A DYNAMIC APPROACH TO THE EDUCATION-INVESTMENT DECISION STRATEGY USING OPTIMAL CONTROL THEORY

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Abstract

An important aspect to expand the production of human capital is the properly investment decision strategy in educational matters. Thus, a foregone income of thousands of euros per individual are fluently spent both by him and by public government into that specific direction. The purpose of this study is to analytically determine the optimal lifetime path of education (or equivalently, let us say: the stock of knowledge) for an average individual (particularly someone with limited income) and the public policy implications of his decision. Thus, an optimal control theory model to the education – investment decision strategy that maximizes the present value of future earnings for an individual is fully developed. The formulation of this model is quite general including several inputs variables, assuming only the rate of schooling as the control variable. Finally, an illustrative application is presented. In that application, it is considered a special case of the famous Cobb – Douglas production function.

Keywords: Modelling, Deterministic Optimal Control Theory, Allocation of Resources; Human Capital, Education – Investment decision.

Presenting Author's biography

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1 Introduction

Last decades, in the global economies, the multi – scale expansion of scientific and technical knowledge has inevitably raised the productivity of labor and other important inputs in the complex chain of production. Furthermore and more relatively to the systematic applications; the deeper knowledge, the value of education, the schooling and on – the – job training (i.e. the further growth of scientific and technical knowledge) have become embodied in people – scientists, scholars, technicians, managers and other contributors.

Consequently, the lifetime path of education is become one of the most important investments in human capital's growth. In that direction, several empirical studies have showed that good college training, strong influence of family and lifetime schooling have raised sharply a persons' income, see for instance [1], [15]. Thus, the earnings of more educated people are almost always well above average [13], although the gains are generally larger in less – developed countries [1].

Into our micro – macro economical environment, the investment of an individual in educational matters is a very important, meanwhile such an expensive, aspect. For instance, the foregone income alone for the years of primary and secondary (high school) education is likely to be about two hundred thousands euros per student. However, the internal rate of return of this investment has been estimated to be in the rage of ten to twenty, or more in some developing countries, see for instance [3], [10] and [16]. Furthermore, according to [19], [14] the generous investment in education restricts effectively the poverty, as the income earnings are strongly related to the educational level of the individuals. It is also noted that the poor generally undertake less or (in the majority of cases) none education than the non poor.

The problem of an optimal lifestyle investment in the limited stock of human capital, i.e. the optimal stock of education (knowledge), has been a subject in the literature of economical mathematics for many decades. Analytically, the problem is to determine formalistically (and not merely empirically) the optimal lifetime path of education policy of an average individual, who can split up its time into learning and working, and it is a subject to negative income by taxation, and by the cost of learning education, as well.

Actually, the above interesting problem can be easily reformulated into an optimal control problem. Although optimal control theory was developed by engineers in order to investigate the properties of dynamic systems of difference or differential equations, it has also been applied to financial problems. Tustin, see [20], was the first to spot a possible analogy between the industrial and engineering processes and post – war macroeconomic policy – making (see, [12] for further historical details). Furthermore, the research work [2] is one of the earliest applications of optimal control theory that was devoted to this topic.

Meanwhile, in the middle of the previous century, different kinds of models following that point of view have been developed to analyse and maximize several kinds of objective functions, respectively. See the pioneer work [10], [18], [19], [4], [11] and various other extensions of these models.

In this paper we theoretically derive an optimal path of education (knowledge) investment or, in other words, schooling resources allocation over time for an individual. Since the economical and social environment is continually changing, we should also consider the education – investment decision as a dynamic process over the course of a life time. In this framework some assumptions must also be made. Thus, as in many developing models, see [2], [19], and [11], the individual purchases knowledge solely for its investment value, i.e. the consumption aspects of education is definitely ignored. Moreover, the individual is assumed to act in an optimal way, i.e. it is interesting in purchasing education as long as its incremental value is greater then its respective cost. At last but not at least, two individuals are having the same formal schooling, celeris paribus, they may have quite different levels of acquired knowledge. Since no readily data are available, social indicators such as sex, colour, labour nationality or immigration are not taking into consideration.

Furthermore, the income function used in this paper is enlarged from earlier studies to include simultaneously profits from risk – free investments (i.e. T-bills, cash accounts etc), from the level of education – including the possibility of the individual to participate in different financed projects or simply to obtain a scholarship, and the experience of individual (i.e. its age), as well. Additionally, it is stretched out that we consider the income function as a dynamic process over the course of a life time.

The next section develops the basic continuous – time model in implicit form and obtains some theoretical results. Section 3 provides a very interesting case which is mainly constrained by a special case of the famous Cobb – Douglas production function. Thus, a linear case is fully investigated. By using empirical results, an optimal investment – education decision strategy is eventually derived in the Section 4. Conclusions and further research proposals are provided in the Section 5.

2 Development of the optimal dynamic model in continuous – time framework

This section proceeds with the analytical presentation of the proposed model, transferring the entire discussion and motivation of the previous section

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into mathematical equations. Firstly, the necessary symbols and the respective notation are defined keeping in mind the continuous – time framework of our analysis. For the better understanding of the model, it is important to construct the involved equations in a more general format, following as closed as it is possible [19], [17] and [11] research works.

Obviously, the stock of human capital (knowledge) embodied in an individual may change over the period of schooling, see [5]. Since the schooling increases the education level, the lack of it may allow the education to decline.

Analytically, let $x(t) \in C^1(\mathbb{R})$ denote the level of education (i.e. the stock of human capital – knowledge) and $u(t):[0,T] \to [0,1]$, which is a sufficiently differentiable function, denote the investment into that stock. Equivalently, the fraction of time devoted to work is $1-u(t)$. The change in human capital over time is then given by a weakly non linear ordinary differential equation:

$$\dot{x}(t) = -a(t)x(t) + f(t, x(t), u(t))$$

$$x(0) = x_o \tag{1}$$

where $a(t):[0,T] \to U \subset \mathbb{R}$ the depreciation rate of education and $f(\cdot):[0,T]\times\mathbb{R}\times[0,1] \to \mathbb{R}$ which represents the production of human capital [1], [8] are also sufficiently differentiable equations. Note that the deduction factor $a(t)$ varies with the different type of education (i.e. medical, mathematical vs. technical etc), and the different economical – political – social circumstances (i.e. in the western vs. sub – African countries). Moreover, the source of this depreciation may be either from forgetting or from technological obsolescence, or both, see [18].

Furthermore, it is assumed that the potential money income that can be earned by an individual is mainly a function of his level of education x, the age t and the risk – free interest rate $r > 0$ is fixed. Thus, we obtain

$$\dot{y}(t) = ry(t) + (1-u(t))h_1(t, x(t)) + u(t)h_2(t, x(t))$$

$$y(0) = y_o \tag{2}$$

This differential function implies that money income is evaluated by the risk – free investment (i.e. T-bills, cash accounts etc), by the function

$$h_1(\cdot):[0,T]\times\mathbb{R} \to \mathbb{R}$$

for the time which is being spent at work, $1-u(t)$ and by the function

$$h_2(\cdot):[0,T]\times\mathbb{R} \to \mathbb{R}$$

for the time which is being invested in schooling through a scholarship, or his participation into a research programme. Note that it is assumed that part – time work is equally paid as the full – time work.

The direct cost of education is assumed to be linearly related to the proportion invested to knowledge, i.e. $g_1(\cdot):[0,1] \to \mathbb{R}$ and to the level of education x, $g_1(\cdot):\mathbb{R} \to \mathbb{R}$. This yields

$$c(t) = g_1(u)\cdot g_2(x) \tag{3}$$

A further realistic consideration is the income tax policy, which is surely correlated to the earning and to the different expenditures, as the following expression is devoted

$$T(t) = \tau_o + \tau_1 y(t) - \tau_2 c(t) \tag{4}$$

Analytically, the constant number τ_o can be a negative or positive number analogously to the present and proposed welfare system, see [19]. Moreover, expression (4) takes into consideration a percentage, τ_1, of the actual income diminished by a percentage, τ_2, of the direct cost of education. However, in practice, the actual tax system is somewhat much more complex because it takes into consideration several others inputs such as capital gains, medical deductions, number of infants etc. Although, a number of empirical studies (e.g. [3] etc) have found that the tax system is approximately proportional, which gives support to expression (4).

Now, in the same point of view as in research work [19] [16], and [11] it can be assumed that the objective function is to optimize (maximize, in this case) the discounted present value of future income stream. The expression under parentheses in the objective function (5) is the net cash flow at time $t \in [0,T]$. Additionally, it can be also strength out that the controlled interval period is $[0,T]$ (e.g. 0: the starting of working and T: the year of retirement) and the discount rate r is constant and equal to the premium of a $T-$period government (risk – free) bond. Thus

$$\max_u \left\{ \int_0^T e^{-rt}\left(y(t) - T(t) - c(t)\right)dt \right\} \tag{5}$$

Actually, the individual follows a time – path of education (through seminars, attainting MSc courses or doing MBA etc) into that period in order to maximize the value of (5). Of course, the investment into the knowledge stock via the rate $u(t)$ has a limited range, between 0 and 1, since he can not obtain schooling at a negative rate or more than full time.

Historically, the maximum principle, formulated and derived by Potryagin and his group in the 1950s, is truly a milestone of optimal control theory, see [21] for more details. It states that any optimal control along with the optimal state trajectory must solve the so – called (extended) Hamiltonian system, which is a two – point boundary value problem (and can also called a forward – backward differential equation), plus a maximum condition of a function called the Hamiltonian.

Thus, the Hamiltonian function is given by expression (6)

$$\begin{aligned} H(t,x,y,u,p,q) &= e^{-rt}\left(y(t)-T(t)-c(t)\right) \\ &\quad + p(t)\dot{x}(t)+q(t)\dot{y}(t) \end{aligned} \tag{6}$$

$$(t,x,y,u,p,q):[0,T]\times\mathbb{R}\times\mathbb{R}\times[0,1]\times\mathbb{R}\times\mathbb{R}$$

where the shadow prices p and q of the level of education and the potential money income, respectively are the solution of

$$\dot{p}(t) = -H_x\left(t,x(t),y(t),u(t),p(t),q(t)\right) \tag{7}$$

$$\dot{q}(t) = -H_y\left(t,x(t),y(t),u(t),p(t),q(t)\right) \tag{8}$$

at a. e. $t\in[0,T]$.

Furthermore, since the objective function is the maximization of cognitive knowledge, x, at the end of the period, the following transversely condition applies

$$p(T)=0 \tag{9}$$

and additionally,

$$q(0)=q_o \tag{10}$$

The condition for optimality is

$$\begin{aligned} &H\left(t,x(t),y(t),u(t),p(t),q(t)\right) \\ &= \max_{u\in[0,1]} H\left(t,x(t),y(t),u,p(t),q(t)\right) \end{aligned} \tag{11}$$

In practice, the maximization of the criterion is achieved if the control be chosen to maximize the Hamiltonian at each point in time. Thus, the necessary first – order condition is

$$H_u = 0 \tag{12}$$

Note that time dependency (t) of the variables is omitted for notational convenience. By substituting expressions (3) and (4) into (5) it is derived

$$\max_u\left\{\int_0^T e^{-rt}\left((1-\tau_1)y-\tau_o-(1-\tau_2)c\right)dt\right\} \tag{13}$$

So, from expression (7) and (8) by using the reformed Hamiltonian equation (14)

$$\begin{aligned} H(t,x,y,u,p,q) &= e^{-rt}\left((1-\tau_1)y-\tau_o-(1-\tau_2)c\right) \\ &\quad + p\{-ax+f\}+q\{ry+(1-u)h_1+uh_2\} \end{aligned} \tag{14}$$

it is obtained

$$\dot{p} = (a-f_x)p+(1-\tau_2)e^{-rt}g_1g_{2x}-q\left(h_{1x}+u\left(h_{2x}-h_{1x}\right)\right) \tag{15}$$

and

$$\dot{q} = -rq-(1-\tau_2)e^{-rt} \tag{16}$$

Moreover, through the expression (12) it is taken

$$-(1-\tau_2)e^{-rt}g_{1u}g_2+pf_u=0$$

or equivalently

$$f_u/g_{1u} = (1-\tau_2)e^{-rt}g_2/p \tag{17}$$

The Hamiltonian (14) and the co state variables p and q should be analysed by taking into consideration the several economical – social interpretations. According to (15) the first co state variable which reflects the level of education per individual is very complicated, as many parameters get involved. Although, the rate $a-f_x$ decreases the ordinary linear differential equation of function p if the marginal productivity of human capital exceeds the rate of depreciation of knowledge, see also [17]. Moreover, the second co state variable which reflects the potential money income per individual is simpler and depends mainly on the discount rate r. It is intuitively clear that the income is decreasing by the increasing of the discount rate.

After these preliminary results and comments, the properties of the optimal investment policy can be determined. Differentiate (17) with respect to time and substitute the necessary equations to obtain

$$\begin{aligned} &\dot{p}f_u+pf_{uu}\dot{u}+pf_{ux}\dot{x}+pf_{ut} \\ &+(1-\tau_2)re^{-rt}g_{1u}g_2-(1-\tau_2)e^{-rt}g_{1uu}\dot{u}g_2-(1-\tau_2)e^{-rt}g_{1u}g_{2x}\dot{x} \\ &=\dot{q}(h_1-h_2)+q(h_{1t}-h_{2t})+q(h_{1x}-h_{2x})\dot{x} \end{aligned}$$

or the equivalent non – linear partial differential equation (18)

$$\begin{aligned} &\left[pf_{uu}-(1-\tau_2)e^{-rt}g_{1uu}g_2\right]\dot{u} \\ &=\left[q(h_{1x}-h_{2x})-pf_{ux}+(1-\tau_2)e^{-rt}g_{1u}g_{2x}\right]\dot{x}+\dot{q}(h_1-h_2) \\ &\quad +q(h_{1t}-h_{2t})-\dot{p}f_u-pf_{ut}-(1-\tau_2)re^{-rt}g_{1u}g_2 \end{aligned} \tag{18}$$

Given the functions involved in expression above, the time path of education along the turnpike can be found using (18), and the sufficient boundary conditions.

Finally, the Hamiltonian may be interpreted as the net "profit" at time t from the net investment in human capital. Moreover, by taking also into consideration the above complicated expression (18), a much insightful view for the percentage of education that someone should invest into that stock is derived in order to maximize his "profit". Equivalently, the fraction of time devoted to work, $1-u$, is also obtained. In the next section a particular production function, f is used.

3 A special case: Cobb – Douglas production function

In economics, the Cobb – Douglas functional expression of productivity is widely used to represent the strong relation of an output to inputs. This functional expression has been firstly used in [6] as a law of production, but as it is mentioned in [7], it was already known by Pareto, several decades before.

So, in this section a special case of function f is considered; i.e. the famous Cobb – Douglas production function, as it has already been used in [19], [17] and [10] research works which are relative to our paper.

Thus, it is assumed that

$$f(t,x,u)=b(t)u^{\beta}(t)x^{\gamma}(t) \tag{19}$$

where $\beta,\ \gamma\in\mathbb{R}$

Obviously, the expression (1) obtain the following form

$$\dot{x}(t)=-a(t)x(t)+b(t)u^{\beta}(t)x^{\gamma}(t) \tag{20}$$

Moreover, it is assumed

$$h_1(t,x(t))=a_1(t)x(t) \text{ and } h_2(t,x(t))=a_2(t)x(t)$$

Where, the coefficients parameters $a_1(t)$ and $a_2(t)$ are t – continuous functions.

So, the expression (2) is transposed to the linear equation

$$\begin{aligned}\dot{y}(t)&=r(t)y(t)+a_1(t)x(t)\\&-\big(a_1(t)-a_2(t)\big)u(t)x(t)\end{aligned} \tag{21}$$

Following the [17], the cost of education is assumed to have the expression

$$c(t)=u(t)\cdot g(x) \tag{22}$$

Then, by using the expression (17) and noting that time dependency (t) of the variables is also omitted for notational convenience, it is derived a complicated expression for the controller

$$u=\left[\frac{pb\beta}{1-\tau_2}e^{rt}\frac{x^{\gamma-1}}{g(x)}\right]^{\frac{1}{1-\beta}} \tag{23}$$

where $\beta,\ \gamma,\ \tau_2,\ r$ are constant, b is a function of t, and $p,\ x$ are the solution of (15) and (20) respectively.

Now, by substituting the expression above (23) into (20) and (15), the following strong non linear system should be solved.

$$\dot{x}=-ax+\left(\frac{\beta}{1-\tau_2}\right)^{\frac{\beta}{1-\beta}}b^{\frac{1}{1-\beta}}e^{\frac{\beta r}{1-\beta}t}p^{\frac{\beta}{1-\beta}}\frac{x^{\frac{\beta}{1-\beta}(\gamma-1)+\gamma}}{g^{\frac{\beta}{1-\beta}}(x)} \tag{24}$$

$$\begin{aligned}&\dot{p}=ap-a_1qx\\&-\left\{\begin{aligned}&\gamma\left[\frac{\beta}{1-\tau_2}e^{rt}\frac{1}{g(x)}\right]^{\beta}\\&+\left[(1-\tau_2)e^{-rt}g_x+(a_1-a_2)x\right]\end{aligned}\right\}\left[\frac{\beta be^{rt}}{1-\tau_2}\right]^{\frac{1}{1-\beta}}\cdot\\&\cdot\left[\frac{x^{\gamma-1}}{g(x)}\right]^{\frac{1}{1-\beta}}p^{\frac{1}{1-\beta}}\end{aligned} \tag{25}$$

where $\beta,\ \gamma\in\mathbb{R}$ and q is the solution of ordinary linear differential equation (16). The solution of such non linear systems is far beyond the main target of this paper. However, in order to obtain some insightful comments we denote $\beta=\gamma=1$.

Thus, expression (19) is rewritten

$$f(t,x,u)=b(t)u(t)x(t)$$

Moreover, we denote $c(t)=cu(t)x(t)$, where c is constant.

Then, by taking also into consideration the expression (17), it is derived.

$$p(t)=\frac{c}{b(t)}(1-\tau_2)e^{-rt} \tag{26}$$

and by using the differential equation (15)

$$u(t)=\frac{\dot{p}(t)-a(t)p(t)+a_1(t)q(t)}{(1-\tau_2)ce^{-rt}-b(t)p(t)-\big(a_1(t)-a_2(t)\big)q(t)} \tag{27}$$

The solution of (16) is given by (28)

$$q(t)=\big(q_o-(1-\tau_1)t\big)e^{-rt} \tag{28}$$

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and the controller, i.e. the fraction of time invested into education is

$$u(t)=\frac{a_1(t)\left(q_o-(1-\tau_1)t\right)-\dfrac{c}{b(t)}(1-\tau_2)\left(r-a(t)\right)}{(1-\tau_2)(1-c)-(a_1-a_2)\left(q_o-(1-\tau_1)t\right)} \tag{29}$$

Thus, through the equation (29) we can efficiently control the pattern of human capital (i.e. knowledge) in order to maximize the financial profits of the individuals.

Moreover, according to the expression (20), the change in human capital over time is then obtained by the following linear ordinary linear equation

$$\dot{x}(t)=\left[b(t)u(t)-a(t)\right]x(t) \tag{30}$$

Obviously, the above results are apparently interesting and very helpful to practitioners, as well.

In the following section an interesting numerical application is analytically presented.

4 Using some empirical numerical results to obtain the optimal education – investment decision.

In general, the form of the optimal solution is the initial full time schooling, i.e. $u(t)=1$ (scaled down to part time if consumption requirements are sufficient high) followed by alternating periods of part – time schooling and zero schooling, with the last period prior to retirement one of zero schooling. However in this numerical application, we are mainly being interested about the period of part time schooling.

Analytically, the observed relations on income of age and education are used to develop the optimal allocation of effort between work (i.e. employment) and education (i.e. lifetime schooling). Moreover, it is underlying that our results are based on the expression (29) and its parameters relatively involved.

Furthermore, we would like to mention that the key factor for optimal allocation policy is the measure of the depreciation rate of education. This estimation is remarkably important; since it could answer several serious questions which are naturally derived, see [9]. To be honest, fairly little things are known about that rate. However, quite recently a simple methodology to the determination of the depreciation rate is proposed by [9] and it has been applying into real data sets of Great Britain, see [13]. In this numerical application we use the results of the empirical application mentioned above for the population of Great Britain (for more details, see [9]).

The findings in [9] suggest that the rate of depreciation is 11 – 17% per year. That quite high depreciation rates (compare them with the applications of [19]) emphasize more the importance of lifetime learning.

Before we go further it is important to determine the values of the variables which are taken into consideration in expression (29):

$a(t)$: The depreciation rate of education, $t\in[0,T]$. According to the results of [9] it takes values into the interval $[11\%,17\%]$, see expression (1).

$a_1(t)$: The increment of earnings through work, see expressions (2) and (21). Suppose it is stable, 3% increment for each year.

$a_2(t)$: The increment of earnings through scholarships, or participation into a research programme, see expressions (2) and (21). Suppose it is almost unchangeable, 0.5%, i.e. almost no serious increment at all.

T: The end of the time period, i.e. the year of retirement. In our application, it is 35 years of full time work.

r: The risk – free interest rate.

c: The proportion of cost for the relative education, see expressions (3) and $c(t)=cu(t)x(t)$. This proportion depends of the quality of education, i.e. the cost of attainting seminar, doing MSc courses or MBA is quite different. In our application, it is supposed that the three choices mentionned above of training are the only available. Thus, we consider the proportion of cost to be equal to 1/3.

τ_1: The percentage of the actual income, see expression (4). It is suppose to be stable and it takes the value of 10%.

τ_2: The percentage of direct cost of education, see also expression (4). It is also suppose to be stable and it takes the value of 8%. Obviously, it can be either smaller or greater of τ_1, as a mere conesquence of the government policy. In the particular case that $\tau_2>\tau_1$, the government provides extra tax motivation for lifelong learning. In our example the tax policy does not provide any extra motivation.

$b(t)$: Without further details, we suppose that the t – continuous parameter of the production function f is constant and equal to 1, see expression (19). Obviously, it can be any smooth real function which is really feasible or, in practice, it depends upon the empirical data of each special problem.

Finally, since we have assumed that at the beginning of the part – time period, i.e. at time $t = 0$, the proportion $u(t)$ is equal to 1, we can obtain

$$u(0) = \frac{a_1 q_o - c(1-\tau_2)(r - a(t))}{(1-\tau_2)(1-c) - (a_1 - a_2)(q_o)}$$

or equivalently

$$q_o = \frac{(1-\tau_2)\left[1 - c(1 + a(t) - r)\right]}{2a_1 - a_2}$$

Thus, we conclude the discussion above by presenting the following collective Table.

Table 1

Application Parameters

a1 = 3%	T = 35	τ1 = 10%	c = 1/3	r = 4%
a2 = 0.5%	t < T	τ2 = 8%	b(t) = 1	a in [11%,17%]

Now, in the figure 1, the values of our controller $u(t)$ are observed for the different depreciation rates. It is clear that the depreciation rates have a large impact on the fraction of time invested into education. The larger the depreciation rate is the more time should be spent in schooling. This result is apparently obvious.

Moreover, the following comment is easily derived; since someone has a depreciation rate of 11%, it needs to spend almost one third of his time in schooling in order to receive optimal income results.

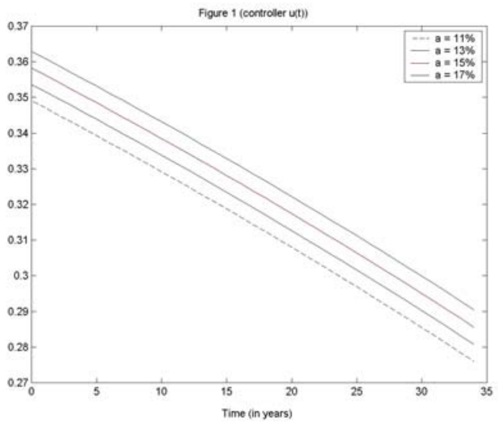

Figure 1: The fraction of time invested into education for different depreciation rates.

The next figure shows something really astonishing. Firstly, we have computed the interest rate of income return for various values. Then, by using an average deprecation rate of 13%, we observe that our optimal controller is become a strongly decreasing function for the different values of interest rate. Thus, the more someone earns the less time should be spent in schooling.

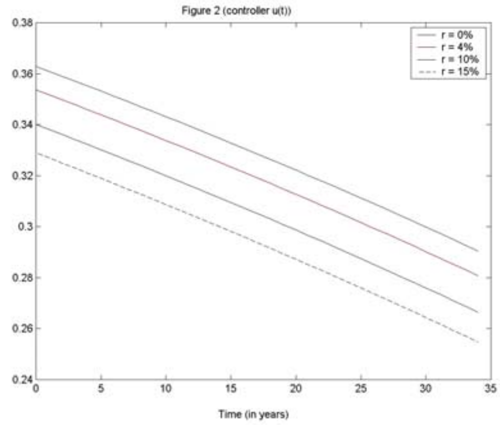

Figure 2: The fraction of time invested into education for different interest rate values.

Finally, it turns out that the increasing of the depreciation rate also increases the fraction of time invested into education (i.e. knowledge), see (29). The opposite direction is derived when the interest rate is increased.

5 Conclusions

The purpose of this paper is the introduction of an optimal control theory model to the education – investment decision strategy that maximizes the present value of future earnings for an individual. The formulation of this model is quite general including several inputs variables, assuming only the rate of schooling as the control variable.

Moreover, by using the Potryagin maximum principle and the relative Hamiltonian function, some very general results for the determination of the time path education – pattern, and the optimal lifetime policy are derived.

Further, more practical and straightforward results are obtained when a special case productivity function, the famous Cobb – Douglas, is introduced.

The results may be summarized as follows:

a) An analytic control function for the exact determination of the fraction of the time invested into education is derived. The formula, although complicated – since it considers several parameters, is very insightful and presents the efficient way to spend our time between job and schooling.

b) For lower (or higher) depreciation rates, the optimal pattern is a full time education, i.e. $u(t) = 1$, for the very first years, followed by a period of education maintenance via higher (lower) part time education (for instance, in our application is almost one third of the time), and finished by zero education for retired persons. The results are analogous to [19].

c) Moreover, it appears that the interest rate of return decreases the optimal pattern and obviously it follows an opposite monotonicity with the depreciation rate.

It should also be emphasized that the paper uses a quite recently method, see [9], to measure the rate of depreciation, and the application bases on real data, as well.

Finally, we should stress two possible directions for further research. The first one considers the same problem with a generalization as regard the number of individuals, the inputs parameters and consequently expansion of the number of the control parameters. The second direction considers the introduction of stochastic point of view for the Cobb – Douglas production function, the income function, the taxation etc. This approach transforms the deterministic optimal allocation problem into a stochastic optimal control framework. For those two projects, there is some research work in progress.

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