MODELLING OF LEADER-FOLLOWER SPACECRAFT FORMATIONS IN 6DOF

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Abstract

This paper concerns modelling of relative translation and rotation of formation flying spacecraft in Earth orbits. A nonlinear 6 degree-of-freedom (DOF) mathematical model of relative translation and rotation in a leader-follower formation consisting of two spacecraft is derived, with a formulation similar to general Euler-Lagrange system. The model of relative translation is based on the laws of Newton, specifically Newton's law of gravitational attraction from which the two-body problem can be derived. The relative rotation is based on Euler's momentum equations and the attitude is represented by Euler parameters, or unit quaternions. Based on the models of relative translation and rotation, the total 6DOF model is derived. The model is referenced both in a leader orbit coordinate system and in an Earth-fixed inertial coordinate system. The leader orbit coordinate system is located in the centre of mass of the leader spacecraft, whereas the Earth-fixed inertial coordinate system is located in the centre of the Earth. The rotation matrices between the different coordinate systems are presented. The Earth-fixed model is based on work in the field of marine control systems. The system properties for both the models are presented, where properties like symmetry, skew-symmetry and positive definiteness of matrices can be incorporated into the stability analysis when designing control systems. These system properties represent physical properties of the system. Furthermore, simulations of the model referred to leader orbit coordinate system are presented where the impact of perturbing forces and torques are illustrated. The perturbing forces and torques considered in the simulation are due to atmospheric drag and the oblateness of the Earth.

Keywords: Leader-follower spacecraft formation, mathematical modelling, relative translation and rotation, Earth orbit, 6 degree-of-freedom

Presenting Author's Biography

Lisa Maria Svendsen is from Norway. She has a bachelor degree in Electrical Engineering with a specialization in telecommunication at Sør-Trøndelag University College in Trondheim, Norway. Furthermore, she has a master degree in Space Technology at Narvik University College in Narvik, Norway. After graduating she started working at Triad AS, Norway. Triad AS works with scientific research in various fields of physical science where the main field is the characterization of general targets and geophysical phenomena by electromagnetic and acoustic waves.

1 Introduction

Spacecraft formations has been a subject of many research studies in recent years, and can be defined as a set of more than one spacecraft in close flight whose dynamic states are coupled through a common control law for translational and/or rotational motion (*cf.* [1, 2]). Previous definitions of formation flying have not clearly differentiated it from constellations, so the definition of formation flying can be extended to include that at least one of the spacecraft must track a desired state relative to another spacecraft, and the tracking control law must at the minimum depend upon the state of this other spacecraft [3]. The control law that satisfies the latter condition is called a formation tracking control law [1]. Spacecraft flying in formation are revolutionizing our way of performing space-based operations, and bring out several advantages in space mission accomplishment, as well as new opportunities and applications for such missions. The concept makes the way for new and better applications in space industry, such as improved monitoring of the Earth and its surrounding atmosphere, geodesy, deep-space imaging and exploration and even in-orbit spacecraft servicing and maintenance.

However, the advantages of using spacecraft formations come at a cost of increased complexity and technological challenges. Formation flying introduces a control problem with strict and time-varying boundaries on spacecraft reference trajectories, and requires detailed knowledge and tight control of relative distances and velocities for participating spacecraft. One of the main challenges is the need of a more advanced control algorithms for controlling the formation and to avoid collisions, leading to a requirement of dynamically synchronized control (*cf.* [4]) of relative position and attitude between the members of the formation.

1.1 Previous work

Both linear and nonlinear models of formation dynamic have been developed for formation maintenance. Most previous studies on spacecraft formation problems have been based on decoupled translational and rotational models where various control methods are utilized, with main focus on 3DOF translational motion including disturbances. The Clohessy-Wiltshire equations have been widely used to model the leader-follower relative position dynamics of formation flying systems and are a linear approximation of the nonlinear dynamics originating from the two-body problem based on the laws of Newton and Kepler. A model based on the Clohessy-Wiltshire equations assumes spacecraft with small relative distance in a circular orbit with no orbital perturbations and includes no higher order non-linear terms [5]. Another, yet similar, modelling approach, known as the Lawden equations [6] or Tschauner-Hempel equations [7], is an extension to the elliptical Keplerian orbits which also do not consider orbital perturbations. Originally both models were presented for solutions to the problem of orbital rendezvous [8], but have later been used to describe the control problem of spacecraft formation flying. A different alternative modelling approach is using orbit element differences ([9, 10]). Orbit element differences originate from Lagrange and Gauss equations where the orbit of each spacecraft is described by orbit parameters. The spacecraft will deviate from their desired orbit due to orbital perturbations, hence the name of the modelling approach. The advantage with the orbit element differences model is that the spacecraft are controlled relative to their natural orbits but the requirement of orbit determination and global positioning can be very computationally demanding.

Research of rotational spacecraft control has also been a topic the past decade, however, natural extension to dynamics and control for 6DOF coupled translation and rotation has received scant attention in current literature, except some recent studies, (*cf.* [11, 12, 13]). To provide optimal and robust control for spacecraft flying in formation a detailed mathematical model is very important. A total model derived with 6DOF will be more accurate than separate/decoupled 3DOF models, since translation and rotation in spacecraft formations are coupled through actuators and external disturbances. A model based on 3DOF translation of motion does not consider the influence of angular motion of the spacecraft body relative to the Earth and the other spacecraft in formation. The model should also include terms of perturbations due to external disturbances caused by atmospheric drag, solar drag and variations in the gravity field of the Earth. Including such terms of perturbations will provide a more accurate model and thereby make the system more fuel-efficient, since the need for adjustment and corrections of the spacecraft‘s deviation from its orbit will be less.

2 Reference frame

To describe the translation and attitude motion dynamics of a spacecraft, different reference frames need to be presented.

2.1 Earth-Centred Inertial frame

The Earth-centred inertial (ECI) frame, denoted $\mathcal{F}_i$, has its origin in the centre of the Earth. The z_i axis is directed along the axis of rotation of the Earth toward the celestial North Pole. The x_i axis is pointing in the direction of the vernal equinox, Υ, which is the vector pointing from the centre of the sun toward the centre of the Earth during the vernal equinox. Finally the y_i axis completes a right handed orthogonal frame. This frame is non-rotating and assumed fixed in space, *i.e.* it is an inertial frame in which Newton‘s laws of motion apply. See Figure 1 for a graphical description of the frame.

2.2 Leader orbit reference frame

This reference frame, denoted $\mathcal{F}_l$, has its origin in the centre of mass of the leader spacecraft. The basis vectors are denoted $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_h$, where $\mathbf{e}_r$ is parallel to the vector $\mathbf{r}_l$ pointing from the centre of the earth to the centre of mass of the leader spacecraft. The $\mathbf{e}_h$ axis is parallel to the orbit momentum vector pointing in the orbit normal direction, and the $\mathbf{e}_\theta$ axis completes a right-handed orthogonal frame, as shown in Figure 1.

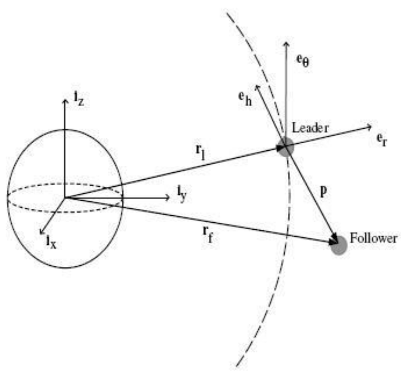

Fig. 1 Reference coordinate frames

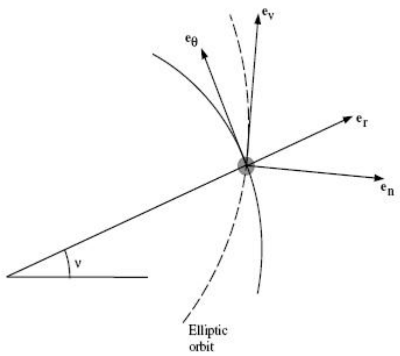

Fig. 2 Auxiliary vectors for the leader orbit reference frame

This frame is often referred to as a local-vertical-local-horizontal (LVLH) reference frame because it tracks the local horizontal plane. This plane is spanned by $\mathbf{e}_\theta$ and $\mathbf{e}_h$, and the local vertical direction, *i.e.* $\mathbf{e}_r$ where these basis vectors are defined as [14]

$$\mathbf{e}_r = \frac{\mathbf{r}_l}{r_l} \qquad \mathbf{e}_\theta = \mathbf{e}_h \times \mathbf{e}_r \qquad \mathbf{e}_h = \frac{\mathbf{h}}{h}$$

where $\mathbf{h} = \mathbf{r}_l \times \dot{\mathbf{r}}_l = \mathbf{r}_l \times \mathbf{v}_l$ is the angular momentum vector of the orbit and $h = |\mathbf{h}|$. The average angular velocity of the reference frame is $\omega_{il} = \dot{\nu}\mathbf{e}_h$ where ν is the true anomaly of the orbit of the leader spacecraft. Assuming a circular orbit results in $\mathbf{e}_\theta$ parallel to the velocity vector of the spacecraft, and the $\mathcal{F}_l$ frame rotates relative to the $\mathcal{F}_i$ frame with an angular velocity of approximately

$$\omega_o \approx \sqrt{\frac{\mu}{r_l^3}}$$

where μ denotes the geocentric gravitational constant of the Earth and r_l is the distance between the frame origin and the centre of the Earth.

In addition, two auxiliary vectors can be defined, $\mathbf{e}_v$ and $\mathbf{e}_n$, where $\mathbf{e}_v$ is pointing in the direction of the velocity vector of the spacecraft and $\mathbf{e}_n$ completes a orthogonal system with $\mathbf{e}_v$ and $\mathbf{e}_h$, which gives $\mathbf{e}_n = \mathbf{e}_v \times \mathbf{e}_h$, see Figure 2. If the leader spacecraft orbit is assumed to be circular then $\mathbf{e}_v = \mathbf{e}_\theta$ and $\mathbf{e}_n = \mathbf{e}_r$.

2.3 Follower orbit reference frame

The follower orbit reference frame, denoted $\mathcal{F}_f$ has its origin in the centre of mass of the follower spacecraft. The vector from the centre of the Earth to the centre of the follower orbit frame is denoted $\mathbf{r}_f$. The position of the follower spacecraft relative to the leader spacecraft can be given by $\mathbf{p} = \mathbf{r}_f - \mathbf{r}_l$ where $\mathbf{p} = [x \quad y \quad z]^T$. The relative orbit position vector can be expressed in the $\mathcal{F}_l$ frame as

$$\mathbf{p} = \mathbf{r}_f - \mathbf{r}_l = x\mathbf{e}_r + y\mathbf{e}_\theta + z\mathbf{e}_h \,.$$

For a graphical description of the frame, see Figure 1.

2.4 Body-fixed reference frame

The body-fixed reference frame has the origin located in the centre of mass of the spacecraft and is denoted $\mathcal{F}_{lb}$ and $\mathcal{F}_{fb}$ for leader spacecraft and follower spacecraft, respectively. The basis vectors are fixed in the spacecraft and coincide with its principal axes of inertia. The axes are denoted x_{lb}, y_{lb} and z_{lb} for the leader-spacecraft, and x_{fb}, y_{fb} and z_{fb} for the follower spacecraft. Rotation about these axes are defined as roll, pitch and yaw, respectively [15]. The spacecraft‘s attitude with respect to any reference frame can be defined by a direction cosine matrix, by its quaternion vector $\mathbf{q}$ or by the Euler angles [16].

2.5 Coordinate frame transformation

The orientation of the spacecraft can be presented in different frames, and in the following we define the transformations between the different frames.

2.5.1 Rotation from ECI to the leader orbit frame

The rotation from the Earth-centred inertial frame $\mathcal{F}_i$ to the leader orbit frame $\mathcal{F}_l$ can be described by three consecutive rotations, where the total rotation matrix can be written as [16]

$$\mathbf{R}_i^l = \mathbf{R}_z(\omega + \nu)\mathbf{R}_x(i)\mathbf{R}_z(\Omega)$$

where ν is the true anomaly defined as the angle between the major axis pointing to the perigee and the radius vector from the prime focus F (in this case: the Earth) to the moving body, in this case the leader spacecraft [16]. The term ω is the argument of perigee of the leader orbit, $\omega + \nu$ is the argument of the location of the leader spacecraft from the ascending node, i is the inclination of the leader orbit and Ω is the right ascension of the ascending node of the leader orbit. The inverse rotation $\mathbf{R}_l^i = \left(\mathbf{R}_i^l\right)^{-1} = \left(\mathbf{R}_i^l\right)^T$ is given by

$$\mathbf{R}_l^i = \mathbf{R}_z^T(\Omega)\mathbf{R}_x^T(i)\mathbf{R}_z^T(\omega + \nu)\,.$$

2.5.2 Orbit frame transformation

The orbit frame transformation describes the transformation between the orbit plane acceleration vector

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components. The acceleration of the leader spacecraft can be written as

$$\begin{aligned}\mathbf{a} =& a_r\mathbf{e}_r + a_\theta\mathbf{e}_\theta + a_h\mathbf{e}_h \\ =& a_n\mathbf{e}_n + a_v\mathbf{e}_v + a_h\mathbf{e}_h\end{aligned} \quad (1)$$

using both the original and auxiliary vectors described in Section 2.2. Furthermore, the spacecraft velocity vector can be expressed as [14]

$$\mathbf{v}_s = \dot{\mathbf{r}}\frac{\mu}{h}\left((e\sin\nu)\mathbf{e}_r + (\frac{p}{r})\mathbf{e}_\theta\right)$$

where μ is the geocentric gravitational constant of the Earth, h is the magnitude of the angular momentum, e is the eccentricity and $p = \frac{h^2}{\mu}$ is the semi-latus rectum of the spacecraft orbit. As described in Section 2.2 the vector $\mathbf{e}_v$ is pointing along the velocity vector $\mathbf{v}_s$, and this relationship can be expressed as [17]

$$\mathbf{e}_v = \frac{\mathbf{v}_s}{|\mathbf{v}_s|} = \frac{h}{pv}\left(e\sin\nu\mathbf{e}_r + \frac{p}{r}\mathbf{e}_\theta\right) \quad (2)$$

The vector $\mathbf{e}_n$ is defined normal to $\mathbf{e}_v$ and $\mathbf{e}_h$, and hence

$$\mathbf{e}_n = \mathbf{e}_v \times \mathbf{e}_h = \frac{h}{p\nu}\left(\frac{p}{r}\mathbf{e}_r - e\sin\nu\mathbf{e}_\theta\right) \quad (3)$$

The transformation between the orbit plane acceleration vector components can now be derived. Insertion of the equations (2)-(3) into (1) gives

$$\begin{aligned}a_r\mathbf{e}_r + a_\theta\mathbf{e}_\theta + a_h\mathbf{e}_h =& \left(a_n\frac{h}{p\nu}\frac{p}{r} + a_v\frac{h}{p\nu}e\sin\nu\right)\mathbf{e}_r \\ &+ \left(a_v\frac{h}{p\nu}\frac{p}{r} - a_n\frac{h}{p\nu}e\sin\nu\right)\mathbf{e}_\theta + (a_h)\,\mathbf{e}_h\,.\end{aligned}$$

By comparing both sides of the sign of equality the relationship can be written as

$$\begin{bmatrix}a_r\\a_\theta\\a_h\end{bmatrix} = \frac{h}{p\nu}\begin{bmatrix}\frac{p}{r} & e\sin\nu & 0\\ -e\sin\nu & \frac{p}{r} & 0\\ 0 & 0 & \frac{pv}{h}\end{bmatrix}\begin{bmatrix}a_n\\a_v\\a_h\end{bmatrix},$$

where the transformation is given by the rotation matrix $\mathbf{C}_a^l$ written as

$$\mathbf{C}_a^l = \frac{h}{p\nu}\begin{bmatrix}\frac{p}{r} & e\sin\nu & 0\\ -e\sin\nu & \frac{p}{r} & 0\\ 0 & 0 & \frac{pv}{h}\end{bmatrix}.$$

Note that $\mathbf{C}_a^l$ is not a proper rotation matrix since $\det\mathbf{C}_a^l = 1 + e^2 + 2e\cos\nu \neq 1$. The inverse transformation is given by

$$\begin{bmatrix}a_n\\a_v\\a_h\end{bmatrix} = \frac{h}{p\nu}\begin{bmatrix}\frac{p}{r} & -e\sin\nu & 0\\ e\sin\nu & \frac{p}{r} & 0\\ 0 & 0 & \frac{pv}{h}\end{bmatrix}\begin{bmatrix}a_r\\a_\theta\\a_h\end{bmatrix}.$$

2.5.3 Body frame rotation

Body frame rotation describes rotation from an orbit frame, denoted $\mathcal{F}_o$, to a body frame, $\mathcal{F}_b$. The rotation expressed in terms of the Euler parameters is

$$\mathbf{R}_o^b = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3] = \mathbf{I}+2\eta\mathbf{S}(\epsilon)+2\mathbf{S}^2(\epsilon)$$

where the elements $\mathbf{c}_i$ are the directional cosine vectors. The column vectors are cosines of the angle between the two frames, hence the name of the elements $\mathbf{c}_i$. Furthermore, the inverse rotation is given by

$$\mathbf{R}_b^o = (\mathbf{R}_o^b)^{-1} = (\mathbf{R}_o^b)^T = \mathbf{R}_e^T(\eta, \epsilon)\,.$$

3 Modelling

In this section a mathematical model of the relative translation and rotation between the leader and follower spacecraft is derived and presented. The model is referenced both in the leader orbit frame $\mathcal{F}_l$ and the Earth inertial reference frame $\mathcal{F}_i$. When the context is sufficiently explicit, we may omit to write arguments of functions, vectors or matrices.

3.1 Kinematics

The kinematic differential equations can be found from the Euler parameters given by the scalar η and the vector ϵ defined by [15]

$$\dot{\eta}=-\frac{1}{2}\epsilon^T\omega_{ob}^b \qquad \dot{\epsilon}=\frac{1}{2}\left[\eta\mathbf{I} + \mathbf{S}(\epsilon)\right]\omega_{ob}^b$$

and can be written in terms of the unit quaternion as

$$\dot{\mathbf{q}} = \begin{bmatrix}\dot{\eta}\\ \dot{\epsilon}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-\epsilon^T\\ \eta\mathbf{I} + \mathbf{S}(\epsilon)\end{bmatrix}\omega_{ob}^b = \mathbf{T}(\mathbf{q})\omega_{ob}^b \quad (4)$$

where $\omega_{ob}^b = \mathbf{T}^T(\mathbf{q})\dot{\mathbf{q}}$ is the angular velocity vector in body frame relative to the orbit reference frame represented in body frame and ω_{bo}^o denotes the angular velocity vector in orbit frame relative body frame decomposed in orbit frame. A useful relation can be stated as [18]

$$\dot{\mathbf{q}} = \mathbf{T}(\mathbf{q})\omega_{ob}^b = \frac{1}{2}\mathbf{A}(\omega_{ob}^b)\mathbf{q}$$

where

$$\mathbf{A}(\omega_{ob}^b) = \begin{bmatrix}0 & -\left(\omega_{ob}^b\right)^T\\ \omega_{ob}^b & -\mathbf{S}(\omega_{ob}^b)\end{bmatrix} \in SS(4) \quad (5)$$

i.e. is skew-symmetrical. The time derivative of the angular velocity can be written as $\dot{\omega}_{ob}^b = \mathbf{T}^T(\mathbf{q})\ddot{\mathbf{q}}$ where one of the properties of $\mathbf{T}(\mathbf{q})$ is defined as [19]

$$\frac{d}{dt}\left(\mathbf{T}^T(\mathbf{q})\dot{\mathbf{q}}\right) = \mathbf{T}^T(\mathbf{q})\ddot{\mathbf{q}} \quad (6)$$

since $\dot{\mathbf{T}}^T(\mathbf{q})\dot{\mathbf{q}} = \mathbf{T}^T(\dot{\mathbf{q}})\dot{\mathbf{q}} = \mathbf{0}$.

3.2 The N-body problem

Most analysis of celestial and spacecraft orbit dynamics are based on Newton‘s laws [16], and Newton‘s law of gravitational attraction is used for close formation flying consisting of N bodies. The sum of forces acting on the i‘th body is [16]

$$\mathbf{F}_i = \mathbf{G}\sum_{j=1}^{n} \frac{m_i m_j}{r_{ij}^3}(\mathbf{r}_j - \mathbf{r}_i), \quad i \neq j$$

where m is the mass of a body, G is the universal constant of gravitation is given by $G = 6.669 \cdot 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$ and $(\mathbf{r}_j - \mathbf{r}_i)$ is the vector from body i to body j. The distance between any two particles with mass m_i and m_j is written as

$$r_{ij} = |\mathbf{r}_j - \mathbf{r}_i| \ .$$

Application of Newton‘s second law of motion gives N vector differential equations

$$\frac{d^2\mathbf{r}_i}{dt^2} = \mathbf{G}\sum_{j=1}^{n} \frac{m_j}{r_{ij}^3}(\mathbf{r}_j - \mathbf{r}_i), \quad i \neq j \ . \qquad (7)$$

From (7) the fundamental differential equation for a two-body problem can be derived

$$\frac{d^2\mathbf{r}}{dt^2} + \frac{\mu}{r^3}\mathbf{r} = \mathbf{0} \qquad (8)$$

where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ is the relative position of masses and $\mu = G(m_1 + m_2)$ where m_1 and m_2 is the masses of two respective bodies.

3.2.1 Formation dynamics

Equation (8) describes the orbit dynamics for a spacecraft under ideal condition, that is with no external disturbances. The equation can be augmented to include terms due to external disturbance, $\mathbf{f}_{dl}$ and $\mathbf{f}_{df}$, and control input vectors, $\mathbf{f}_{al}$ and $\mathbf{f}_{af}$, from actuators onboard the spacecraft, leader and follower respectively. Force terms due to disturbance can be aerodynamic disturbances, gravitational forces from other bodies, *i.e.* third-body attractions, solar radiation, magnetic fields etc. Equation (8) can then be expressed for the leader spacecraft with these including terms as

$$\ddot{\mathbf{r}}_l = -\frac{\mu}{r_l^3}\mathbf{r}_l + \frac{\mathbf{f}_{dl}}{m_l} + \frac{\mathbf{f}_{al}}{m_l} \qquad (9)$$

where $\mu = G(M_e + m_l)$, and for the follower as

$$\ddot{\mathbf{r}}_f = -\frac{\mu}{r_f^3}\mathbf{r}_f + \frac{\mathbf{f}_{df}}{mf} + \frac{\mathbf{f}_{af}}{m_f} \qquad (10)$$

where $\mu = G(M_e + m_f)$. The parameter M_e denotes the mass of the Earth, m_l is the mass of the leader spacecraft and m_f is the mass of the follower spacecraft. The masses of the spacecraft, m_f and m_l, are assumed to be small relative to the mass of the Earth M_e, that is $\mu \approx GM_e$. The second order derivative of the relative orbit position vector $\mathbf{p} = \mathbf{r}_f - \mathbf{r}_l$ is given by (9) and (10) as

$$\begin{aligned}\ddot{\mathbf{p}} =& \ddot{\mathbf{r}}_f - \ddot{\mathbf{r}}_l \\ =& -\frac{\mu}{r_f^3}\mathbf{r}_f + \frac{\mathbf{f}_{df}}{m_f} + \frac{\mathbf{f}_{af}}{m_f} + \frac{\mu}{r_l^3}\mathbf{r}_l - \frac{\mathbf{f}_{dl}}{m_l} - \frac{\mathbf{f}_{al}}{m_l}\end{aligned}$$

and after some rearranging the second order derivative can be written as

$$\begin{aligned}m_f\ddot{\mathbf{p}} = & -m_f\mu\Big(\frac{\mathbf{r}_l + \mathbf{p}}{(r_l + p)^3} - \frac{\mathbf{r}_l}{r_l^3}\Big) \\ & + \mathbf{f}_{af} + \mathbf{f}_{df} - \frac{m_f}{m_l}(\mathbf{f}_{al} + \mathbf{f}_{dl}) \qquad (11)\end{aligned}$$

3.3 Referenced in a leader orbit frame

3.3.1 Relative translation

The inertial position equation for the follower spacecraft is

$$\mathbf{r}_f = \mathbf{r}_l + \mathbf{p} = (r_l + x)\mathbf{e}_r + y\mathbf{e}_\theta + z\mathbf{e}_h$$

and its second order derivative can be written as

$$\begin{aligned}\ddot{\mathbf{r}}_f =& (\ddot{r}_l + \ddot{x})\mathbf{e}_r + 2(\dot{r}_l + \dot{x})\dot{\mathbf{e}}_r + (r_l + x)\ddot{\mathbf{e}}_r + \ddot{y}\mathbf{e}_\theta \\ & + 2\dot{y}\dot{\mathbf{e}}_\theta + y\ddot{\mathbf{e}}_\theta + \ddot{z}\mathbf{e}_h + 2\dot{z}\dot{\mathbf{e}}_h + z\ddot{\mathbf{e}}_h \ . \qquad (12)\end{aligned}$$

The location of a moving body can be described of its angular deviation from the major axis [16]. By using the true anomaly, ν, the relationship between basis vectors, $\mathbf{e}_r$ and $\mathbf{e}_\theta$, in the $\mathcal{F}_l$ frame can be written as [14]

$$\dot{\mathbf{e}}_r = \mathbf{e}_\theta\dot{\nu} \quad \text{and} \quad dot\mathbf{e}_\theta = -\mathbf{e}_r\dot{\nu} \ . \qquad (13)$$

and moreover, the second derivatives of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are

$$\ddot{\mathbf{e}}_r = \ddot{\nu}\mathbf{e}_\theta - \dot{\nu}^2\mathbf{e}_r, \quad ddot\mathbf{e}_\theta = -\ddot{\nu}\mathbf{e}_\theta - \dot{\nu}^2\mathbf{e}_r \ . \quad (14)$$

The vector $\mathbf{e}_h$ points in the orbit normal direction and since the motion is in a plane, which means no out-of-plane motion exists [17], then $\dot{\mathbf{e}}_h = \ddot{\mathbf{e}}_h = 0$. Insertion of the equations (13)-(14) into (12) gives

$$\begin{aligned}\ddot{\mathbf{r}}_f =& (\ddot{r}_l + \ddot{x} - 2\dot{y}\dot{\nu} - \dot{\nu}^2(r_l + x) - y\ddot{\nu})\mathbf{e}_r + \ddot{z}\mathbf{e}_h \\ & + (\ddot{y} + 2\dot{\nu}(\dot{r}_l + \dot{x}) + \ddot{\nu}(r_l + x) - y\dot{\nu}^2)\mathbf{e}_\theta \ . \qquad (15)\end{aligned}$$

The position of the leader spacecraft, $\mathbf{r}_l$, can be expressed as

$$\mathbf{r}_l = r_l\mathbf{e}_r$$

where the second order derivative of the position is written as

$$\ddot{\mathbf{r}}_l = \ddot{r}_l\mathbf{e}_r + 2\dot{r}_l\dot{\mathbf{e}}_r + r_l\ddot{\mathbf{e}}_r \ . \qquad (16)$$

Insertion of $\dot{\mathbf{e}}_r$ and $\ddot{\mathbf{e}}_r$ into equation (16) gives

$$\ddot{\mathbf{r}}_l = (\ddot{r}_l - \dot{\nu}^2)\mathbf{e}_r + (2\dot{r}_l\dot{\nu} + r_l\ddot{\nu})\mathbf{e}_\theta \qquad (17)$$

expressed with the basis vectors. The second order derivative of the relative position orbit vector $\mathbf{p}$ can be derived by subtracting equation (17) from equation (15) as

$$\begin{aligned}\ddot{\mathbf{p}} =& \ddot{\mathbf{r}}_f - \ddot{\mathbf{r}}_l = (\ddot{x} - 2\dot{y}\dot{\nu} - \dot{\nu}^2x - y\ddot{\nu})\mathbf{e}_r \\ & + (\ddot{y} + 2\dot{\nu}\dot{x} + \ddot{\nu}x - y\dot{\nu}^2)\mathbf{e}_\theta + \ddot{z}\mathbf{e}_h \ . \qquad (18)\end{aligned}$$

The position dynamics

A model of relative position dynamics can now be presented. Insertion of equation (18) into equation (11) gives the nonlinear position dynamics as

$$m_f\ddot{\mathbf{p}} + \mathbf{C}_t(\dot{\nu})\dot{\mathbf{p}} + \mathbf{D}_t(\dot{\nu}, \ddot{\nu}, r_f)\mathbf{p} + \mathbf{n}_t(r_l, r_f) = \mathbf{F}_a + \mathbf{F}_d \quad (19)$$

where

$$\mathbf{C}_t(\dot{\nu}) = 2m_f\dot{\nu}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a skew-symmetric Coriolis-like matrix,

$$\mathbf{D}_t(\dot{\nu}, \ddot{\nu}, r_f)\mathbf{p} = m_f \begin{bmatrix} \frac{\mu}{r_f^3} - \dot{\nu}^2 & -\ddot{\nu} & 0 \\ \ddot{\nu} & \frac{\mu}{r_f^3} - \dot{\nu}^2 & 0 \\ 0 & 0 & \frac{\mu}{r_f^3} \end{bmatrix} \mathbf{p}$$

may be viewed as a time-varying potential force term [17] and

$$\mathbf{n}_t(r_l, r_f) = m_f\mu \begin{bmatrix} \frac{r_l}{r_f^3} - \frac{1}{r_l^2} \\ 0 \\ 0 \end{bmatrix}$$

The relative actuator force $\mathbf{F}_a$ and perturbation force $\mathbf{F}_d$ is given by

$$\mathbf{F}_a = \mathbf{f}_{af} - \frac{m_f}{m_l}\mathbf{f}_{al} \qquad \mathbf{F}_d = \mathbf{f}_{df} - \frac{m_f}{m_l}\mathbf{f}_{dl}$$

respectively.

3.3.2 Relative rotation

Attitude kinematics

The kinematic differential equation of the leader spacecraft in its orbit frame $\mathcal{F}_l$ can be found from equation (4) as

$$\dot{\mathbf{q}}_l = \begin{bmatrix} \dot{\eta}_l \\ \dot{\epsilon}_l \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -\epsilon_l^T \\ \eta_l\mathbf{I} + \mathbf{S}(\epsilon_l) \end{bmatrix} \omega_{l,lb}^{lb}$$

where $\omega_{l,lb}^{lb}$ is the angular velocity of the $\mathcal{F}_{lb}$ frame relative to the $\mathcal{F}_l$ frame, represented in $\mathcal{F}_{lb}$. The kinematic differential equation of the follower spacecraft is analogously found to be

$$\dot{\mathbf{q}}_f = \begin{bmatrix} \dot{\eta}_f \\ \dot{\epsilon}_f \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -\epsilon_f^T \\ \eta_f\mathbf{I} + \mathbf{S}(\epsilon_f) \end{bmatrix} \omega_{f,fb}^{fb}$$

where $\omega_{f,fb}^{fb}$ is the angular velocity of the $\mathcal{F}_{fb}$ frame relative to the $\mathcal{F}_f$ frame, represented in $\mathcal{F}_{fb}$.

Attitude dynamics

With the assumption of rigid body movement, the rotational motion of the leader can be expressed, including terms due to perturbations and actuators, as [16]

$$\mathbf{J}_l\dot{\omega}_{i,lb}^{lb} = -\mathbf{S}(\omega_{i,lb}^{lb})\mathbf{J}_l\omega_{i,lb}^{lb} + \tau_d^{lb} + \tau_a^{lb} \quad (20)$$

$$\omega_{l,lb}^{lb} = \omega_{i,lb}^{lb} + \omega_o\mathbf{c}_2 \,. \quad (21)$$

The parameter $\omega_{i,lb}^{lb}$ is the angular velocity of $\mathcal{F}_{lb}$ relative to the $\mathcal{F}_i$ frame represented in the $\mathcal{F}_{lb}$ frame. The term $\omega_{l,lb}^{lb}$ denotes the angular velocity of $\mathcal{F}_{lb}$ relative to the $\mathcal{F}_l$ frame represented in $\mathcal{F}_{lb}$. The term $\mathbf{J}_l$ is the inertia matrix of the leader spacecraft, ω_o is the orbit angular velocity, τ_d^{lb} is the total disturbance torque, τ_a^{lb} is the actuator torque and $\mathbf{c}_2$ is the directional cosine vector. The angular velocity of the spacecraft relative to the $\mathcal{F}_i$ frame expressed in the $\mathcal{F}_{lb}$ frame is

$$\omega_{i,lb}^{lb} = \omega_{i,l}^{lb} + \omega_{l,lb}^{lb} = \mathbf{R}_o^b\omega_{i,l}^l + \omega_{l,lb}^{lb}$$

where $\mathbf{R}_o^b = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3]$ is the rotation matrix. The orbit frame revolves relative to the $\mathcal{F}_i$ frame with the angular velocity ω_o which yields $\omega_{i,l}^l = [0 \quad -\omega_o \quad 0]^T$. This gives the angular velocity in the body frame relative to the $\mathcal{F}_i$ frame as $\omega_{i,lb}^{lb} = \omega_{l,lb}^{lb} - \omega_o\mathbf{c}_2$ where $\omega_o = \sqrt{\mu/r_l^3}$. Analogously the rotational motion of the follower spacecraft can be written as

$$\mathbf{J}_f\dot{\omega}_{i,fb}^{fb} = -\mathbf{S}(\omega_{i,fb}^{fb})\mathbf{J}_f\omega_{i,fb}^{fb} + \tau_d^{fb} + \tau_a^{fb} \quad (22)$$

$$\omega_{f,fb}^{fb} = \omega_{i,fb}^{fb} + \omega_o\mathbf{c}_2 \quad (23)$$

where the $\omega_{i,fb}^{fb}$ is the angular velocity of $\mathcal{F}_{fb}$ relative to the $\mathcal{F}_i$ frame presented in the $\mathcal{F}_{fb}$ frame. The term $\omega_{f,fb}^{fb}$ denotes the angular velocity of $\mathcal{F}_{fb}$ relative to the $\mathcal{F}_f$ frame represented in $\mathcal{F}_{fb}$. The term $\mathbf{J}_f$ is the inertia matrix of the follower spacecraft, τ_d^{fb} is the total disturbance torque, τ_a^{fb} is the actuator torque and $\mathbf{c}_2$ is the directional cosine vector.

Relative attitude

The attitude kinematics of the follower spacecraft relative to the leader spacecraft can be described by the quaternion product [15]

$$\mathbf{q}_a = \mathbf{q}_f \otimes \bar{\mathbf{q}}_l = \begin{bmatrix} \eta_a \\ \epsilon_a \end{bmatrix} = \begin{bmatrix} \eta_f\eta_l + \epsilon_f^T\epsilon_l \\ \eta_l\epsilon_f - \eta_f\epsilon_l - \mathbf{S}(\epsilon_f)\epsilon_l \end{bmatrix}$$

where $\bar{\mathbf{q}}_l$ is the inverse unit quaternion. The kinematic differential equation of relative attitude can then be written as

$$\dot{\mathbf{q}}_a = \begin{bmatrix} \dot{\eta}_a \\ \dot{\epsilon}_a \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -\epsilon_a^T \\ \eta_a\mathbf{I} + \mathbf{S}(\epsilon_a) \end{bmatrix} \omega_{lb,fb}^{fb}$$

where

$$\omega_{lb,fb}^{fb} = \omega_{i,fb}^{fb} - \mathbf{R}_{lb}^{fb}\omega_{i,lb}^{lb} \quad (24)$$

is the angular velocity between the leader body reference frame, $\mathcal{F}_{lb}$, and the follower body reference frame, $\mathcal{F}_{fb}$. The parameter $\omega_{i,fb}^{fb}$ denotes the angular velocity of the $\mathcal{F}_{fb}$ frame relative to the $\mathcal{F}_i$ frame represented in the $\mathcal{F}_{fb}$ frame and $\omega_{i,lb}^{lb}$ is the angular velocity of the $\mathcal{F}_{lb}$ frame relative to the $\mathcal{F}_i$ frame represented in $\mathcal{F}_{lb}$. The rotation matrix $\mathbf{R}_{lb}^{fb}$ describes rotation from the $\mathcal{F}_{lb}$ frame to the $\mathcal{F}_{fb}$ frame and expresses $\omega_{i,lb}^{lb}$ in the $\mathcal{F}_{fb}$ frame. The relative dynamics can be expressed by differentiating (24) and multiplying with $\mathbf{J}_f$ as

$$\mathbf{J}_f\dot{\omega}_{lb,fb}^{fb} = \mathbf{J}_f\dot{\omega}_{i,fb}^{fb} - \mathbf{J}_f\dot{\mathbf{R}}_{lb}^{fb}\omega_{i,lb}^{lb} - \mathbf{J}_f\mathbf{R}_{lb}^{fb}\dot{\omega}_{i,lb}^{lb} \,. \quad (25)$$

Since

$$\dot{\mathbf{R}}_{lb}^{fb} = -\mathbf{S}(\omega_{lb,fb}^{fb})\mathbf{R}_{lb}^{fb}$$

and

$$\mathbf{S}(\omega_{lb,fb}^{fb})\omega_{i,lb}^{fb} = -\,\mathbf{S}(\omega_{i,lb}^{fb})\omega_{lb,fb}^{fb}$$

equation (25) can be written as

$$\begin{aligned}\mathbf{J}_f\dot{\omega}_{lb,fb}^{fb} =& \mathbf{J}_f\dot{\omega}_{i,fb}^{fb} \\ &- \mathbf{J}_f\mathbf{S}(\omega_{i,lb}^{fb})\omega_{lb,fb}^{fb} - \mathbf{J}_f\mathbf{R}_{lb}^{fb}\dot{\omega}_{i,lb}^{lb}. \quad (26)\end{aligned}$$

Insertion of (20)-(22) into (26) results in

$$\begin{aligned}\mathbf{J}_f\dot{\omega}_{lb,fb}^{fb} =& \mathbf{J}_f\mathbf{R}_{lb}^{fb}\mathbf{J}_l^{-1}\mathbf{S}\left(\omega_{i,lb}^{lb}\right)\mathbf{J}_l\omega_{i,lb}^{lb} + \tau_d^{fb} + \tau_a^{fb} \\ &- \mathbf{J}_f\mathbf{S}\left(\mathbf{R}_{lb}^{fb}\omega_{i,lb}^{lb}\right)\omega_{lb,fb}^{fb} - \mathbf{J}_f\mathbf{R}_{lb}^{fb}\mathbf{J}_l^{-1}\tau_d^{lb} \\ &- \mathbf{J}_f\mathbf{R}_{lb}^{fb}\mathbf{J}_l^{-1}\tau_a^{lb} - \mathbf{S}\left(\omega_{i,fb}^{fb}\right)\mathbf{J}_f\omega_{i,fb}^{fb}\end{aligned}$$

where

$$\omega_{i,fb}^{fb} = \omega_{lb,fb}^{fb} + \omega_{i,lb}^{fb} = \omega_{lb,fb}^{fb} + \mathbf{R}_{lb}^{fb}\omega_{i,lb}^{lb}.$$

The relative rotation with perturbation forces and torques can now be expressed as

$$\begin{aligned}\mathbf{J}_f\dot{\omega}_{lb,fb}^{fb} =& -\mathbf{J}_f\mathbf{S}(\mathbf{R}_{lb}^{fb}\omega_{i,lb}^{lb})\omega_{lb,fb}^{fb} + \boldsymbol{\Upsilon}_d + \boldsymbol{\Upsilon}_a \\ &- \mathbf{S}(\omega_{lb,fb}^{fb}+\mathbf{R}_{lb}^{fb}\omega_{i,lb}^{lb})\mathbf{J}_f(\omega_{lb,fb}^{fb} + \mathbf{R}_{lb}^{fb}\omega_{i,lb}^{lb}) \\ &+ \mathbf{J}_f\mathbf{R}_{lb}^{fb}\mathbf{J}_l^{-1}\mathbf{S}(\omega_{i,lb}^{lb})\mathbf{J}_l\cdot\omega_{i,lb}^{lb} \quad (27)\end{aligned}$$

where $\boldsymbol{\Upsilon}_d = \tau_d^{fb} - \mathbf{J}_f\mathbf{R}_{lb}^{fb}\mathbf{J}_l^{-1}\tau_d^{lb}$ are the relative disturbance torques and $\boldsymbol{\Upsilon}_a = \tau_a^{fb} - \mathbf{J}_f\mathbf{R}_{lb}^{fb}\mathbf{J}_l^{-1}\tau_a^{lb}$ are the relative actuator torques. Furthermore, the second term in the equation 27 can be rewritten as

$$\begin{aligned}&\mathbf{S}\left(\omega_{lb,fb}^{fb} + \mathbf{R}_{lb}^{fb}\omega_{i,lb}^{lb}\right)\mathbf{J}_f\left(\omega_{lb,fb}^{fb} + \mathbf{R}_{lb}^{fb}\omega_{i,lb}^{lb}\right) = \\ &\mathbf{S}\left(\mathbf{R}_{lb}^{fb}\omega_{i,lb}^{lb}\right)\mathbf{J}_f\omega_{lb,fb}^{fb} + \mathbf{S}\left(\mathbf{R}_{lb}^{fb}\omega_{i,lb}^{lb}\right)\mathbf{J}_f\mathbf{R}_{lb}^{fb}\omega_{i,lb}^{lb} \\ &- \mathbf{S}\left(\mathbf{J}_f\left(\omega_{lb,fb}^{fb} + \mathbf{R}_{lb}^{fb}\omega_{i,lb}^{lb}\right)\right)\omega_{lb,fb}^{fb}.\end{aligned}$$

The relative rotation dynamics can now be written as

$$\begin{aligned}\mathbf{J}_f\dot{\omega}_{lb,fb}^{fb} +& \mathbf{C}_r(\omega_{lb,fb}^{fb})\omega_{lb,fb}^{fb} \\ &+ \mathbf{n}_r(\omega_{lb,fb}^{fb}) = \boldsymbol{\Upsilon}_d + \boldsymbol{\Upsilon}_a \quad (28)\end{aligned}$$

where

$$\begin{aligned}\mathbf{C}_r(\omega_{lb,fb}^{fb}) =& \mathbf{J}_f\mathbf{S}(\mathbf{R}_{lb}^{fb}\omega_{i,lb}^{lb}) + \mathbf{S}(\mathbf{R}_{lb}^{fb}\omega_{i,lb}^{lb})\mathbf{J}_f \\ &- \mathbf{S}(\mathbf{J}_f(\omega_{lb,fb}^{fb} + \mathbf{R}_{lb}^{fb}\omega_{i,lb}^{lb}))\end{aligned}$$

is a skew-symmetric matrix, and

$$\begin{aligned}\mathbf{n}_r(\omega_{lb,fb}^{fb}) =& \mathbf{S}\left(\mathbf{R}_{lb}^{fb}\omega_{i,lb}^{lb}\right)\mathbf{J}_f\mathbf{R}_{lb}^{fb}\omega_{i,lb}^{lb} \\ &- \mathbf{J}_f\mathbf{R}_{lb}^{fb}\mathbf{J}_l^{-1}\mathbf{S}\left(\omega_{i,lb}^{lb}\right)\mathbf{J}_l\omega_{i,lb}^{lb} \quad (29)\end{aligned}$$

is a nonlinear term [17].

3.3.3 The 6DOF model

In this section the 6DOF model of relative translation and rotation in a leader-follower spacecraft structure referenced in the leader-orbit coordinate system is presented. The state vectors representing the relative position and attitude dynamics cam be defined as

$$\mathbf{x}_1 = \begin{bmatrix}\mathbf{p} \\ \mathbf{q}_a\end{bmatrix},\ \mathbf{x}_2 = \begin{bmatrix}\mathbf{v} \\ \omega_{lb,fb}^{fb}\end{bmatrix}$$

where $\mathbf{p}$ is the relative orbit position vector describing the relative position between the leader and follower spacecraft, $\mathbf{q}_a$ is the unit quaternion describing the relative attitude between the leader and follower spacecraft, $\mathbf{v}$ describes the relative velocity and $\omega_{lb,fb}^{fb}$ describes the relative angular velocity. Combining the models of relative translation and rotation from (19) and (28), respectively, results in the total model for relative translation and rotation with 6DOF coupled through external disturbances and onboard actuators, written as

$$\dot{\mathbf{x}}_1 = \boldsymbol{\Lambda}(\mathbf{x}_1)\mathbf{x}_2 \quad (30)$$

$$\mathbf{M}_f\dot{\mathbf{x}}_2 + \mathbf{C}\mathbf{x}_2 + \mathbf{D}\mathbf{x}_1 + \mathbf{n} = \mathbf{U} + \mathbf{W} \quad (31)$$

where

$$\boldsymbol{\Lambda}(\mathbf{x}_1) = \begin{bmatrix}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\begin{bmatrix}-\epsilon^T \\ \eta\mathbf{I} + \mathbf{S}(\epsilon)\end{bmatrix}\end{bmatrix}$$

is the coupling term between the first and second order dynamics [17],

$$\mathbf{M}_f = \begin{bmatrix}m_f\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_f\end{bmatrix} = \mathbf{M}_f^T$$

is a symmetric positive definite matrix consisting of the mass and moments of inertia of the follower spacecraft,

$$\begin{aligned}\mathbf{C}(\dot{\nu}, \omega_{lb,fb}^{fb}) &= \begin{bmatrix}\mathbf{C}_t(\dot{\nu}) & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_r(\omega_{lb,fb}^{fb})\end{bmatrix} \in SS(6) \\ &= -\,\mathbf{C}^T(\dot{\nu}, \omega_{lb,fb}^{fb}) \quad (32)\end{aligned}$$

is a skew-symmetric Coriolis-like damping matrix,

$$\mathbf{D}(\dot{\nu}, \ddot{\nu}, r_f) = \begin{bmatrix}\mathbf{D}_t(\dot{\nu}, \ddot{\nu}, r_f) & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}$$

is a spring coefficient matrix,

$$\mathbf{n}(r_l, r_f, \omega_{lb,fb}^{fb}) = \begin{bmatrix}\mathbf{n}_t(r_l, r_f) \\ \mathbf{n}_r(\omega_{lb,fb}^{fb})\end{bmatrix}$$

is a composite nonlinear term, and

$$\mathbf{U} = \begin{bmatrix}\mathbf{F}_a \\ \boldsymbol{\Upsilon}_a\end{bmatrix} \quad \text{and} \quad \mathbf{W} = \begin{bmatrix}\mathbf{F}_d \\ \boldsymbol{\Upsilon}_d\end{bmatrix}$$

are relative actuator inputs and orbital perturbations, respectively. It should be noted that the relative rotation and translation of the leader and the follower spacecraft is uncoupled when orbital perturbations and actuator torques are neglected.

The system properties

The model of the nonlinear dynamics of 6DOF is similar to many 6DOF mechanical systems. The system properties are extensively exploited in the control literature, particularly in the control of robot manipulators (*cf.* [18]). System properties like symmetry, skew-symmetric and positiveness of matrices can be incorporated into the stability analysis when designing control system, and represents physical properties of the system [20]. The system inertia matrix $\mathbf{M}_f$ and the Coriolis-like damping matrix $\mathbf{C}(\dot{\nu}, \omega^{fb}_{lb,fb})$ of the leader-fixed vector representation have following advantageous properties. The rigid-body system inertia matrix $\mathbf{M}_f$ satisfies [20]

$$\mathbf{M}_f = \mathbf{M}_f^T > \mathbf{0}_{6\times6} \quad \text{and} \quad \dot{\mathbf{M}}_f = \mathbf{0}_{6\times6}\,.$$

The Coriolis-centripetal matrix $\mathbf{C}$ can always be parameterized such that [20]

$$\mathbf{C} = -\mathbf{C}^T$$

when the system inertia matrix $\mathbf{M}$ is defined as

$$\mathbf{M} = \mathbf{M}^T = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} > \mathbf{0}$$

where $\mathbf{M}_{21} = \mathbf{M}_{12}^T$. The Coriolis-like matrix of the system in this paper is skew-symmetrical, *i.e.*

$$\mathbf{C}\left(\dot{\nu}, \omega^{fb}_{lb,fb}\right) = -\mathbf{C}^T(\dot{\nu}, \omega^{fb}_{lb,fb}) \tag{33}$$

since $\mathbf{C}_t(\dot{\nu})$ and $\mathbf{C}_r(\omega^{fb}_{lb,fb})$ are both skew-symmetric matrices.

3.4 Referenced in Earth-fixed frame

3.4.1 Relative translation

The transformation of the relative translational velocity expressed in the $\mathcal{F}_l$ frame and the $\mathcal{F}_i$ frame is given as $\dot{\mathbf{p}}^i = \mathbf{R}^i_l \mathbf{v}$ where $\mathbf{R}^i_l$ denotes the rotation matrix which rotates the linear velocity vector from $\mathcal{F}_l$ to the $\mathcal{F}_i$ frame. The term $\mathbf{v}$ denotes the linear velocity vector in the leader-fixed reference frame $\mathcal{F}_l$ and $\dot{\mathbf{p}}^i$ denotes the linear velocity in the $\mathcal{F}_i$ frame. The rotation matrix $\mathbf{R}^i_l$ is given by inverse rotation of $\mathbf{R}^l_i$ given as

$$\mathbf{R}^i_l = \left(\mathbf{R}^l_i\right)^{-1} = \left(\mathbf{R}^l_i\right)^T = \mathbf{R}_z^T(\Omega)\mathbf{R}_x^T(i)\mathbf{R}_z^T(\omega+\nu)\,.$$

3.4.2 Relative rotation

The relationship between the relative body-fixed angular velocity vector ω and angular velocity in the $\mathcal{F}_i$ frame can be written as [20] $\dot{\mathbf{q}}^i = \mathbf{T}(\mathbf{q})\omega^b$ where

$$\mathbf{T}(\mathbf{q}) = \frac{1}{2}\begin{bmatrix} -\epsilon^T \\ \eta\mathbf{I} + \mathbf{S}(\epsilon) \end{bmatrix}$$

denotes the transformation matrix in terms of quaternion $\mathbf{q} = \begin{bmatrix}\eta & \epsilon^T\end{bmatrix}^T$ and $\mathbf{T}^T(\mathbf{q})\mathbf{T}(\mathbf{q}) = \frac{1}{4}\mathbf{I}_{3\times3}$. The term ω denotes the angular velocity of the $\mathcal{F}_{fb}$ frame relative the to $\mathcal{F}_{lb}$ frame decomposed in the $\mathcal{F}_{fb}$ frame, such that $\omega = \omega^{fb}_{lb,fb}$.

3.4.3 The 6DOF model

The kinematic equations of motion can be expressed as [20]

$$[\dot{\mathbf{p}} \;\; \dot{\mathbf{q}}] = \begin{bmatrix} \mathbf{R}^i_l & \mathbf{0}_{3\times3} \\ \mathbf{0}_{4\times3} & \mathbf{T}(\mathbf{q}) \end{bmatrix}\begin{bmatrix} \mathbf{v} \\ \omega \end{bmatrix} \Leftrightarrow \dot{\gamma} = \mathbf{J}\mathbf{x}_2$$

where $\mathbf{J} \in \mathbb{R}^{7\times6}$ is a non-quadratic transformation matrix, $\mathbf{x}_2 \in \mathbb{R}^6$ denotes the state vector and $\gamma = \begin{bmatrix}\mathbf{p}^T & \mathbf{q}^T\end{bmatrix}^T \in \mathbb{R}^7$ denotes the relative position and attitude referenced in the $\mathcal{F}_i$ frame. The Earth-fixed representation can be obtained by applying the following kinematic transformations [20]

$$\mathbf{J}^\dagger = \left(\mathbf{J}^T\mathbf{J}\right)^{-1}\mathbf{J}^T = \begin{bmatrix} \left(\mathbf{R}^i_l\right)^T & \mathbf{0}_{3\times4} \\ \mathbf{0}_{3\times3} & \mathbf{T}(\mathbf{q})^T \end{bmatrix} \tag{34}$$

such that $\mathbf{J}^\dagger\mathbf{J} = \mathbf{I}_{6\times6}$, where the 6×7 matrix $\mathbf{J}^\dagger \in \mathbb{R}^{6\times7}$ is the left pseudo-inverse of the matrix $\mathbf{J}$. Now, the relationship between the state vector $\mathbf{x}_2$ and the vector γ can be written as

$$\dot{\gamma} = \mathbf{J}\mathbf{x}_2 \Leftrightarrow \mathbf{x}_2 = \mathbf{J}^\dagger\dot{\gamma} \tag{35}$$

where the time derivative of the state vector $\mathbf{x}_2$ is

$$\dot{\mathbf{x}}_2 = \dot{\mathbf{J}}^\dagger\dot{\gamma} + \mathbf{J}^\dagger\ddot{\gamma}\,. \tag{36}$$

By substituting the equations (35) and (36) into the dynamic equations of motion (31) decomposed in the leader-fixed frame and multiplying each side with the transpose inverse kinematic matrix, we obtain

$$\begin{aligned} \mathbf{J}^{\dagger T}\mathbf{M}_f\left(\dot{\mathbf{J}}^\dagger\dot{\gamma} + \mathbf{J}^\dagger\ddot{\gamma}\right) &+ \mathbf{J}^{\dagger T}\mathbf{C}(\dot{\nu},\omega^{fb}_{lb,fb})\mathbf{J}^\dagger\dot{\gamma} \\ +\mathbf{J}^{\dagger T}\mathbf{D}(\dot{\nu},\ddot{\nu},r_f)\gamma &+ \mathbf{J}^{\dagger T}\mathbf{n}(r_l,r_f,\omega^{fb}_{lb,fb}) \\ &= \mathbf{J}^{\dagger T}\mathbf{U} + \mathbf{J}^{\dagger T}\mathbf{W}\,. \end{aligned}$$

Finally, the total model referenced in the $\mathcal{F}_i$ frame can be written as

$$\mathbf{M}_\gamma\ddot{\gamma} + \mathbf{C}_\gamma\dot{\gamma} + \mathbf{D}_\gamma\gamma + \mathbf{n}_\gamma = \mathbf{J}^{\dagger T}\mathbf{U} + \mathbf{J}^{\dagger T}\mathbf{W}$$

where $\mathbf{M}_\gamma(\gamma) = \mathbf{J}^{\dagger T}\mathbf{M}_f\mathbf{J}^\dagger$ is the system inertia matrix, the term

$$\mathbf{C}_\gamma(\dot{\nu},\omega^{fb}_{lb,fb},\gamma) = \mathbf{J}^{\dagger T}\left[\mathbf{C}(\dot{\nu},\omega^{fb}_{lb,fb})\mathbf{J}^\dagger + \mathbf{M}_f\dot{\mathbf{J}}^\dagger\right]$$

denotes the Coriolis-like damping matrix, whereas

$$\mathbf{D}_\gamma(\dot{\nu},\omega^{fb}_{lb,fb},\gamma) = \mathbf{J}^{\dagger T}\mathbf{D}(\dot{\nu},\ddot{\nu},r_f)\mathbf{J}^\dagger$$

denotes the spring coefficient matrix,

$$\mathbf{n}_\gamma(r_l,r_f,\omega^{fb}_{lb,fb}) = \mathbf{J}^{\dagger T}\mathbf{n}(r_l,r_f,\omega^{fb}_{lb,fb})$$

denotes a composite nonlinear term, $\mathbf{J}^{\dagger T}\mathbf{U}$ is the input torque/force and $\mathbf{J}^{\dagger T}\mathbf{W}$ contains the relative orbital perturbations.

System properties of the ECI Vector representation

It can be shown that the matrices of the system have the following important properties where the inertia matrix of the system has the property [18]

$$\mathbf{M}_\gamma(\gamma) = \mathbf{M}_\gamma^T(\gamma) > \mathbf{0}\ \forall\ \gamma \neq \beta \begin{bmatrix}\mathbf{0}\\ \mathbf{q}\end{bmatrix} \in \mathbb{R}^7$$

and is positive semi-definite, which yields

$$\mathbf{y}^T\mathbf{M}_\gamma(\gamma)\mathbf{y} > 0, \quad \forall\ \mathbf{y} \neq \beta \begin{bmatrix}\mathbf{0}\\ \mathbf{q}\end{bmatrix}, \quad \beta \in \mathbb{R}\,.$$

The derivative of the inertia matrix with respect to time can be written as

$$\dot{\mathbf{M}}_\gamma = \dot{\mathbf{J}}^{\dagger T}\mathbf{M}_f\mathbf{J}^\dagger + \mathbf{J}^{\dagger T}\mathbf{M}_f\dot{\mathbf{J}}^\dagger$$

as opposed to the inertia matrix in the leader-fixed representation where $\dot{\mathbf{M}}_f = 0$. The Coriolis-like damping matrix can be shown to have important property. It should be noted that $\mathbf{C}_\gamma(\dot{\nu}, \omega_{lb,fb}^{fb}, \gamma)$ will not be skew-symmetrical although $\mathbf{C}(\dot{\nu}, \omega_{lb,fb}^{fb})$ is skew-symmetric, that is

$$\mathbf{C}_\gamma(\dot{\nu}, \omega_{lb,fb}^{fb}, \gamma) \neq -\mathbf{C}_\gamma^T(\dot{\nu}, \omega_{lb,fb}^{fb}, \gamma)\,.$$

A useful property including $\mathbf{C}_\gamma(\dot{\nu}, \omega_{lb,fb}^{fb}, \gamma)$ is

$$\left[\dot{\mathbf{M}}_\gamma - 2\mathbf{C}_\gamma\right]^T = -\left[\dot{\mathbf{M}}_\gamma - 2\mathbf{C}_\gamma\right] \qquad (37)$$

i.e. skew-symmetric, and hence

$$\mathbf{s}^T\left[\dot{\mathbf{M}}_\gamma - 2\mathbf{C}_\gamma\right]\mathbf{s} = \mathbf{0}\,.$$

This property can be shown as follows [18]:

$$\begin{aligned}\dot{\mathbf{M}}_\gamma - 2\mathbf{C}_\gamma \\ &= \dot{\mathbf{J}}^{\dagger T}\mathbf{M}_f\mathbf{J}^\dagger + \mathbf{J}^{\dagger T}\mathbf{M}_f\dot{\mathbf{J}}^\dagger \\ &\quad - 2\mathbf{J}^{\dagger T}\mathbf{C}\mathbf{J}^\dagger - 2\mathbf{J}^{\dagger T}\mathbf{M}_f\dot{\mathbf{J}}^\dagger \\ &= \dot{\mathbf{J}}^{\dagger T}\mathbf{M}_f\mathbf{J}^\dagger - \mathbf{J}^{\dagger T}\mathbf{M}_f\dot{\mathbf{J}}^\dagger - 2\mathbf{J}^{\dagger T}\mathbf{C}\mathbf{J}^\dagger\end{aligned}$$

and hence (37) follows from the fact that $\mathbf{C}^T\left(\dot{\nu}, \omega_{lb,fb}^{fb}\right) = -\mathbf{C}\left(\dot{\nu}, \omega_{lb,fb}^{fb}\right)$. Furthermore, we find that

$$\dot{\mathbf{M}}_\gamma = \mathbf{C}_\gamma + \mathbf{C}_\gamma^T$$

since $\dot{\mathbf{M}}_\gamma(\gamma) = \dot{\mathbf{M}}_\gamma^T(\gamma)$.

4 Simulations

In this Section simulations of the model referred to the leader orbit coordinate system is presented where the impact of the perturbing forces and torques are illustrated. The formation consists of two spacecraft, a leader and a follower. The simulations are performed using a Runge-Kutta ODE (ordinary differential equations) solver, namely ODE45. The perturbations acting on the follower spacecraft considered in the simulation are based on standard perturbation models. Although perturbations due to several disturbing forces are acting on the spacecraft, only atmospheric drag and the oblateness of the Earth are included. Perturbations due to atmospheric drag have a considerable effect on spacecraft located in altitudes below 500 kilometres, and the drag will be larger for a spacecraft with low mass, large area or low height. Perturbations due to solar radiation and the third body effect are not considered due to the fact that the effects of these perturbations vary based on the location of the Sun and other celestial bodies. In our simulation scenario the follower spacecraft is located 10 m behind the leader spacecraft in the along-track direction and is assumed to have the same initial values as the leader spacecraft, that is initial orbit velocity and attitude. The initial values for the leader and follower spacecraft referenced in the $\mathcal{F}_i$ frame are given as

$$\mathbf{x}_l = \begin{bmatrix}6621 \cdot 10^3 & 0 & 0 & 1 & 0 & 0 & 0\end{bmatrix}^T\,.$$

The leader spacecraft is assumed perfectly controlled in an elliptic orbit with an altitude of 250 km at perigee with inclination 10° and eccentricity of 0.2. The masses of the spacecraft are assumed to be 100 kg and constant. Furthermore, the inertia matrix for both spacecraft are assumed to be

$$\mathbf{J}_l = \mathbf{J}_f = \begin{bmatrix}0.06 & 0 & 0\\ 0 & 0.06 & 0\\ 0 & 0 & 0.0030\end{bmatrix} \mathrm{kgm}^2\,.$$

The relative desired position and attitude are given as

$$\mathbf{x}_d = \begin{bmatrix}\mathbf{p}_d\\ \mathbf{q}_d\end{bmatrix} = \begin{bmatrix}100 & 70 & 50 & 0 & 1 & 0 & 0\end{bmatrix}^T\,.$$

The position and velocity of the follower spacecraft relative to the leader spacecraft are shown in Figure 3, and the relative attitude and angular velocity are shown in Figure 4.

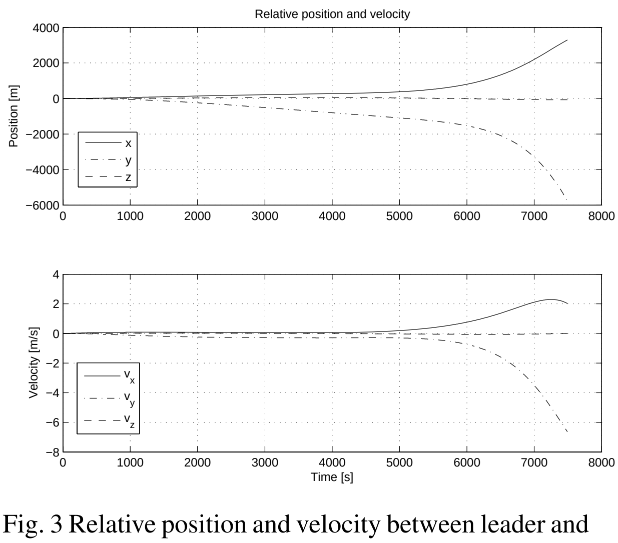

Fig. 3 Relative position and velocity between leader and follower spacecraft.

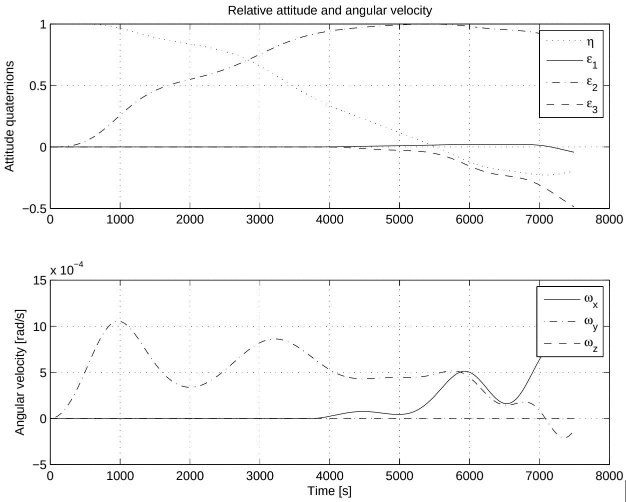

Fig. 4 Relative attitude and angular velocity between leader and follower spacecraft.

If no orbital perturbations were present, the relative position and attitude would be constant. Hence, the perturbing forces and torques can be seen from the figures to have a large impact on the system states.

5 Conclusion

In this paper we have presented a 6DOF mathematical Euler-Lagrange model formulation of relative translation and rotation in a leader-follower formation consisting of two spacecraft. The model is referenced both in the leader-fixed system and in the Earth-fixed inertial coordinate system, and system properties like symmetry, skew-symmetric and positiveness of matrices for both the models was presented. Simulation results for the leader-fixed representation were presented, where orbital perturbations due to atmospheric drag and gravity variation were included in the simulation. The simulation results show that the orbital perturbations have a strong impact on the spacecraft.

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