MODELING AND SIMULATION OF FRACTIONAL SYSTEMS USING ORTHOGONAL FUNCTIONS

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Abstract

In this paper, a new method for modeling linear fractional order systems is proposed. the developments are carried out in the Laplace domain using the transfer function representation. In fact, the main idea here is to approximate a fractional transfer function by an integer one with appropriate order. Obviously, this result would be useful thereafter for simulation and analysis of such dynamic systems. This technique uses the generalized operational matrices of integration and differentiation of orthogonal basis which computes rigourously the analytical Riemann-Liouville integral and derivative. Generally, the main advantage of using this mathematical tool is that it allows to transform the analytical fractional differential calculus into an algebraic one relatively easier to solve. Particularly in this paper, the well known Block Pulse orthogonal functions are used. Their remarkable upper triangular operational matrices of both fractional integration and differentiation, unlike those of orthogonal polynomials, reduces considerably the calculus with no significant loss of precision. Indeed, the wanted transfer function parameters are found algebraically simply with a least square algorithm formulation. The effectiveness of this method is shown through a set of numerical examples. Comparison study with some rigorous works in literature, namely the recursive poles and zeros approximation technique, is also presented.

Keywords: Modeling, Fractional systems, Linear systems, Orthogonal functions, Generalized matrices of integration and differentiation.

Presenting Author's Biography

Mohamed karim Bouafoura was born in Tunis in 1980. He received the diploma in Electrical Engineering from the National School of Engineers of Sfax (ENIS) in 2003 and the master degree in Automatic Control and Signal Processing from the National School of Engineers of Tunis (ENIT) in 2005. He is currently working toward Ph.D. degree in Electrical Engineering at National School of Engineers of Tunis. His research interests include robust control, fractional systems and controllers, orthogonal functions and generalized operational matrices.

1 Introduction

It was shown in the literature that several real physical systems, considered in various fields, for example, in geology [1], in electromagnetism [2] or in thermal transfer [3], are better described by fractional differential equations where one calls for the concept of non-integer derivation and/or integration, this operator allows in particular, to highlight the long memory or the hereditary behavior of these systems. In this context, many works [4, 5, 6] were dedicated to modeling, parametrical identification and simulation of such systems.

Often, simulation, especially in the time domain, is computed with some difficulties of implementation. In the last few years, several numerical algorithms were proposed in order to simulate the fractional systems, one may distinguish two types of methods: the direct method and the indirect one [7].

The idea of the direct method is based on the approximation of the non-integer operator of derivation by an integer model. Various techniques are of use, namely, the techniques based on the approximation of the non-integer operator by an integer discrete-time model which is obtained by a continuous-discrete type transformation. The transformations mostly in use are the Euler, Tustin, Simpson and Al-alaoui [8, 9] techniques. An other technique is based on the approximation of the fractional operator bounded in a frequency interval by an integer model in the continuous time domain obtained by recursive distribution of zeros and poles [10, 11].

However, the simulation of continuous fractional systems through an indirect method is obtained by the use of an operator or a specific representation of the considered system. Afterwards the operator of non-integer derivation or integration is replaced by its rational approximation. In literature, among these methods one can quote the approach of the diffusive representation [12], the method based on the compagnon form in the state-space representation [13] and the methods which use the development of the transfer function of the considered system in modal form, Maclaurin series or in continuous fraction expansion [5].

The concern of control engineers and researchers is to bring back the fractional model to an integer one, sometimes even with significant order. The computed integer model is aimed to follow as well as possible the wanted fractional dynamic and thus allowing its simulation and analysis. In this paper, a new method is proposed. It allows the approximation of fractional system (the order of derivation considered is real) by a continuous linear system with the help of orthogonal functions. This technique primarily calls for the orthogonal Block Pulse basis and the related operational matrices of integration and derivation generalized to fit the Riemann-Liouville integral and derivative formulas.

This paper is organized as follows: In the section 2, fractional systems and the mathematical tools necessary for their representation are introduced. Section 3 is dedicated to the presentation of the generalized operational matrices. The method suggested for modeling of the fractional systems is developed in section 4, and finally the last section is reserved for the exposure of simulation examples showing the availability of this method.

2 Fractional linear models

The fractional integral (Riemann-Liouville integral) of a function $f(t)$ is defined by [14]:

$$\left(I_{t_0}^{\alpha} f\right)(t) \triangleq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \qquad (1)$$

where $t > t_0$ and α is a real positive order, and $\Gamma(\alpha)$ is the Euler Gamma function:

$$\Gamma(\alpha) = \int_{0}^{\infty} e^{-x} x^{\alpha-1} dx \qquad (2)$$

Several definitions for the fractional derivative already exist, one might mention, the Riemann-Liouville [15], the Grünwald-Letnikov [14] and Caputo definitions [16]. In the developments below, the fractional derivative definition considered is the Riemann-Liouville formula given by [15]:

$$D_{t_0}^{\alpha} f(t) \triangleq \left(\frac{d}{dt}\right)^{m} \left(I_{t_0}^{m-\alpha} f(t)\right), t > t_0 \qquad (3)$$

where m is the smallest integer greater or equal to α. The Laplace transform of the integral of $f(t)$ is given by:

$$\mathscr{L}\left\{I_0^{\alpha} f(t)\right\} = \frac{1}{s^{\alpha}} F(s) \qquad (4)$$

In addition, when

$$f(0) = D_0^1 f(0) = \ldots = D_0^{\infty} f(0) = 0,$$

the Laplace transform of the generalized derivative may be expressed as follows [17]:

$$\mathscr{L}\left\{D_0^{\alpha} f(t)\right\} = s^{\alpha} F(s) \qquad (5)$$

Starting from the defintion (4), a Single Input Single Output (SISO) linear time invariant system, relaxed at $t = 0$, may be described by the following differential equation:

$$\sum_{i=0}^{n} a_i D_0^{\alpha_i} y(t) = \sum_{j=0}^{m} b_j D_0^{\beta_j} u(t) \tag{6}$$

As a consequence system (6) could be represented in the Laplace domain by the transfer function:

$$\begin{aligned} H(s) &= \frac{Y(s)}{U(s)} \\ &= \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \ldots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \ldots + a_0 s^{\alpha_0}} \end{aligned} \tag{7}$$

where α_i and β_i are arbitrary real positive orders (a commensurate order is not needed here), $u(t)$ and $y(t)$ are respectively the input and the output of the system.

3 Orthogonal functions and generalized operational matrices

The application of orthogonal functions for the modeling and the control of linear systems was initially approached by Chen and Hsaio [18] with the use of Walsh functions as a tool of approximation. Works using the block pulse functions were also elaborated [19], and later, better results were obtained by using a set of orthogonal polynomials generated from the family of Jacobi polynomials. Among them the most used ones are those of Laguerre, Legendre, Chebychev and Hermite [20, 21]. Other techniques calling for the orthogonal functions were also extended to the analysis of nonlinear systems [22].

In the framework of analysis of fractional systems, several works have been interested in the generalization of the algebraic derivation and integration operator of some orthogonal basis previously mentioned [23, 24, 25]. In our work, the choice was made on the use of the operational matrices relating to the block pulse functions. This is due to the best quality of approximation of the integrals and derivatives in the non-integer case [25] and also the remarkable triangular forms of their generalized operational matrices.

The block pulse functions constitute a complete set of orthogonal functions and can be defined over the time interval $[0, T]$ by:

$$\psi_i(t) = \begin{cases} 1 & \frac{i-1}{M} T \le t \le \frac{i}{M} T \\ 0 & elsewhere \end{cases} \tag{8}$$

where M is the number of elementary functions to use. Any function $f(t)$ integrable on $[0, T]$, can be expanded on the block pulse basis as follows:

$$f(t) \cong \mathrm{f}^T \psi_{(M)}(t) = \sum_{i=1}^{M} f_i \psi_i(t) \tag{9}$$

where

$$\mathrm{f}^T = [f_1 f_2 \ldots f_M]$$

$$\psi_{(M)}^T(t) = [\psi_1(t)\, \psi_2(t) \ldots \psi_M(t)]$$

$$\begin{aligned} f_i &= \frac{M}{T} \int_0^T f(t)\psi_i(t)dt \\ &= \frac{M}{T} \int_{[(i-1)/M]T}^{(i/M)T} f(t)\psi_i(t)dt \end{aligned} \tag{10}$$

the block pulse operational matrix of integration [19] is given by:

$$\int_0^t \psi_{(M)}(\tau)\, d\tau \cong F_1 \psi_{(M)}(t) \tag{11}$$

where

$$F_1 = \frac{T}{2M} \begin{pmatrix} 1 & 2 & \ldots & 2 \\ 0 & 1 & \ldots & 2 \\ \vdots & \ddots & \ddots & 2 \\ 0 & \ldots & 0 & 1 \end{pmatrix} \tag{12}$$

moreover,

$$\underbrace{\int_0^t \cdots \int_0^t}_{N\ times} \psi_{(M)}(\tau)\, d\tau \cong F_1^N \psi_{(M)}(t), \tag{13}$$

where N is a positive integer. The generalization of this result to a real positive order according to the Riemann-Liouville definition had been already proposed by Wang [25]:

$$\begin{aligned} (I_0^\alpha \psi_{(M)})(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \psi_{(M)}(\tau) d\tau \\ &\cong F_\alpha \psi_{(M)}(t) \end{aligned} \tag{14}$$

with

$$F_\alpha = \left(\frac{T}{M}\right)^\alpha \frac{1}{\Gamma(\alpha+2)} \begin{pmatrix} f_1 & f_2 & f_3 & \ldots & f_M \\ 0 & f_1 & f_2 & \ldots & f_{M-1} \\ \vdots & \ddots & f_1 & \ldots & f_{M-2} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 & f_1 \end{pmatrix} \tag{15}$$

where

$$f_1 = 1, f_p = p^{\alpha+1} - 2(p-1)^{\alpha+1} + (p-2)^{\alpha+1} \quad (16)$$

As a consequence, the analytical expression of generalized integral given by (1), may be found algebraically, that is to say:

$$(I_0^\alpha f)(t) \cong \mathrm{f}^{\mathrm{T}} F_\alpha \psi_{(M)}(t) \quad (17)$$

then, the generalized operational matrix of derivation can be also computed simply by inverting the matrix of integration [25]:

$$\begin{aligned} G_\alpha &= F_\alpha^{-1} \\ &= \left(\tfrac{M}{T}\right)^\alpha \Gamma(\alpha+2) \\ &\quad \begin{pmatrix} g_1 & g_2 & g_3 & \cdots & g_M \\ 0 & g_1 & \cdots & \cdots & g_{M-1} \\ \vdots & \ddots & g_1 & \cdots & g_{M-2} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & g_1 \end{pmatrix} \end{aligned} \quad (18)$$

The coefficients g_i are easily found when solving the following M equations system:

$$F_\alpha . G_\alpha = I_{M\times M} \quad (19)$$

Similarly to the expression (17), the non-integer derivative of a function may be found algebraically as follows:

$$D_0^\alpha f(t) \cong \mathrm{f}^{\mathrm{T}} G_\alpha \psi_{(M)}(t) \quad (20)$$

4 Modeling fractional systems using orthogonal functions

We are interested in this paper in the modeling of fractional invariant systems described by the fractional transfer function (7) by an integer linear invariant system of appropriate order, that is to say:

$$\begin{aligned} H_v(s) &= \tfrac{Y_v(s)}{U_v(s)} \\ &= \tfrac{c_r s^r + c_{r-1} s^{r-1} + \ldots + c_0}{k_q s^q + k_{q-1} s^{q-1} + \ldots + k_0} \end{aligned} \quad (21)$$

where r and q are positive integers.

By applying results exhibited in (17) and (20) to the system input and output as follows:

$$D_0^\alpha y(t) = \mathscr{L}^{-1}(s^\alpha Y(s)) \cong Y_M^T G_\alpha \psi_{(M)}(t) \quad (22)$$

$$D_0^\alpha u(t) = \mathscr{L}^{-1}(s^\alpha U(s)) \cong U_M^T G_\alpha \psi_{(M)}(t) \quad (23)$$

equation (6) may be rewritten under the following form:

$$Y_M^T\, \mathcal{D}\, \psi_{(M)}(t) = U_M^T\, \mathcal{N}\, \psi_{(M)}(t) \quad (24)$$

where $\mathcal{D}$ and $\mathcal{N}$ are square matrices given by:

$$\mathcal{D} = (a_n G_{\alpha_n} + a_{n-1} G_{\alpha_{n-1}} + \ldots + a_0 G_{\alpha_0})$$

$$\mathcal{N} = (b_m G_{\beta_m} + b_{m-1} G_{\beta_{m-1}} + \ldots + b_0 G_{\beta_0})$$

which yields

$$Y_M^T = U_M^T\, \mathcal{N}\, \mathcal{D}^{-1} \quad (25)$$

Similarly to the developments above, let us apply an inverse Laplace transformation to equation (21), the expansion of the system input and output over the orthogonal basis as in (22) and (23) gives directly:

$$Y_{vM}^T = U_{vM}^T\, \mathcal{N}_v\, \mathcal{D}_v^{-1} \quad (26)$$

where $\mathcal{D}_v$ and $\mathcal{N}_v$ are square matrices given by:

$$\mathcal{N}_v = (c_r G_1^r + c_{r-1} G_1^{r-1} + \ldots + c_0 I_{M\times M})$$

$$\mathcal{D}_v = (k_q G_1^q + k_{q-1} G_1^{q-1} + \ldots + k_0 I_{M\times M})$$

systems H and H_v are equivalent if:

$$\mathcal{N}\mathcal{D}^{-1} = \mathcal{N}_v \mathcal{D}_v^{-1} \quad (27)$$

In equation (27), the left hand member being totally known, we may represent it by a matrix . The relation (27) is written now as follows:

$$P = \mathcal{N}_v \mathcal{D}_v^{-1} \quad (28)$$

Bearing in mind that our goal is to identify the parameters of the integer transfer function $H_v(s)$, while fixing k_0 at an arbitrary value, the system (28) can be easily transformed into the following form:

$$W.\theta = X \quad (29)$$

where θ is a vector containing all unknown variables:

$$\theta^T = [\ k_q \ \ \ldots \ \ k_1 \ \ c_r \ \ \ldots \ \ c_0\ ] \quad (30)$$

Noticing that all matrices used in the developed calculus when solving system (27) keep their upper triangular form even after some algebraic operations like inversion, addition or multiplication of these matrices. In fact they remain all similar to the generalized operational matrix of integration F_α, for example we have:

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1M} \\ 0 & p_{11} & p_{12} & \cdots & p_{1(M-1)} \\ \vdots & \ddots & p_{11} & \cdots & p_{1(M-2)} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & p_{11} \end{pmatrix} \quad (31)$$

where the matrix P appears in (28).

Then, for any arbitrary k_0 chosen, W and X in (29) are given respectively by equations (32) and (33).

The notation $(Matrix)_{ij}$ in (32) indicates the element referred by the row i and the column j of $Matrix$.

and

$$X = \begin{pmatrix} -k_0.p_{11} \\ -k_0.p_{12} \\ \vdots \\ \vdots \\ -k_0.p_{1M} \end{pmatrix} \quad (33)$$

the obtained system (29) can be solved by the least square algorithm, if $M \geq dim\theta$, we find directly:

$$\theta = \left(W^T W\right)^{-1} W^T X \quad (34)$$

A direct interesting application consists in finding the expression of the approximation of a pure non-integer differentiator, since this expression will be useful to draw up a direct simulation scheme [5] of the fractional systems where each non-integer differentiator in the considered fractional transfer function is replaced by its rational expression. The system to solve is a particular case of the general case defined by (29), thus the new system is given by:

$$W_d.\theta = X_d \quad (35)$$

Considering a real positive order of derivation δ, it would be sufficient to replace the matrix P by G_δ, then matrix (32) is now transformed as shown in (36).

and

$$X_d = \begin{pmatrix} -k_0.(G_\delta)_{11} \\ -k_0.(G_\delta)_{12} \\ \vdots \\ \vdots \\ -k_0.(G_\delta)_{1M} \end{pmatrix} \quad (37)$$

In the same way, it would be possible to determine the expression of the approximation of a γ order pure non-integer integrator. This is may be leaded simply by substituting G_δ with F_γ in (36) and (37) to find the parameters of the required transfer function.

5 Simulation examples

5.1 Method's validation

In order to validate our method, let us propose in this part a series of simulations of simple systems, using the orthogonal functions (OF), whose the checking of results consists in comparing the found step responses either with a well known response of an integer order system, a known time-domain analytical response or with the result of another method of fractional order systems simulations which its validity had been already proven in the literature.

As a first example, the linear time invariant system described by the following transfer function is considered:

$$H_1(s) = \frac{6s+10}{s^3+5s^2+3s+1}$$

the simulation of it step response using orthogonal functions (OF) is obtained through:

$$y_1(t) \cong Y_{1M}.\psi_{(M)}(t)$$

where

$$Y_{1M} = U_M\left(6G_1+10\right)\left(G_1^3+5G_1^2+3G_1+I_{M\times M}\right)^{-1}$$

and

$$U_M = \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \end{bmatrix}$$

with $M = 27$ and $T = 20$, the obtained result is exposed in the figure below:

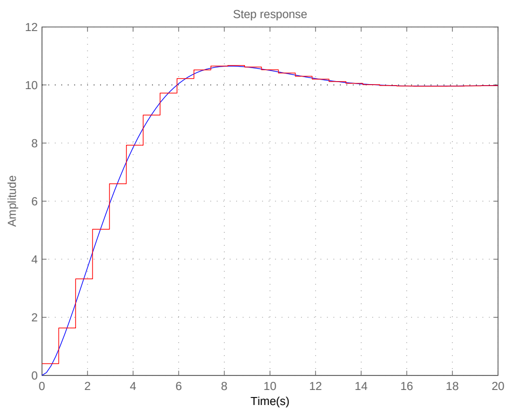

Fig. 1 Step response of the integer transfer function $H_1(s)$.

As second example, a non integer pure integrator is represented by its transfer function:

$$H_2(s) = \frac{1}{s^{0.25}}$$

$$W = \begin{pmatrix} (PG_1^q)_{11} & \left(PG_1^{q-1}\right)_{11} & \cdots & (PG_1)_{11} & -(G_1^r)_{11} & -\left(G_1^{r-1}\right)_{11} & \cdots & -(G_1)_{11} & -1 \\ (PG_1^q)_{12} & \left(PG_1^{q-1}\right)_{12} & \cdots & (PG_1)_{12} & -(G_1^r)_{12} & -\left(G_1^{r-1}\right)_{12} & \cdots & -(G_1)_{12} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (PG_1^q)_{1M} & \left(PG_1^{q-1}\right)_{1M} & \cdots & (PG_1)_{1M} & -(G_1^r)_{1M} & -\left(G_1^{r-1}\right)_{1M} & \cdots & -(G_1)_{1M} & 0 \end{pmatrix} \tag{32}$$

$$W_d = \begin{pmatrix} (G_\delta G_1^q)_{11} & \left(G_\delta G_1^{q-1}\right)_{11} & \cdots & (G_\delta G_1)_{11} & -(G_1^r)_{11} & -\left(G_1^{r-1}\right)_{11} & \cdots & -(G_1)_{11} & -1 \\ (G_\delta G_1^q)_{12} & \left(G_\delta G_1^{q-1}\right)_{12} & \cdots & (G_\delta G_1)_{12} & -(G_1^r)_{12} & -\left(G_1^{r-1}\right)_{12} & \cdots & -(G_1)_{12} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (G_\delta G_1^q)_{1M} & \left(G_\delta G_1^{q-1}\right)_{1M} & \cdots & (G_\delta G_1)_{1M} & -(G_1^r)_{1M} & -\left(G_1^{r-1}\right)_{1M} & \cdots & -(G_1)_{1M} & 0 \end{pmatrix} \tag{36}$$

knowing that the analytical impulse response of fractional pure integrator of a real order σ is defined by [15]:

$$h(t) = \frac{t^{\sigma-1}}{\Gamma(\sigma)} \tag{38}$$

the analytical step response of $H_2(s)$ is dictated by:

$$\begin{aligned} y_2(t) &= \mathscr{L}^{-1}\left(\tfrac{1}{p}H_2(s)\right) \\ &= \mathscr{L}^{-1}\left(\tfrac{1}{s^{1.25}}\right) = \tfrac{1}{\Gamma(1.25)}t^{0.25} \end{aligned}$$

the approximation through an expansion over the orthogonal basis yields:

$y_2(t) \cong Y_{2M}.\psi_{(M)}(t)$, with $Y_{2M} = U_M.F_{0.25}$

taking $M = 27$ and $T = 100$, the simulation result is shown in figure2.

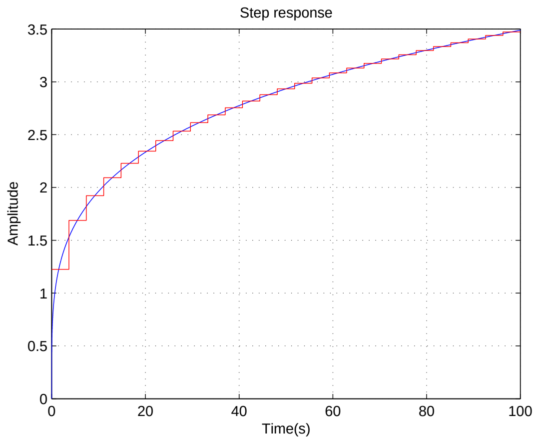

Fig. 2 Step response of the fractional pure integrator $H_2(s)$.

Let us consider the following non-integer transfer function as a last example in this subsection:

$$H_3(s) = \frac{1}{s^{0.5} + s^{0.2} + 1}$$

In this example, the simulations based on the use of orthogonal functions is compared to those computed with a direct simulation method, here the direct method of use is the one proposed by Oustaloup [11], where a frequency-band fractional differentiator is firstly considered and then approximated by recursive poles and zeros over an arbitrary bounded frequency interval. This technique will be adopted to compare it again with the rest of simulations which will be presented in the next subsection.

for $M = 27$ and $T = 10$, the result found is illustrated in figure3 below:

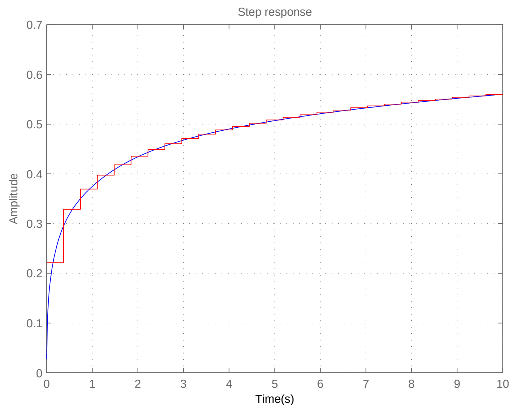

Fig. 3 Step response of the fractional transfer function $H_3(s)$.

5.2 Main method's results

The results presented in the previous subsection showed the effectiveness of the orthogonal functions and their generalized operational matrices in simulating the integer and fractional systems. This preliminary test brought a quite satisfactory result and then allows us to advance towards our main purpose which is modeling fractional systems by reduced integer order transfer functions.

By considering the non-integer pure differentiator given by its transfer function:

$$H_4\left(s\right) = s^{0.25}$$

and for $M = 16$ and $T = 100$, $r = q = 3$ and $k_0 = 1$, solving (35) gives:

$$H_{4v}\left(s\right) = \frac{590.3s^3 + 240.5s^2 + 21.37s + 0.2491}{605.5s^3 + 350.1s^2 + 45.53s + 1}$$

The step responses of both analytical expression and the integer transfer function approximation using Orthogonal function (OF) of the considered system are compared in figure 4.

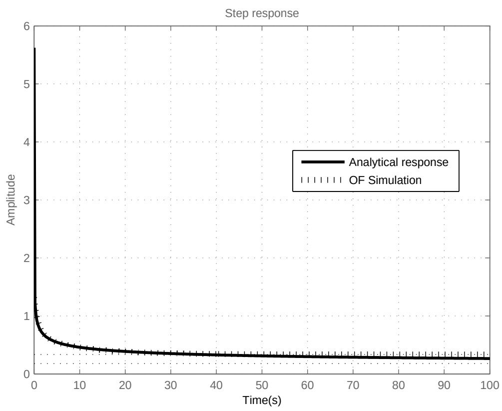

Fig. 4 step response of the fractional pure differentiator $H_4(s)$.

In the same way, a fractional pure integrator is considered as follows:

$$H_5\left(s\right) = \frac{1}{s^{0.6}}$$

with $M = 16$ and $T = 100$, $r = q = 3$ and $k_0 = 1$, solving (35) where $G_{0.6}$ is replaced by $F_{0.6}$, leads to the following transfer function:

$$H_{5v}\left(s\right) = \frac{6413s^3 + 6651s^2 + 1204s + 36.1}{8172s^3 + 2879s^2 + 191.4s + 1}$$

Simulation results of the pure fractional integrator are presented in figure 5.

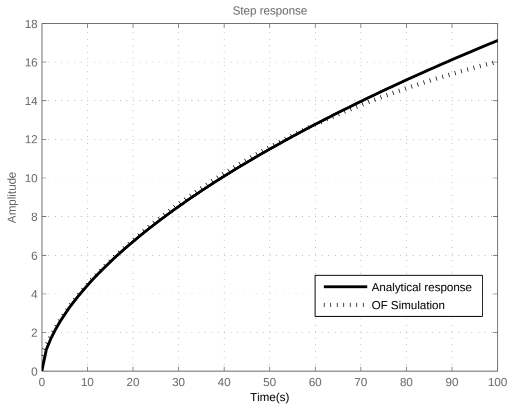

Fig. 5 Step response of the fractional pure integrator $H_5(s)$.

Let us reconsider now the system described by $H_3(s)$. taking $M = 16$ and $T = 20$, $r = q = 5$ and $k_0 = 1$, solving (29) provides the transfer function H_{3v}.

$$H_{3v}\left(s\right) = \frac{0.447s^5+4.56s^4+15.28s^3+19.23s^2+7.9s+0.65}{2.158s^5+19.06s^4+51.59s^3+51.39s^2+16.44s+1}$$

Comparison of the step responses of H_3 and H_{3v} is illustrated in figure 6.

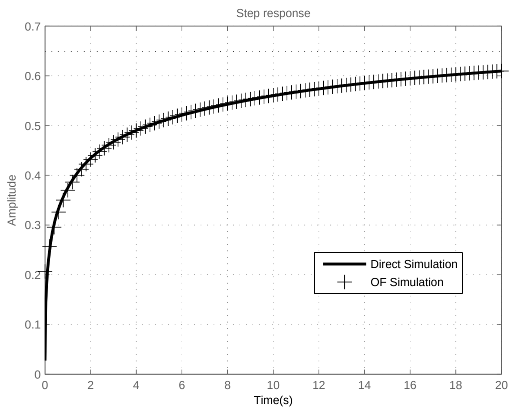

Fig. 6 Step response of the fractional transfer function $H_3(s)$.

In this last example, the non-integer transfer function below is considered:

$$H_6\left(s\right) = \frac{1 + s^{0.4}}{1 + s^{0.4} + s^{1.4}}$$

let $M = 16$ and $T = 20$, $r = q = 4$ and $k_0 = 1$, solving (29) allows us to find the parameters of the required integer transfer function:

$$H_{6v}(s) = \frac{-0.266s^4+0.552s^3+4.133s^2+4.491s+1.005}{0.397s^4+2.99s^3+6.075s^2+4.739s+1}$$

In figure 7, the simulation results of this last example are exposed.

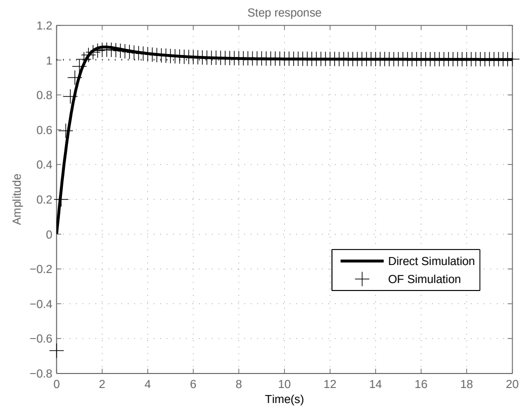

Fig. 7 Step response of the fractional order transfer function $H_6(s)$.

6 Conclusion

In this paper, a new method for modeling and simulation of fractional systems was introduced. It calls for the orthogonal functions as a tool of approximation. The use of the operational matrices of derivation and integration generalized to fit the Riemann-Liouville integral and derivative allows us to transform the non-integer differential equations into linear algebraic equations much more simple to solve. Indeed, it would be possible with the help of this technique to model a non-integer system by an integer transfer function having a reduced order over a chosen time interval. The results obtained with the block pulse basis, compared to other simulation method, namely the direct Oustaloup's method, were satisfactory. This is primarily due to the fidelity of the approximation of fractional Riemann-Liouville integral and derivative. This quite general method, applied by using other orthogonal basis, especially orthogonal polynomials is still effective but would not lead to valid numerical results only if the relative generalized algebraic operator (operational matrix of non-integer integration and derivative) ensures the best approximation of the generalized integral and derivative.

Also it should be mentioned that precautions must be taken when using operational matrices of derivation. Some risks of divergences are considered when using an order of derivation and/or a number of elementary functions rather high.

7 References

[1] B. Mandelbrot. *the fractal geometry of nature*. CA:Freeman, San Fransisco, 1982.

[2] N. Engheta. On fractional calculus and fractional multipoles in electromagnetism. *IEEE trans. Ant. Prop.*, 44, no.4:554–566, 1996.

[3] O. Cois. *systèmes linéaires non entiers et identification par modèle non entier: application en thermique*. PhD thesis, Université Bordeaux I, 2002.

[4] M. Mbarek, N. Benhadj Braiel, and M. Benrejeb. Sur la modélisation des systèmes d'ordre fractionnaires. In *3ème colloque mghrébin sur les modèles numériques de l'ingénieur*, Tunis, 1991.

[5] M. Aoun. *Systèmes linéaires non entiers et identification par bases orthogonales non entières*. PhD thesis, Université Bordeaux I, France, 2005.

[6] L. Lelay. *Identification fréquentielle et temporelle par modèles non entiers*. PhD thesis, Université Bordeaux I, France, 1998.

[7] M. Aoun, R. Malti, F. Levron, and A. Oustaloup. Numerical simulation of fractionl systems: An overview of existing methods and improvements. *International Journal of Nonlinear Dynamics and Chaos in Engineering systems, special issue on fractional derivatives and their applications*, 38:117–131, 2004.

[8] Y.Q. Chen and K.L. Moore. Discretization schemes of fractional order differentiators and integrators. *IEEE Trans. Circuits Syst. I*, 49(3):363–367, 2002.

[9] I. Podlubny, I. Petras, B.M. Vinagre, P. O'Leary, and L. Dorcak. Analogue realizations of fractional order controllers. *Nonlinear Dynamics*, 38, no.1-4:281–296, 2002.

[10] A. Oustaloup, F. Levron, and B. Mathieu. Frequency-based complex noninteger differentiator characterization and synthesis. *IEEE Trans. Circuits Syst. I*, 47:25–39, Janvier 2000.

[11] A. Oustaloup. *La dérivation non entière: théorie, synthèse et applications*. Paris, 1995.

[12] G. Montseny. Diffusive representation of pseudo-differential time-operators. In *fractional differential systems: Models, Methods ans Applications*, volume 83, pages 159–175, Paris, 1998.

[13] T. Poinot and J.C. Trigeassou. A method for modelling and simulation of fractional systems. *Signal processing*, 83:2319–2333, 2003.

[14] S.G. Samko, A.A. Kilbas, and O.I. Marichev. *fractional integrals ands derivatives: Theory and applications*. Gordon and Breach Science Publishers, Amesterdam, 1993.

[15] K.S. Miller and B. Ross. *An introduction to the fractional calculus and fractional differential equations*. A wiley-Interscience publication, San Fransisco, 1993.

[16] I. Podlubny. *Fractional differential equations*. San Diego, 1999.

[17] K.B. Oldham and J. Spanier. *the fractional calculus*. New York and London, 1974.

[18] C.F. Chen and C.H. Hsiao. Design of piecewise constant gains for optimal control via walsh functions. *IEEE Trans. Autom. Contr.*, 20, no.5:596–603, 1975.

[19] L.S. Shih, C.K. Yeung, and B.G. Mc Inis. Solution of state-space equations via block-pulse functions. *International Journal of Contr.*, 28:383–392, 1978.

[20] R.Y. Chang and M.L. Wang. Legendre polynomials approximation to dynamic linear state equations with initial or boundary value conditions. *International Journal of Contr.*, 43, no.1:215–232, 1984.

[21] P.N. Paraskevopoulos. Chebychev series approach to system identification, analysis and optimal control. *Journal of the Franklin Institute*, 316:135–157, 1983.

[22] N. Benhadj Braiek. *Application des fonctions de walsh et les fonctions modulatrices à la modélisation des systèmes continus non linéaires*. PhD thesis, Université des sciences et techniques de Lille, Flandres Artois, 1990.

[23] C.F. Chen, Y.T. Tsay, and T.T. Wu. Walsh operational matrices of fractional calculus and their applications to distributed systems. *Journal of the Franklin Institute*, 303, no.3:267–284, 1977.

[24] R.Y. Chang, K.C. Chen, and M.L. Wang. Modified laguerre operational matrices for fractional calculus and applications. *International Journal of Systems Science*, 16, no.9:1163–1172, 1985.

[25] C.H. Wang. On the generalization of block pulse operational matrices for fractional calculus and applications. *Journal of the Franklin Institute*, 315, no.2:91–102, 1983.

