive realization of the transfer function (4) if all coefficients of its numerator and denominator are nonnegative.

If the assumptions of Theorem 2 are satisfied then a positive realization can be found by the use of the following procedure.

Procedure.

Step 1. Write the transfer function $T(s,z)$ in the form (5) and the equations (6).

Step 2. Using (6) draw the state variable diagram shown in Fig. 1 taking into account Fig. 2.

Step 3. Choose the state variables and write equations (7) and (8).

Step 4. Using (7) – (9) find the desired realization (11) of transfer function (4).

Example 1. Using the Procedure find positive realization of the transfer function of

$$T(s,z) = \frac{b_{11}(w)sz + b_{10}(w)s + b_{01}z + b_{00}(w)}{sz - a_{10}(w)s - a_{01}z - a_{00}(w)} \quad (12)$$

where

$b_{11}(w) = b_{11}^1 w + b_{11}^0$, $b_{10}(w) = b_{10}^1 w + b_{10}^0$,
$b_{00}(w) = b_{00}^1 w + b_{00}^0$, $a_{10}(w) = a_{10}^1 w + a_{10}^0$,
$a_{00}(w) = a_{00}^1 w + a_{00}^0$ and $w = e^{-sd}$.

Step 1.
Multiplying the numerator and denominator of transfer function (12) by $s^{-1}z^{-1}$ we obtain

$$T(s,z) = \frac{b_{11}(w) + b_{10}(w)z^{-1} + b_{01}s^{-1} + b_{00}(w)s^{-1}z^{-1}}{1 - a_{10}(w)z^{-1} - a_{01}s^{-1} - a_{00}(w)s^{-1}z^{-1}} = \frac{Y}{U} \quad (13)$$

Defining

$$E = \frac{U}{1 - a_{10}(w)z^{-1} - a_{01}s^{-1} - a_{00}(w)s^{-1}z^{-1}} \quad (14)$$

from (14) and (13) we obtain

$$\begin{aligned} E &= U + (a_{10}(w)z^{-1} + a_{01}s^{-1} + a_{00}(w)s^{-1}z^{-1})E \\ Y &= (b_{11}(w) + b_{10}(w)z^{-1} + b_{01}s^{-1} + b_{00}(w)s^{-1}z^{-1})E \end{aligned} \quad (15)$$

Taking into account that $b_{ij}(w)$, $a_{ij}(w)$ for $i = 1,...,n$; $j = 1,...,m$ we obtain

$$\begin{aligned} E &= U + [(a_{10}^0 z^{-1} + a_{01}^0 s^{-1} + a_{00}^0 s^{-1}z^{-1}) \\ &\quad + (a_{10}^1 z^{-1} + a_{00}^1 s^{-1}z^{-1})w]E \\ Y &= [(b_{11}^0 + b_{10}^0 z^{-1} + b_{01}^0 s^{-1} + b_{00}^0 s^{-1}z^{-1}) \\ &\quad + (b_{11}^1 + b_{10}^1 z^{-1} + b_{00}^1 s^{-1}z^{-1})w]E \end{aligned} \quad (16)$$

Step 2.
Using (16) we may draw the state variable diagram shown in Fig. 3.

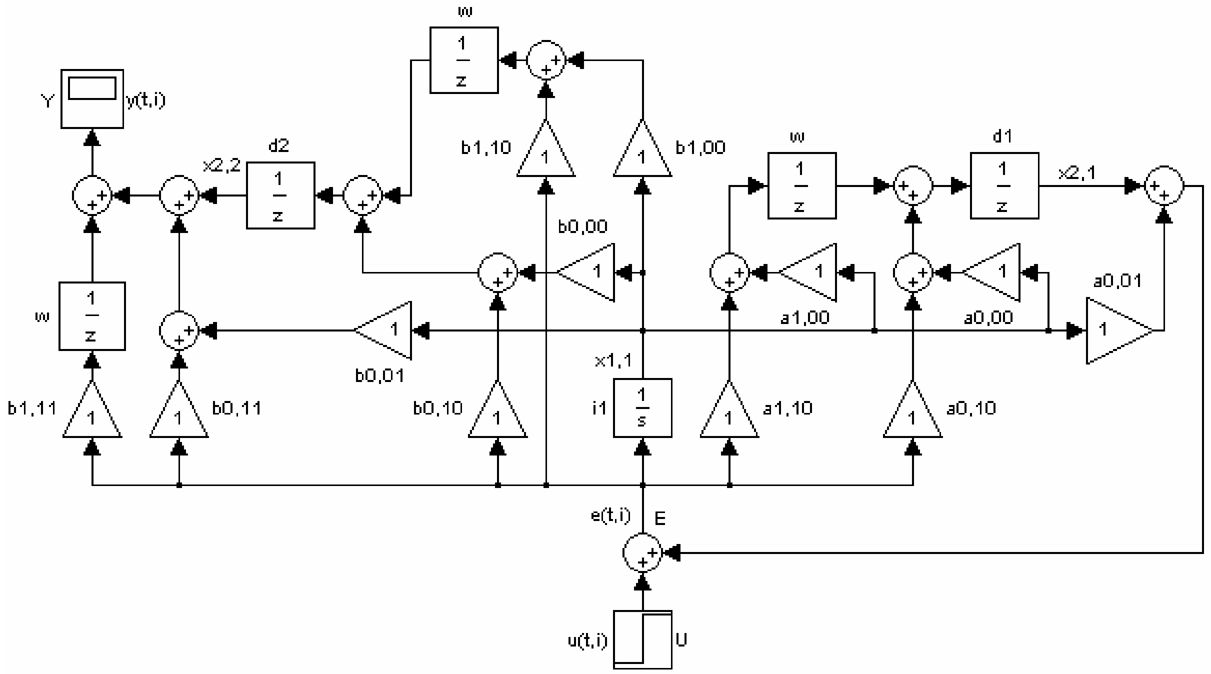

Fig. 3 MATLAB/SIMULINK state variable diagram for transfer function Eq. (13).

Step 3.
As a state variable we choose the outputs of integrators ($x_{1,1}(t,i)$) and of delay elements ($x_{2,1}(t,i)$, $x_{2,2}(t,i)$). Using the state variables diagram (Fig. 3) we can write the following differential and difference equations

$$\begin{aligned}
&\dot{x}_{1,1}(t,i) = e(t,i)\\
&x_{2,1}(t,i+1) = a_{00}^0 x_{1,1}(t,i) + a_{00}^1 x_{1,1}(t-d,i)\\
&\qquad + a_{10}^0 e(t,i) + a_{10}^1 e(t-d,i-1)\\
&x_{2,2}(t,i+1) = b_{00}^0 x_{1,1}(t,i) + b_{00}^1 x_{1,1}(t-d,i)\\
&\qquad + b_{10}^0 e(t,i) + b_{10}^1 e(t-d,i-1)\\
&y(t,i) = x_{2,2}(t,i) + b_{01}^0 x_{1,1}(t,i) + b_{11}^0 e(t,i)\\
&\qquad + b_{11}^1 e(t-d,i-1)
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
&e(t,i) = x_{2,1}(t,i) + a_{01}^0 x_{1,1}(t,i) + u(t,i)\\
&e(t-d,i-1) = x_{2,1}(t,i-1) + a_{01}^0 x_{1,1}(t-d,i)\\
&\qquad + u(t-d,i-1)
\end{aligned} \tag{18}$$

Substituting (18) into (17) we obtain

$$\begin{aligned}
&\dot{x}_{1,1}(t,i) = a_{01}^0 x_{1,1}(t,i) + x_{2,1}(t,i) + u(t,i)\\
&x_{2,1}(t,i+1) = (a_{00}^0 + a_{01}^0 a_{10}^0) x_{1,1}(t,i) + a_{10}^0 x_{2,1}(t,i)\\
&\qquad + a_{10}^0 u(t,i) + (a_{00}^1 + a_{01}^0 a_{10}^1) x_{1,1}(t-d,i)\\
&\qquad + a_{10}^1 x_{2,1}(t,i-1) + a_{10}^1 u(t-d,i-1)\\
&x_{2,2}(t,i+1) = (b_{00}^0 + a_{01}^0 b_{10}^0) x_{1,1}(t,i) + b_{10}^0 x_{2,1}(t,i)\\
&\qquad + b_{10}^0 u(t,i) + (b_{00}^1 + a_{01}^0 b_{10}^1) x_{1,1}(t-d,i)\\
&\qquad + b_{10}^1 x_{2,1}(t,i-1) + b_{10}^1 u(t-d,i-1)\\
&y(t,i) = (b_{01}^0 + a_{01}^0 b_{11}^0) x_{1,1}(t,i) + b_{11}^0 x_{2,1}(t,i)\\
&\qquad + x_{2,2}(t,i) + b_{11}^0 u(t,i) + (a_{01}^0 b_{11}^1) x_{1,1}(t-d,i)\\
&\qquad + b_{11}^1 x_{2,1}(t,i-1) + x_{2,2}(t,i-1)\\
&\qquad + b_{11}^1 u(t-d,i-1)
\end{aligned} \tag{19}$$

Step 4.
Defining

$$x_1(t,i) = [x_{1,1}(t,i)],\quad x_2(t,i) = \begin{bmatrix} x_{2,1}(t,i)\\ x_{2,2}(t,i)\end{bmatrix} \tag{20}$$

we can write the equations (19) in the form

$$\begin{aligned}
&\begin{bmatrix}\dot{x}_1(t,i)\\ x_2(t,i+1)\end{bmatrix} = A^0\begin{bmatrix}x_1(t,i)\\ x_2(t,i)\end{bmatrix} + A^1\begin{bmatrix}x_1(t-d,i)\\ x_2(t,i-1)\end{bmatrix}\\
&\qquad + B^0 u(t,i) + B^1 u(t-d,i-1)\\
&y(t,i) = C^0\begin{bmatrix}x_1(t,i)\\ x_2(t,i)\end{bmatrix} + C^1\begin{bmatrix}x_1(t-d,i)\\ x_2(t,i-1)\end{bmatrix}\\
&\qquad + D^0 u(t,i) + D^1 u(t-d,i-1)
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
&A^0 = \begin{bmatrix}A_{11}^0 & A_{12}^0\\ A_{21}^0 & A_{22}^0\end{bmatrix} = \begin{bmatrix} a_{01}^0 & 1 & 0\\ a_{00}^0 + a_{01}^0 a_{10}^0 & a_{10}^0 & 0\\ b_{00}^0 + a_{01}^0 b_{10}^0 & b_{10}^0 & 0\end{bmatrix},\\
&A^1 = \begin{bmatrix}A_{11}^1 & A_{12}^1\\ A_{21}^1 & A_{22}^1\end{bmatrix} = \begin{bmatrix} 0 & 0 & 0\\ a_{00}^1 + a_{01}^0 a_{10}^1 & a_{10}^1 & 0\\ b_{00}^1 + a_{01}^0 b_{10}^1 & b_{10}^1 & 0\end{bmatrix},\\
&B^0 = \begin{bmatrix}B_1^0\\ B_2^0\end{bmatrix} = \begin{bmatrix}1\\ a_{10}^0\\ b_{10}^0\end{bmatrix},\quad B^1 = \begin{bmatrix}B_1^1\\ B_2^1\end{bmatrix} = \begin{bmatrix}0\\ a_{10}^1\\ b_{10}^1\end{bmatrix},\\
&C^0 = [C_1^0 \quad C_2^0] = [b_{01}^0 + b_{11}^0 a_{01}^0 \quad b_{11}^0 \quad 1],\\
&C^1 = [C_1^1 \quad C_2^1] = [a_{01}^0 b_{11}^1 \quad b_{11}^1 \quad 1],\\
&D^0 = [b_{11}^0],\quad D^1 = [b_{11}^1]
\end{aligned} \tag{22}$$

4 Generalization for MIMO systems

Consider the m-inputs and p-outputs 2D hybrid linear system with h delays (1) with the proper transfer matrix

$$T(s,z) = \begin{bmatrix} T_{11}(s,z) & \dots & T_{1m}(s,z)\\ \vdots & \vdots & \vdots\\ T_{p1}(s,z) & \dots & T_{pm}(s,z)\end{bmatrix} \in R^{p\times m}(s,z) \tag{12}$$

where

$$T_{rl}(s,z) = \frac{\sum_{i=0}^{n_{rl}}\sum_{j=0}^{m_{rl}} b_{i,j}^{rl}(w) s^i z^j}{s^{n_{rl}} z^{m_{rl}} - \left(\sum_{\substack{i=0\\ i+j\neq n_{rl}+m_{rl}}}^{n_{rl}}\sum_{j=0}^{m_{rl}} a_{i,j}^{rl}(w) s^i z^j\right)} \tag{13}$$

for $r = 1,2,\dots,p;\ l = 1,2,\dots,m$

and $b_{ij}^{rl}(w) = b_{ij}^{h,rl} w^h + \dots + b_{ij}^{1,rl} w + b_{ij}^{0,rl}$,
$a_{ij}^{rl}(w) = a_{ij}^{h,rl} w^h + \dots + a_{ij}^{1,rl} w + a_{ij}^{0,rl}$ for $i = 0,1,\dots,n_{kl}$;
$j = 0,1,\dots,m_{rl}$, and coefficients $b_{ij}^{rl}(w),\ a_{ij}^{rl}(w)$ for $i = 0,1,\dots,n_{rl}-1$; $j = m_{rl}$ are equal to
$b_{im_{rl}}^{rl}(w) = b_{im_{rl}}^{0,rl},\ a_{im_{rl}}^{rl}(w) = a_{im_{rl}}^{0,rl}$.

It is well-known [5] that the 2D transfer matrix (12) can be always written in the form

$$\begin{aligned}
T(s,z) &= \begin{bmatrix} \frac{n_{11}(s,z)}{d_1(s,z)} & \dots & \frac{n_{1m}(s,z)}{d_m(s,z)}\\ \vdots & \vdots & \vdots\\ \frac{n_{p1}(s,z)}{d_1(s,z)} & \dots & \frac{n_{pm}(s,z)}{d_m(s,z)}\end{bmatrix}\\
&= \begin{bmatrix} \frac{N_1(s,z)}{d_1(s,z)} & \dots & \frac{N_m(s,z)}{d_m(s,z)}\end{bmatrix}
\end{aligned} \tag{14}$$

where

$$
\begin{aligned}
&N_l(s,z) = \left[n_{1l}(s,z) \quad \dots \quad n_{pl}(s,z)\right]^T \\
&n_{rl}(s,z) = b^{rl}_{n_{rl},m_{rl}}(w)s^{n_{rl}}z^{m_{rl}} + b^{rl}_{n_{rl},m_{rl}-1}(w)s^{n_{rl}}z^{m_{rl}-1} \\
&\qquad + \dots + b^{rl}_{11}(w)sz + b^{rl}_{10}(w)s + b^{rl}_{01}(w)z + b^{rl}_{00}(w) \\
&d_l(s,z) = s^{n_l}z^{m_l} - a^l_{n_l,m_l-1}(w)s^{n_l}z^{m_l-1} \\
&\qquad - \dots - a^l_{11}(w)sz - a^l_{10}(w)s - a^l_{01}(w)z - a^l_{00}(w) \\
&b^{rl}_{ij}(w) = b^{h,rl}_{ij}w^h + \dots + b^{1,rl}_{ij}w + b^{0,rl}_{ij} \\
&a^l_{ij}(w) = a^{h,l}_{ij}w^h + \dots + a^{1,l}_{ij}w + a^{0,l}_{ij} \\
&n_l = n_{rl}, \quad m_l = m_{rl}, \\
&r = 1,2,\dots,p; \quad l = 1,2,\dots,m \\
&i = 0,1,\dots,n_{rl}; \quad j = 0,1,\dots,m_{rl}
\end{aligned}
\tag{15}
$$

and $b^{rl}_{im_{rl}}(w) = b^{0,rl}_{im_{rl}}$, $a^{rl}_{im_{rl}}(w) = a^{0,rl}_{im_{rl}}$ for $i = 0,1,\dots,n_{rl}-1$; T denotes the transpose.

In a similar way as for SISO systems, multiplying the numerator and denominator of each element of transfer matrix (14) by $s^{-n_l}z^{-m_l}$ we obtain

$$
\begin{aligned}
&E_l = U_l + \overline{d}_l(s,z)E_l \\
&\begin{bmatrix} Y_1 \\ \vdots \\ Y_p \end{bmatrix} = \begin{bmatrix} \overline{n}_{11}(s,z) & \dots & \overline{n}_{1m}(s,z) \\ \vdots & \vdots & \vdots \\ \overline{n}_{p1}(s,z) & \dots & \overline{n}_{pm}(s,z) \end{bmatrix} \begin{bmatrix} E_1 \\ \vdots \\ E_m \end{bmatrix}
\end{aligned}
\tag{16}
$$

where

$$
\begin{aligned}
&\overline{d}_l(s,z) = a^l_{n_l,m_l-1}(w)z^{-1} + a^l_{n_l-1,m_l}(w)s^{-1} \\
&\qquad + \dots + a^l_{00}(w)s^{-n_l}z^{-m_l} \\
&\overline{n}_{rl}(s,z) = b^{rl}_{n_{rl},m_{rl}}(w) + b^{rl}_{n_{rl},m_{rl}-1}(w)z^{-1} \\
&\qquad + b^{rl}_{n_{rl}-1,m_{rl}}(w)s^{-1} + \dots + b^{rl}_{00}(w)s^{-n_{rl}}z^{-m_{rl}} \\
&r = 1,2,\dots,p; \quad l = 1,2,\dots,m
\end{aligned}
\tag{17}
$$

Similarly as for SISO systems using (16) we may draw a suitable state variable diagram for the MIMO system with the proper transfer matrix (14). Using the state variable diagram we may write the set of differential and difference equations.

Defining vectors

$$
x_1(t-kd,i) = \begin{bmatrix} x_{1,1}(t-kd,i) \\ \vdots \\ x_{1,m}(t-kd,i) \end{bmatrix}, \quad x_2(t,i-k) = \begin{bmatrix} x_{2,1}(t,-ki) \\ \vdots \\ x_{2,m}(t,-ki) \end{bmatrix}
$$

where

$$
\begin{aligned}
&x_{1,l}(t-kd,i) = \begin{bmatrix} x_{1,l,1}(t-kd,i) \\ \vdots \\ x_{1,l,n_l}(t-kd,i) \end{bmatrix}, \\
&x_{2,l}(t,i-k) = \begin{bmatrix} x_{2,k,1}(t,i-k) \\ \vdots \\ x_{2,r,(p+1)m_l}(t,i-k) \end{bmatrix}
\end{aligned}
\tag{18}
$$

for $l = 1,2,\dots,m; \quad k = 0,1,\dots,h;$

$$
\begin{aligned}
&u(t-kd,i-k) = \begin{bmatrix} u_1(t-kd,i-k) \\ \vdots \\ u_m(t-kd,i-k) \end{bmatrix}, \\
&y(t,i) = \begin{bmatrix} y_1(t,i) \\ \vdots \\ y_p(t,i) \end{bmatrix}
\end{aligned}
\tag{19}
$$

we may write the set of equations in the form

$$
\begin{aligned}
&\begin{bmatrix} \dot{x}_1(t,i) \\ x_2(t,i+1) \end{bmatrix} = \sum_{k=0}^{h} \begin{bmatrix} A^k_{11} & A^k_{12} \\ A^k_{21} & A^k_{22} \end{bmatrix} \begin{bmatrix} x_1(t-kd,i) \\ x_2(t,i-k) \end{bmatrix} \\
&\qquad + \sum_{k=0}^{h} B^k u(t-kd,i-k) \\
&y(t,i) = \sum_{k=0}^{h} \begin{bmatrix} C^k_1 & C^k_2 \end{bmatrix} \begin{bmatrix} x_1(t-kd,i) \\ x_2(t,i-k) \end{bmatrix} \\
&\qquad + \sum_{k=0}^{h} D^k u(t-kd,i-k)
\end{aligned}
\tag{20}
$$

where

$$
\begin{aligned}
&A^k_{11} = blockdiag\left[A^{k,1}_{11} \quad \dots \quad A^{k,m}_{11}\right], \\
&A^k_{12} = blockdiag\left[A^{k,1}_{12} \quad \dots \quad A^{k,m}_{12}\right], \\
&A^k_{21} = blockdiag\left[A^{k,1}_{21} \quad \dots \quad A^{k,m}_{21}\right], \\
&A^k_{22} = blockdiag\left[A^{k,1}_{22} \quad \dots \quad A^{k,m}_{22}\right], \\
&B^k = blockdiag\left[\begin{bmatrix} B^{k,1}_1 \\ B^{k,1}_2 \end{bmatrix} \quad \dots \quad \begin{bmatrix} B^{k,m}_1 \\ B^{k,m}_2 \end{bmatrix}\right], \\
&C^k_1 = \begin{bmatrix} C^{k,1}_1 \\ \vdots \\ C^{k,p}_1 \end{bmatrix}, \quad C^k_2 = \begin{bmatrix} C^{k,1}_2 \\ \vdots \\ C^{k,p}_2 \end{bmatrix}, \\
&D^k = \begin{bmatrix} b^{k,11}_{n_{11},m_{11}} & \dots & b^{k,1m}_{n_{1m},m_{1m}} \\ \vdots & \vdots & \vdots \\ b^{k,p1}_{n_{p1},m_{p1}} & \dots & b^{k,pm}_{n_{pm},m_{pm}} \end{bmatrix}
\end{aligned}
\tag{21}
$$

and

$$
\begin{aligned}
&A^{0,l}_{11} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a^{0,l}_{0,m_l} & a^{0,l}_{1,m_l} & a^{0,l}_{2,m_l} & \dots & a^{0,l}_{n_l-1,m_l} \end{bmatrix} \in R^{n_l \times n_l}, \\
&A^{k,l}_{11} = [0] \in R^{n_l \times n_l}, \quad k = 1,2,\dots,h; \\
&A^{0,l}_{12} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix} \in R^{n_l \times (p+1)m_l}, \\
&A^{k,l}_{12} = [0] \in R^{n_l \times (p+1)m_l}, \quad k = 1,2,\dots,h;
\end{aligned}
\tag{22}
$$

$$A_{21}^{k,l}=\begin{bmatrix} \bar{a}_{0,m_l-1}^{k,l} & \bar{a}_{1,m_l-1}^{k,l} & \bar{a}_{2,m_l-1}^{k,l} & \dots & \bar{a}_{n_l-1,m_l-1}^{k,l} \\ \bar{a}_{0,m_l-2}^{k,l} & \bar{a}_{1,m_l-2}^{k,l} & \bar{a}_{2,m_l-2}^{k,l} & \dots & \bar{a}_{n_l-1,m_l-2}^{k,l} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{a}_{00}^{k,l} & \bar{a}_{10}^{k,l} & \bar{a}_{20}^{k,l} & \dots & \bar{a}_{n_l-1,0}^{k,l} \\ \bar{b}_{0,m_l-1}^{k,rl} & \bar{b}_{1,m_l-1}^{k,rl} & \bar{b}_{2,m_l-1}^{k,rl} & \dots & \bar{b}_{n_l-1,m_l-1}^{k,rl} \\ \bar{b}_{0,m_l-2}^{k,rl} & \bar{b}_{1,m_l-2}^{k,rl} & \bar{b}_{2,m_l-2}^{k,rl} & \dots & \bar{b}_{n_l-1,m_l-2}^{k,rl} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{b}_{00}^{k,rl} & \bar{b}_{10}^{k,rl} & \bar{b}_{20}^{k,rl} & \dots & \bar{b}_{n_l-1,0}^{k,rl} \end{bmatrix} \in R^{(p+1)m_l\times n_l}$$

$k=0,1,\dots,h$

$$A_{22}^{0,l}=\begin{bmatrix} a_{n_l,m_l-1}^{0,l} & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ a_{n_l,m_l-2}^{0,l} & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n_l,2}^{0,l} & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ a_{n_l,1}^{0,l} & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ a_{n_l,0}^{0,l} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ b_{n_l,m_l-1}^{0,rl} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ b_{n_l,m_l-2}^{0,rl} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{n_l,2}^{0,rl} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ b_{n_l,1}^{0,rl} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ b_{n_l,0}^{0,rl} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

$\in R^{(p+1)m_l\times(p+1)m_l}$

$$A_{22}^{k,l}=\begin{bmatrix} a_{n_l,m_l-1}^{k,l} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n_l,0}^{k,l} & 0 & \dots & 0 \\ b_{n_l,m_l-1}^{k,rl} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ b_{n_l,0}^{k,rl} & 0 & \dots & 0 \end{bmatrix} \in R^{(p+1)m_l\times(p+1)m_l},$$

$k=1,\dots,h$ (22)

$$B_1^{0,l}=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in R^{n_l\times 1},\quad B_1^{k,l}=[0]\in R^{n_l\times 1},\quad k=1,\dots,h;$$

$$B_2^{k,l}=\begin{bmatrix} a_{n_l,m_l-1}^{k,l} \\ a_{n_l,m_l-2}^{k,l} \\ \vdots \\ a_{n_l,0}^{k,l} \\ b_{n_l,m_l-1}^{k,rl} \\ b_{n_l,m_l-2}^{k,rl} \\ \vdots \\ b_{n_l,0}^{k,rl} \end{bmatrix} \in R^{(p+1)m_l\times 1},\quad k=0,1,\dots,h$$

$$\begin{aligned} C_1^{0,r} &= \begin{bmatrix} C_1^{0,r1} & \dots & C_1^{0,rm} \end{bmatrix}, \\ C_1^{0,rl} &= \begin{bmatrix} \bar{b}_{0,m_l}^{0,rl} & \bar{b}_{1,m_l}^{0,rl} & \dots & \bar{b}_{n_l-1,m_l}^{0,rl} \end{bmatrix} \in R^{1\times n_l}, \\ C_2^{0,r} &= \begin{bmatrix} C_2^{0,r1} & \dots & C_2^{0,rm} \end{bmatrix}, \\ C_2^{0,rl} &= \begin{bmatrix} C_{21}^{0,rl} & \dots & C_{2,p+1}^{0,rl} \end{bmatrix} \in R^{1\times(p+1)m_l}, \\ C_{21}^{0,rl} &= \begin{bmatrix} b_{n_l m_l}^{0,rl} & 0 & \dots & 0 \end{bmatrix} \in R^{1\times m_l}, \\ C_{2,k+1}^{0,rl} &= \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \in R^{1\times m_l} \qquad (22) \\ C_1^{k,r} &= \begin{bmatrix} C_1^{k,r1} & \dots & C_1^{k,rm} \end{bmatrix}, \\ C_1^{k,rl} &= \begin{bmatrix} \bar{c}_{0,m_l}^{k,rl} & \bar{c}_{1,m_l}^{k,rl} & \dots & \bar{c}_{n_l-1,m_l}^{k,rl} \end{bmatrix} \in R^{1\times n_l}, \\ C_2^{k,r} &= \begin{bmatrix} C_2^{k,r1} & \dots & C_2^{k,rm} \end{bmatrix}, \\ C_2^{k,rl} &= \begin{bmatrix} C_{21}^{k,rl} & \dots & C_{2,p+1}^{k,rl} \end{bmatrix} \in R^{1\times(p+1)m_l}, \\ C_{21}^{k,rl} &= \begin{bmatrix} b_{n_l m_l}^{k,rl} & 0 & \dots & 0 \end{bmatrix} \in R^{1\times m_l}, \\ C_{2,k+1}^{k,rl} &= \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \in R^{1\times m_l},\quad k=1,\dots,h \end{aligned}$$

and $\bar{a}_{ij}^{k,l}=a_{ij}^{k,l}+a_{im_l}^{0,l}a_{n_l j}^{k,l}$, $\bar{b}_{ij}^{k,rl}=b_{ij}^{k,rl}+a_{im_l}^{0,l}b_{n_l j}^{k,rl}$, $\bar{c}_{ij}^{k,rl}=a_{im_l}^{0,l}b_{n_l j}^{k,rl}$ for $k=0,1,\dots,h$; $r=1,2,\dots,p$; $l=1,2,\dots,m$; $i=0,1,\dots,n_l-1$; $j=0,1,\dots,m_l-1$.

Summing up the considerations we obtain for the MIMO 2D hybrid linear system with delays the following theorem.

Theorem 3. There exists a positive realization if all coefficients of the numerators and denominators of the transfer matrix (12) are nonnegative.

The procedure given for SISO systems with slight modifications can be also used for finding a positive realization of the transfer matrix (12).

Example 2. Find a positive realization (2) of the proper transfer matrix with h = 1 delays

$$T(s,z)=\begin{bmatrix} \dfrac{b_{11}^{11}(w)sz+b_{10}^{11}(w)s+b_{01}^{11}z+b_{00}^{11}(w)}{sz-a_{10}^{1}(w)s-a_{01}^{1}z-a_{00}^{1}(w)} & \dfrac{b_{21}^{12}(w)s^2z+b_{20}^{12}(w)s^2+b_{11}^{12}sz+b_{10}^{12}(w)s+b_{01}^{12}z+b_{00}^{12}(w)}{s^2z-a_{20}^{2}(w)s^2-a_{11}^{2}sz-a_{10}^{2}(w)s-a_{01}^{2}z-a_{00}^{2}(w)} \\ \dfrac{b_{11}^{21}(w)sz+b_{10}^{21}(w)s+b_{01}^{21}z+b_{00}^{21}(w)}{sz-a_{10}^{1}(w)s-a_{01}^{1}z-a_{00}^{1}(w)} & \dfrac{b_{21}^{22}(w)s^2z+b_{20}^{22}(w)s^2+b_{11}^{22}sz+b_{10}^{22}(w)s+b_{01}^{22}z+b_{00}^{22}(w)}{s^2z-a_{20}^{2}(w)s^2-a_{11}^{2}sz-a_{10}^{2}(w)s-a_{01}^{2}z-a_{00}^{2}(w)} \end{bmatrix} \qquad (22)$$

In this case we have $p=2$ outputs and $m=2$ inputs, coefficients for one delay have the forms $a_{ij}^{l}(w)=a_{ij}^{1l}w+a_{ij}^{0l}$, $b_{ij}^{rl}(w)=b_{ij}^{1rl}w+b_{ij}^{0rl}$, $a_{ij}^{l}=a_{ij}^{0l}$, $b_{ij}^{rl}=b_{ij}^{0rl}$.

Using Procedure we obtain the following.

Step1.

Multiplying numerators and denominator of the first column by $s^{-1}z^{-1}$ and multiplying numerators and denominator of the second column by $s^{-2}z^{-1}$ we obtain

$$T(s,z)=\begin{bmatrix}\dfrac{b_{11}^{11}(w)+b_{10}^{11}(w)z^{-1}+b_{01}^{11}s^{-1}+b_{00}^{11}(w)s^{-1}z^{-1}}{1-a_{10}^{1}(w)z^{-1}-a_{01}^{1}s^{-1}-a_{00}^{1}(w)s^{-1}z^{-1}} & \dfrac{b_{21}^{12}(w)+b_{20}^{12}(w)z^{-1}+b_{11}^{12}s^{-1}+b_{10}^{12}(w)s^{-1}z^{-1}+b_{01}^{12}s^{-2}+b_{00}^{12}(w)s^{-2}z^{-1}}{1-a_{20}^{2}(w)z^{-1}-a_{11}^{2}s^{-1}-a_{10}^{2}(w)s^{-1}z^{-1}-a_{01}^{2}s^{-2}-a_{00}^{2}(w)s^{-2}z^{-1}} \\ \dfrac{b_{11}^{21}(w)+b_{10}^{21}(w)z^{-1}+b_{01}^{21}s^{-1}+b_{00}^{21}(w)s^{-1}z^{-1}}{1-a_{10}^{1}(w)z^{-1}-a_{01}^{1}s^{-1}-a_{00}^{1}(w)s^{-1}z^{-1}} & \dfrac{b_{21}^{22}(w)+b_{20}^{22}(w)z^{-1}+b_{11}^{22}s^{-1}+b_{10}^{22}(w)s^{-1}z^{-1}+b_{01}^{22}s^{-2}+b_{00}^{22}(w)s^{-2}z^{-1}}{1-a_{20}^{2}(w)z^{-1}-a_{11}^{2}s^{-1}-a_{10}^{2}(w)s^{-1}z^{-1}-a_{01}^{2}s^{-2}-a_{00}^{2}(w)s^{-2}z^{-1}}\end{bmatrix}$$

and

$$E_1=U_1+(a_{10}^{1}(w)z^{-1}+a_{01}^{1}s^{-1}+a_{00}^{1}(w)s^{-1}z^{-1})E_1$$
$$E_2=U_2+(a_{20}^{2}(w)z^{-1}+a_{11}^{2}s^{-1}+a_{10}^{2}(w)s^{-1}z^{-1}+a_{01}^{2}s^{-2}+a_{00}^{2}(w)s^{-2}z^{-1})E_2 \quad (23)$$
$$\begin{bmatrix}Y_1\\Y_2\end{bmatrix}=\begin{bmatrix}b_{11}^{11}(w)+b_{10}^{11}(w)z^{-1}+b_{01}^{11}s^{-1}+b_{00}^{11}(w)s^{-1}z^{-1} & b_{21}^{12}(w)+b_{20}^{12}(w)z^{-1}+b_{11}^{12}s^{-1}+b_{10}^{12}(w)s^{-1}z^{-1}+b_{01}^{12}s^{-2}+b_{00}^{12}(w)s^{-2}z^{-1}\\ b_{11}^{21}(w)+b_{10}^{21}(w)z^{-1}+b_{01}^{21}s^{-1}+b_{00}^{21}(w)s^{-1}z^{-1} & b_{21}^{22}(w)+b_{20}^{22}(w)z^{-1}+b_{11}^{22}s^{-1}+b_{10}^{22}(w)s^{-1}z^{-1}+b_{01}^{22}s^{-2}+b_{00}^{22}(w)s^{-2}z^{-1}\end{bmatrix}\begin{bmatrix}E_1\\E_2\end{bmatrix}$$

Step 2.

State variable diagram for (23) has the form shown in Fig. 4.

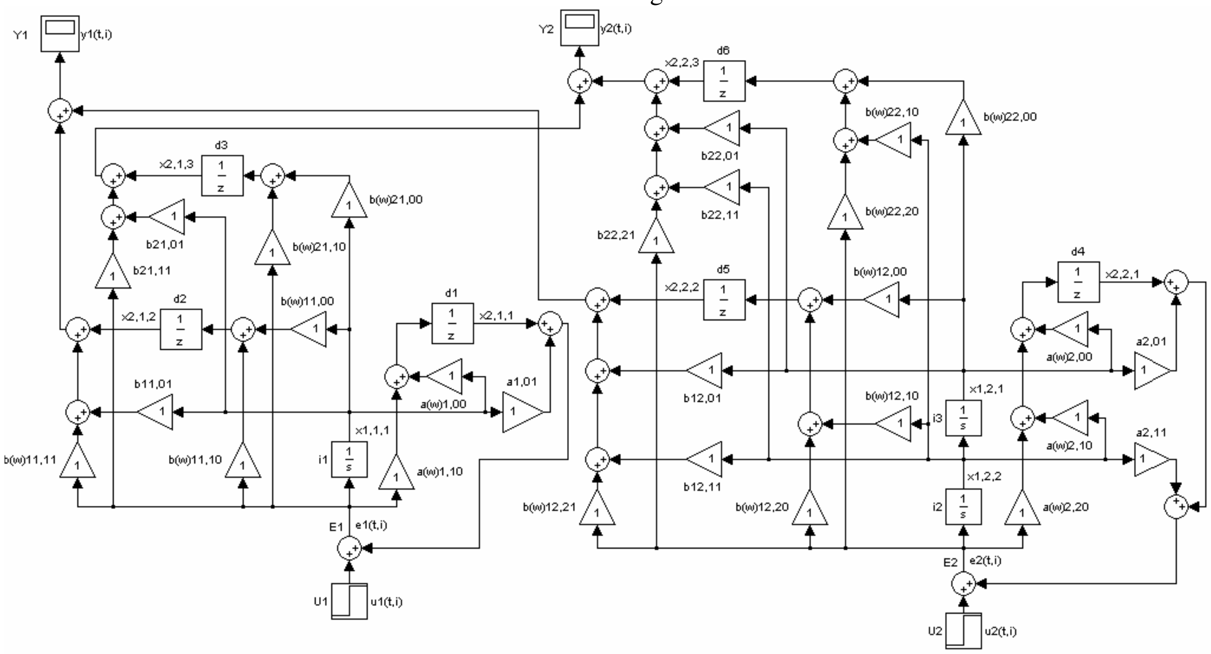

Fig. 4 MATLAB/SIMULINK state variable diagram for transfer function (22).

where coefficients $a_{ij}^{l}(w)$ and $b_{ij}^{rl}(w)$ for one delay have the form shown in Fig. 5

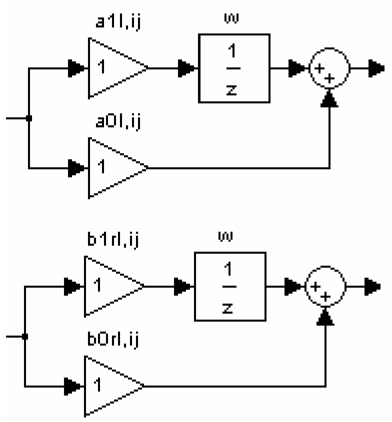

Fig. 5 MATLAB/SIMULINK state variable diagram for transfer for blocks $a_{ij}^{l}(w)$ and $b_{ij}^{rl}(w)$.

Step 3.

Using state variable diagram we can write the following equations

$$\begin{aligned}
\dot{x}_{1,1,1}(t,i)&=e_1(t,i)\\
\dot{x}_{1,2,1}(t,i)&=x_{1,2,2}(t,i)\\
\dot{x}_{1,2,2}(t,i)&=e_2(t,i)\\
x_{2,1,1}(t,i+1)&=a_{00}^{01}x_{1,1,1}(t,i)+a_{00}^{11}x_{1,1,1}(t-d,i)\\
&\quad+a_{10}^{01}e_1(t,i)+a_{10}^{11}e_1(t-d,i-1) \qquad (24)\\
x_{2,1,2}(t,i+1)&=b_{00}^{011}x_{1,1,1}(t,i)+b_{00}^{111}x_{1,1,1}(t-d,i)\\
&\quad+b_{10}^{011}e_1(t,i)+b_{10}^{111}e_1(t-d,i-1)\\
x_{2,1,3}(t,i+1)&=b_{00}^{021}x_{1,1,1}(t,i)+b_{00}^{121}x_{1,1,1}(t-d,i)\\
&\quad+b_{10}^{021}e_1(t,i)+b_{10}^{121}e_1(t-d,i-1)
\end{aligned}$$

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$$
\begin{aligned}
x_{2,2,1}(t,i+1) &= a_{00}^{02} x_{1,2,1}(t,i) + a_{00}^{12} x_{1,2,1}(t-d,i) \\
&+ a_{10}^{02} x_{1,2,2}(t,i) + a_{10}^{12} x_{1,2,2}(t-d,i) \\
&+ a_{20}^{02} e_2(t,i) + a_{20}^{12} e_2(t-d,i-1) \\
x_{2,2,2}(t,i+1) &= b_{00}^{012} x_{1,2,1}(t,i) + b_{00}^{112} x_{1,2,1}(t-d,i) \\
&+ b_{10}^{012} x_{1,2,2}(t,i) + b_{10}^{112} x_{1,2,2}(t-d,i) \\
&+ b_{20}^{012} e_2(t,i) + b_{20}^{112} e_2(t-d,i-1) \\
x_{2,2,3}(t,i+1) &= b_{00}^{022} x_{1,2,1}(t,i) + b_{00}^{122} x_{1,2,1}(t-d,i) \\
&+ b_{10}^{022} x_{1,2,2}(t,i) + b_{10}^{122} x_{1,2,2}(t-d,i) \\
&+ b_{20}^{022} e_2(t,i) + b_{20}^{122} e_2(t-d,i-1) \\
y_1(t,i) &= x_{2,1,2}(t,i) + b_{01}^{011} x_{1,1,1}(t,i) + b_{11}^{011} e_1(t,i) \\
&+ b_{11}^{111} e_1(t-d,i-1) + x_{2,2,2}(t,i) \\
&+ b_{01}^{012} x_{1,2,1}(t,i) + b_{11}^{012} x_{1,2,2}(t,i) \\
&+ b_{21}^{012} e_2(t,i) + b_{21}^{112} e_2(t-d,i-1) \\
y_2(t,i) &= x_{2,1,3}(t,i) + b_{01}^{021} x_{1,1,1}(t,i) + b_{11}^{021} e_1(t,i) \\
&+ b_{11}^{121} e_1(t-d,i-1) + x_{2,2,3}(t,i) \\
&+ b_{01}^{022} x_{1,2,1}(t,i) + b_{11}^{022} x_{1,2,2}(t,i) \\
&+ b_{21}^{022} e_2(t,i) + b_{21}^{122} e_2(t-d,i-1)
\end{aligned} \tag{24}
$$

where

$$
\begin{aligned}
e_1(t,i) &= a_{01}^{01} x_{1,1,1}(t,i) + x_{2,1,1}(t,i) + u_1(t,i) \\
e_2(t,i) &= a_{01}^{02} x_{1,2,1}(t,i) + a_{11}^{02} x_{1,1,2}(t,i) + x_{2,2,1}(t,i) \\
&\quad + u_2(t,i) \\
e_1(t-d,i-1) &= a_{01}^{01} x_{1,1,1}(t-d,i) + x_{2,1,1}(t,i-1) \\
&\quad + u_1(t-d,i-1) \\
e_2(t-d,i-1) &= a_{01}^{02} x_{1,2,1}(t-d,i) + a_{11}^{02} x_{1,1,2}(t-d,i) \\
&\quad + x_{2,2,1}(t,i-1) + u_2(t-d,i-1)
\end{aligned} \tag{25}
$$

Step 4.
Substituting (25) into (24) and taking into account (18) – (21), the desired realization of (22) has the form

$$
\begin{aligned}
A_{11}^0 &= \begin{bmatrix} A_{11}^{01} & 0 \\ 0 & A_{11}^{02} \end{bmatrix}, \quad A_{11}^1 = \begin{bmatrix} A_{11}^{11} & 0 \\ 0 & A_{11}^{12} \end{bmatrix}, \\
A_{12}^0 &= \begin{bmatrix} A_{12}^{01} & 0 \\ 0 & A_{12}^{02} \end{bmatrix}, \quad A_{12}^1 = \begin{bmatrix} A_{12}^{11} & 0 \\ 0 & A_{12}^{12} \end{bmatrix}, \\
A_{21}^0 &= \begin{bmatrix} A_{21}^{01} & 0 \\ 0 & A_{21}^{02} \end{bmatrix}, \quad A_{21}^1 = \begin{bmatrix} A_{21}^{11} & 0 \\ 0 & A_{21}^{12} \end{bmatrix}, \\
A_{22}^0 &= \begin{bmatrix} A_{22}^{01} & 0 \\ 0 & A_{22}^{02} \end{bmatrix}, \quad A_{22}^1 = \begin{bmatrix} A_{22}^{11} & 0 \\ 0 & A_{22}^{12} \end{bmatrix},
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
B^0 &= \begin{bmatrix} B_1^{01} & 0 \\ B_2^{01} & 0 \\ 0 & B_1^{02} \\ 0 & B_2^{02} \end{bmatrix}, \quad B^1 = \begin{bmatrix} B_1^{11} & 0 \\ B_2^{11} & 0 \\ 0 & B_1^{12} \\ 0 & B_2^{12} \end{bmatrix}, \\
C_1^0 &= \begin{bmatrix} C_1^{01} \\ C_1^{02} \end{bmatrix}, \quad C_2^0 = \begin{bmatrix} C_2^{01} \\ C_2^{02} \end{bmatrix}, \\
C_1^1 &= \begin{bmatrix} C_1^{11} \\ C_1^{12} \end{bmatrix}, \quad C_2^1 = \begin{bmatrix} C_2^{11} \\ C_2^{12} \end{bmatrix}, \\
D^0 &= \begin{bmatrix} b_{11}^{011} & b_{21}^{012} \\ b_{11}^{021} & b_{21}^{022} \end{bmatrix}, \quad D^1 = \begin{bmatrix} b_{11}^{111} & b_{21}^{112} \\ b_{11}^{121} & b_{21}^{122} \end{bmatrix}
\end{aligned} \tag{26}
$$

where

$$
\begin{aligned}
A_{11}^0 &= \begin{bmatrix} a_{01}^{01} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & a_{01}^{02} & a_{11}^{02} \end{bmatrix}, \\
A_{12}^0 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \\
A_{21}^0 &= \begin{bmatrix} \bar{a}_{00}^{01} & 0 & 0 \\ \bar{b}_{00}^{011} & 0 & 0 \\ \bar{b}_{00}^{021} & 0 & 0 \\ 0 & \bar{a}_{00}^{02} & \bar{a}_{10}^{02} \\ 0 & \bar{b}_{00}^{012} & \bar{b}_{10}^{012} \\ 0 & \bar{b}_{00}^{022} & \bar{b}_{10}^{022} \end{bmatrix}, \\
A_{22}^0 &= \begin{bmatrix} a_{10}^{01} & 0 & 0 & 0 & 0 & 0 \\ b_{10}^{011} & 0 & 0 & 0 & 0 & 0 \\ b_{10}^{021} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{10}^{02} & 0 & 0 \\ 0 & 0 & 0 & b_{10}^{012} & 0 & 0 \\ 0 & 0 & 0 & b_{10}^{022} & 0 & 0 \end{bmatrix}, \\
A_{11}^1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
A_{12}^1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
A_{21}^1 &= \begin{bmatrix} \bar{a}_{00}^{11} & 0 & 0 \\ \bar{b}_{00}^{111} & 0 & 0 \\ \bar{b}_{00}^{121} & 0 & 0 \\ 0 & \bar{a}_{00}^{12} & \bar{a}_{10}^{12} \\ 0 & \bar{b}_{00}^{112} & \bar{b}_{10}^{112} \\ 0 & \bar{b}_{00}^{122} & \bar{b}_{10}^{122} \end{bmatrix},
\end{aligned} \tag{27}
$$

$$A_{22}^1 = \begin{bmatrix} a_{10}^{11} & 0 & 0 & 0 & 0 & 0 \\ b_{10}^{111} & 0 & 0 & 0 & 0 & 0 \\ b_{10}^{121} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{10}^{12} & 0 & 0 \\ 0 & 0 & 0 & b_{10}^{112} & 0 & 0 \\ 0 & 0 & 0 & b_{10}^{122} & 0 & 0 \end{bmatrix},$$

$$B^0 = \begin{bmatrix} 1 & 0 \\ a_{10}^{01} & 0 \\ b_{10}^{011} & 0 \\ b_{10}^{021} & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & a_{20}^{02} \\ 0 & b_{20}^{012} \\ 0 & b_{20}^{022} \end{bmatrix}, \quad B^1 = \begin{bmatrix} 0 & 0 \\ a_{10}^{11} & 0 \\ b_{10}^{111} & 0 \\ b_{10}^{121} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & a_{20}^{12} \\ 0 & b_{20}^{112} \\ 0 & b_{20}^{122} \end{bmatrix},$$

$$C_1^0 = \begin{bmatrix} \overline{b}_{01}^{011} & \overline{b}_{01}^{012} & \overline{b}_{11}^{012} \\ \overline{b}_{01}^{021} & \overline{b}_{01}^{022} & \overline{b}_{11}^{022} \end{bmatrix},$$

$$C_2^0 = \begin{bmatrix} b_{11}^{011} & 1 & 0 & b_{21}^{012} & 1 & 0 \\ b_{11}^{021} & 0 & 1 & b_{21}^{022} & 0 & 1 \end{bmatrix}, \tag{27}$$

$$C_1^1 = \begin{bmatrix} \overline{c}_{01}^{111} & \overline{c}_{01}^{112} & \overline{c}_{11}^{112} \\ \overline{c}_{01}^{121} & \overline{c}_{01}^{122} & \overline{c}_{11}^{122} \end{bmatrix},$$

$$C_2^1 = \begin{bmatrix} b_{11}^{111} & 1 & 0 & b_{21}^{112} & 1 & 0 \\ b_{11}^{121} & 0 & 1 & b_{21}^{122} & 0 & 1 \end{bmatrix},$$

$$D^0 = \begin{bmatrix} b_{11}^{011} & b_{21}^{012} \\ b_{11}^{021} & b_{21}^{022} \end{bmatrix}, \quad D^1 = \begin{bmatrix} b_{11}^{111} & b_{21}^{112} \\ b_{11}^{121} & b_{21}^{122} \end{bmatrix}$$

and $\overline{a}_{ij}^{k,l} = a_{ij}^{k,l} + a_{im_l}^{0,l} a_{n_l j}^{k,l}$, $\overline{b}_{ij}^{k,rl} = b_{ij}^{k,rl} + a_{im_l}^{0,l} b_{n_l j}^{k,rl}$, $\overline{c}_{ij}^{k,rl} = a_{im_l}^{0,l} b_{n_l j}^{k,rl}$ for $k = 0,1,...,h$; $r = 1,2,..., p$; $l = 1,2,...,m$; $i = 0,1,...,n_l - 1$; $j = 0,1,...,m_l - 1$.

5 Concluding remarks

A method for computation of a positive realization of a given proper transfer matrix of 2D hybrid linear system with delays has been proposed. Sufficient conditions for the existence of a positive realization of a given proper transfer matrix have been established. A procedure for computation of a positive realization has been proposed. An open problem is formulation of the necessary and sufficient conditions for the existence of solution of the positive realization problem for 2D hybrid systems in the general case. Extension of those considerations for 2D hybrid systems described by models with structures similar to the 2D general model [14] or the 2D second Fornasini-Marchesini model [18] are also open problems.

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