ANALYSIS OF MULTI-SERVER QUEUEING SYSTEM WITH SEMI-MARKOVIAN INPUT FLOW AND NEGATIVE CUSTOMERS ACTED UPON QUEUE END

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Abstract

The multi-server queueing system with a finite of an infinite buffer, with semi-Markovian input flow (for positive and negative customers) and with Markovian Service Process (for positive customers) whose the number of the states of the process and the intensities of the transitions between phases depend on the number of the customers in the system is considered. An arriving negative customer kills the one positive customer at the end of the queue. The relations and algorithms for computation of the steady-state probabilities and for calculation of the steady-state distribution of waiting time of positive customer are received. It is shown how the multi-server queueing system with semi-Markovian input flow, the servicing of the phase type and the above mentioned order of act of the negative customers can be bring to the general queuing system.

Keywords: Queuing systems, Negative customers.

Presenting Author's Biography

Alexander V. Pechinkin was born in Moscow, Russia. He graduated from the Lomonosov Moscow State University in 1968. D. Sc. (Phys.-math.), Professor, author of more then 150 publications in the field of probability theory and its applications. At present he is Principal Scientist of Institute of Informatics Problems RAS.

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1 Introduction

In the course of operation of the inforcommunication systems, an arriving customer may discharge unserved from the buffer. This may happen because of different circumstances. For example, a customer may be unserved because of a virus penetrating the system.

The queuing systems and queuing networks with negative customers have gained wide use as the analytical models of the infotelecommunication systems allowing for such effects. The notion of negative customer was introduced to the queuing theory by E. Gelenbe [1, 2]. In the traditional sense, the effect of a negative customer arriving to the queuing system or a node of the queuing network lies in that it "kills" (destroys) one ordinary (positive) queued customer following which both customers immediately discharge the system, the number of waiting positive customers, if any, being reduced by one. Later on Gelenbe (see, for example, [3]) extended the notion of negative customer to the case where it can kill a group of customers or empty the queue completely (disaster). The notions of the flip-flop pushing the positive customer from one network to another and the signal which with the given probability can be either a traditional negative customer or a flip-flop [3] were introduced in the queuing network. A vast bibliography of the publications on the queuing networks and queuing systems with negative customers or the so-called G-networks and G-systems including the existing generalized notions of the negative customer can be found in the reviews [4, 5].

The existing studies on the G-networks are confined mostly to the class of the BCMP-networks [6] and their modifications assuming the Poisson flows of both positive and negative customers and enabling one to determine the multiplicative form of the stationary probability distributions of the system states. Since for the G-systems the assumption of Poisson distribution and special form of the servicing time distribution functions are not mandatory, the publications on them consider queuing systems with more complicated processes of arrival of both the positive and negative customers, as well as sufficiently general servicing processes. The present paper analyses the multi-server G-system. Therefore, we confine ourselves to a brush treatment of the multi-server queuing systems with negative customers.

In [7] consideration was given only to the one-server queuing system with finite buffer and repeated customers to which a flow of the type of marked Markovian arrival process (MMAP) arrives, the servicing times being distributed exponentially. We note that servicing of the repeated customers is similar in a sense to servicing in the multi-server system. An approximate method of calculation of the stationary probability distribution of the system states was obtained in [7] with regard for the two types of negative customers and disasters. The stationary probability distribution of the system states was determined and the nonstationary mode under high load and disasters was studied in [8] for the $M/M/n/0$ system with repeated customers under the assumption that the negative customer kills a random number of customers. The multi-server queuing system with a finite buffer, recurrent input flow, and Markov servicing process of all customers was considered in [9]. At that, the number of process states and the inter-phase transition intensities depend on the number of the customers in the system. It is assumed that the flow of negative customers is Markovian. At that, the arriving negative customer removes a group of positive customers at the queue head. For this queuing system, a recurrent matrix algorithm to calculate the stationary state probabilities was developed in [9].

Paper [10] considers the multi-server queuing system with infinite buffer, Markovian input flow of positive and negative customers, and Markovian (general) servicing process similar to that discussed in [9]. In distinction to [9], the arriving negative customer kills here one positive customer at the end of the queue. We note that the Markov process can be used with some loss of generality as compared with [9] for analysis of the queuing system at the passage from the recurrent flow to the Markov flow. This spares us the need for bulky calculations of the matrices required for analysis of systems by means of the embedded Markov chain introduced in [9]. Thereby, the matrix algorithm obtained in what follows is much more efficient than that of [9].

In the present paper, the multi-server queuing system with infinite buffer, semi-Markovian input flow of positive and negative customers, and Markovian (general) servicing process is under consideration. The arriving negative customer "kills" here one positive customer at the end of the queue.

The main performance characteristics of the system which is a generalization of systems considered in [9] and [10] are determined. We have to note that algorithms developed in the paper could be applicated for practical numerical calculations if one will use exponential models and matrix power series (see, for example, [11]).

2 Description of the Queueing System with Infinite Buffer

First, let us consider a multi-server queueing system with an infinite buffer, semi-Markovian input flow, Markovian service process, and negative customers. Describe the queueing system by the next way.

Semi-Markovian input flow (the generation process) of customers (SM-flow) is defined below. We have semi-Markovian process. This process functions in the finite set of states (phases of servicing) $\{1,\dots,I\}$, $I<\infty$. Distribution of the moments when the process leaps from the i-th phase to the j-th phase is defined by two functions: $a_{1ij}(x)$ and $a_{0ij}(x)$, $i,j=\overline{1,I}$. Under this condition $a_{1ij}(x)$ presents itself the probability that semi-Markovian process leaps from the i-th phase to the j-th phase immediately during time period of x-length less and a positive customer (farther we shall call a positive customer simply by a customer) arrives also. The arriving of the positive (but not negative) customer is the only difference of the function $a_{0ij}(x)$ from

$a_{1ij}(x)$. Hereinafter we shall suppose for simplicity that functions $a_{1ij}(x)$ and $a_{0ij}(x)$ have the derivatives (the densities) $a'_{1ij}(x)$ and $a'_{0ij}(x)$. We shall note the matrices of the elements $a_{1ij}(x)$, $a_{0ij}(x)$, $a'_{1ij}(x)$ and $a'_{0ij}(x)$ through $A_1(x)$, $A_0(x)$, $A'_1(x)$ and $A'_0(x)$.

Let's introduce the next marks:

$$\tilde{a}_i(x) = \sum_{j=1}^{I} [a_{1ij}(x) + a_{0ij}(x)], \;\; i = \overline{1, I},$$

$$\overline{A} = \int_0^\infty x[A'_1(x) + A'_0(x)]\,dx.$$

Also we shall suppose that

$$A(x) = A_1(x) + A_0(x),$$

$$A_k = A_k(\infty), \;\; k = 0, 1,$$

$$A = A(\infty) = A_1 + A_0.$$

It is intended that the matrix A is irreducible and non-periodic, the matrix A_1 is not zero-matrix and all elements of the matrix $\overline{A}$ are finite. Furthermore when we speak about the steady-state distribution per time we shall consider that the customer generation times can not mean jt only, where t is the positive number and $j = 0, 1, \ldots$.

It is possible to find the string vector $\vec{\pi}_{\mathrm{SM}} = (\pi_{\mathrm{SM}1}, \ldots, \pi_{\mathrm{SM}I})$ of the steady-state probabilities of the states for embedded Markov chain of the semi-Markovian input flow from the equilibrium equations system (EES)

$$\vec{\pi}_{\mathrm{SM}} A = \vec{\pi}_{\mathrm{SM}}$$

with the normalization condition

$$\vec{\pi}_{\mathrm{SM}} \vec{1} = 1,$$

where $\vec{1} = (1, \ldots, 1)^T$ is the column vector of unities whose dimension is context-dependent (in the case at hand, it is I).

The steady-state intensities λ_{pos} and λ_{neg} of the input flows of positive and negative customers are defined by formulas

$$\lambda_{\mathrm{pos}} = \frac{1}{\overline{a}} \vec{\pi}_{\mathrm{SM}} A_1 \vec{1}, \quad \lambda_{\mathrm{neg}} = \frac{1}{\overline{a}} \vec{\pi}_{\mathrm{SM}} A_0 \vec{1},$$

where $\overline{a} = \vec{\pi}_{\mathrm{SM}} \overline{A} \vec{1}$ — the mean time between the arrivals customers (of all types) under the steady-state regime of semi-Markovian input flow.

Markovian service process (MSP) of positive customers has the follow sense. There is a positive integer R called the number of servers. Furthermore there is also a set of positive integers (numbers of the servicing phases) J_r, $r = \overline{0, R}$.

If the system has r, $r > R$, customers and the servicing process is in the i-th state, $i = \overline{1, J_R}$, then servicing of one of the R customers will be completed in a "small" time Δ with the probability $\mu_{1,R+1,i,j}\Delta + o(\Delta)$, $j = \overline{1, J_R}$, which is independent on entire process of system operation, the servicing phase will change to the j-th one, the first customer from the queue will taken for servicing, the rest of the customers will shift with retention of their order. Additionally, the servicing phase will change to the j-th with the probability $\mu_{0,R+1,i,j}\Delta + o(\Delta)$, $j = \overline{1, J_R}$, $j \neq i$, which is also independent on the system operation process, but the servicing of any customer will not be completed.

The case of $1 \leq r \leq R$ differs from the above case only that in a "small" time Δ the probability to complete the servicing of customer (all customers are in servers now) is $\mu_{1rij}\Delta + o(\Delta)$, $i = \overline{1, J_r}$, $j = \overline{1, J_{r-1}}$, and that the probability just to change the servicing phase to the j-th one is $\mu_{0rij}\Delta + o(\Delta)$, $i, j = \overline{1, J_r}$, $j \neq i$.

Finally, in the case of $r = 0$ (there are no customers in the system), the servicing phase can change from i-th to j-th in a "small" time Δ with the probability $\mu_{00ij}\Delta + o(\Delta)$, $i, j = \overline{1, J_0}$, $j \neq i$ only.

Additionally, if r, $0 \leq r < R$, customers are in the system and a new (positive) customer arrives, then the servicing phase will change from i-th to j-th with the probability ω_{rij}, $i = \overline{1, J_r}$, $j = \overline{1, J_{r+1}}$, The matrix of the elements ω_{rij} will be denoted by Ω_r. We shall mark that a sum of row elements for matrix Ω_r is equal to unity for any r (the matrix Ω_r is stochastic).

Farther we assume that $J_R = J$ and denote the matrix of the elements μ_{1rij} and μ_{0rij} $r = \overline{1, R}$, by M_{1r} and M_{0r} respectively, the matrices of the elements $\mu_{1,R+1,i,j}$ and $\mu_{0,R+1,i,j}$, by M_1 and M_0 respectively, the matrix of the elements μ_{00ij}, by M_{00}. Also we assume that $\tilde{M} = M_1 + M_0$.

We denote a string vector of steady-state probabilities for the process of changes of the customer servicing phases by $\vec{\pi}_{\mathrm{MSP}} = (\pi_{\mathrm{MSP}1}, \ldots, \pi_{\mathrm{MSP}J})$ under the assumption that there is an infinite number of customers in the system. Then this vector is found from the EES

$$\vec{\pi}_{\mathrm{MSP}} \tilde{M} = \vec{\pi}_{\mathrm{MSP}}$$

with the normalization condition

$$\vec{\pi}_{\mathrm{MSP}} \vec{1} = 1.$$

The negative customers act as follows. If there are $r > R$ (positive) customers in the system, that is, there are queued customers, then the arriving negative customer "kills" the last (positive) queued customer, and these both customers leave the system. If there are $r \leq R$ (positive) customers in the system, that is, there is no queue, then the arriving negative customer just leaves the system without any influence on the system.

Farther we shall suppose that $\rho < 1$, where ρ obeyed the formula

$$\rho = \frac{\lambda_{\mathrm{pos}}}{\lambda_{\mathrm{neg}} + \vec{\pi}_{\mathrm{MSP}} M_1 \vec{1}},$$

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is called the system load. Furthermore we shall consider the parameters of Markovian process to be given by the way that all appearing embedded Markov chains will be irreducible

3 Steady-State Distribution of the Queue

Let's denote the notation.

Let us mark by $B_{kl}(x),\ k \geq 0,\ l = \overline{0, \min\{k, R\}}$, the matrix, whose element $B_{klij}(x),\ i = \overline{1, J_k},\ j = \overline{1, J_l}$, is the conditional probability that there will l customers exactly at the moment x in the system and the servicing phase will the j-th, provided there are k customer at the moment 0, the servicing phase is the i-th and during the period of x-length none of customers will have arrived in the system.

Let $B_k(x),\ \ k \geq 0$, is the matrix whose element $B_{kij}(x),\ i, j = \overline{1, J}$, is the conditional probability that exactly k customers will be served during the period of x-length and the servicing process passes on the j-th phase, provided there are $k + R$ or more customers at the moment 0, there is the i-th servicing phase and during the period of x-length any customers will not arrive in the system.

Let $\tilde{B}_k(x),\ \ k \geq 1$, is the matrix whose element $B_{kij}(x),\ i, j = \overline{1, J}$, is the conditional probability that during the period of x-length k or more customers will be served and at the moment when the k-th customer complete the servicing the process of the servicing will pass on the j-th phase provided there are $k + R$ or more customers at the moment 0, there is the i-th servicing phase and during the period of x-length any customers will not arrive in the system.

The matrices $B_k(x)$ and $B_{kl}(x)$ satisfy the recurrent relations

$$B_{kk}(x) = e^{M_{0k}x},\ \ k = \overline{0, R},$$

$$B_{kl}(x) = \int_0^x e^{M_{0k}y} M_{1k} B_{k-1,l}(x - y)\, dy,$$

$$k = \overline{1, R},\ \ l = \overline{0, k - 1},$$

$$B_0(x) = e^{M_0 x},$$

$$B_k(x) = \int_0^x B_{k-1}(y) M_1 e^{M_0(x-y)} dy,\ \ k \geq 1,$$

$$\tilde{B}_k(x) = \int_0^x B_{k-1}(y) M_1 dy,\ \ k \geq 1,$$

$$B_{kl}(x) = \int_0^x B_{k-R-1}(y) M_1 B_{Rl}(x - y)\, dy,$$

$$k \geq R + 1,\ \ l = \overline{0, R},$$

and the matrix $B^*_{kl}(x)$ is of form

$$B^*_{kl}(x) = B_{kl}(x)\Omega_l.$$

To find the steady-state distribution of the customers number in the queueing system we shall use an embedded Markov chain formed by the numbers of (positive) customers after the arrivals of any customers directly and by phases of the customer generation in these moments.

An element $B_{kl},\ \ k, l \geq 0$, of the matrix B of interphase transitions probabilities for embedded Markov chain has the form

$$B_{kl} = 0,\ \ k \geq 0,\ \ l \geq k + 2,$$

$$B_{k+l,k+1} = B_l,\ \ k \geq R + 1,\ \ l \geq 0,$$

$$B_k = \int_0^\infty A'_1(x) \otimes B_k(x)\, dx,\ \ k = 0, 1,$$

$$B_k = \int_0^\infty \big[A'_1(x) \otimes B_k(x) +$$

$$A'_0(x) \otimes B_{k-2}(x)\big]\, dx,\ \ k \geq 2,$$

$$B_{k,R+1} = \int_0^\infty \big[A'_1(x) \otimes B_{kR}(x) +$$

$$A'_0(x) \otimes B_{k-R-2}(x)\big]\, dx,\ \ k \geq R + 2,$$

$$B_{k,R+1} = \int_0^\infty A'_1(x) \otimes B_{kR}(x)\, dx,\ \ k = R, R + 1,$$

$$B_{kR} = \int_0^\infty \big[A'_1(x) \otimes B^*_{k,R-1}(x) + A'_0(x) \otimes$$

$$B_{k-R-1}(x)\ + A'_0(x) \otimes B_{kR}(x)\big]\, dx,\ \ k \geq R + 1,$$

$$B_{RR} = \int_0^\infty \big[A'_1(x) \otimes B^*_{R,R-1}(x) +$$

$$A'_0(x) \otimes B_{RR}(x)\big]\, dx,$$

$$B_{k,k+1} = \int_0^\infty A'_1(x) \otimes B^*_{kk}(x)\, dx,\ \ k = \overline{0, R - 1},$$

$$B_{kl} = \int_0^\infty \big[A'_1(x) \otimes B^*_{k,l-1}(x) + A'_0(x) \otimes B_{kl}(x)\big]\, dx,$$

$$k \geq 1,\ \ l = \overline{1, \min(R - 1, k)},$$

$$B_{k0} = \int_0^\infty A'_0(x) \otimes B_{k0}(x)\, dx,\ \ k \geq 0,$$

where "$\otimes$" is the symbol of Kronecker matrix product.

We denote by $\vec{p}_r,\ \ r \geq 0$, the string vector whose coordinates p_{rn}, where $n = \overline{1, IJ_r}$ for $r = \overline{0, R - 1}$ and $n = \overline{1, IJ}$ for $r \geq R$, are the steady-state probabilities that there are r customers in the system and the phases of the processes of customer generation and servicing are i and j respectively. To apply Kronecker matrix product farther it is marked here that $n = (\vec{e}_i' \otimes \vec{e}_j'')\vec{r}_r$, $\vec{r}_r = (1, 2, \ldots, IJ_r)^T$ is column vector of size IJ_r or IJ, and $\vec{e}_i'$ and $\vec{e}_j''$ are the string vectors of sizes I and J_r or J, where the i-th and j-th coordinates are ones, the rest coordinates are zeros.

To find of the steady-state probabilities for embedded Markov chain the EES is of the form

$$\vec{p}_0 = \sum_{r=0}^{\infty} \vec{p}_r B_{r0}, \tag{1}$$

$$\vec{p}_r = \sum_{n=r-1}^{\infty} \vec{p}_n B_{nr}, \;\; r = \overline{1, R+1}, \tag{2}$$

$$\vec{p}_r = \sum_{n=r-1}^{\infty} \vec{p}_n B_{r-n-1}, \;\; r \geq R+2,$$

with the normalization condition

$$\sum_{r=0}^{\infty} \vec{p}_r \vec{1} = 1. \tag{3}$$

Under $r > R+1$ the steady-state probabilities $\vec{p}_r$ is of the form (see [12])

$$\vec{p}_r = \vec{p}_{R+1} G^{r-R-1}, \;\; r > R+1, \tag{4}$$

where the matrix G is the unique solution of the equation

$$G = \sum_{k=0}^{\infty} G^k B_k, \tag{5}$$

whose all eigenvalues are less than one.

Supposing that

$$G_{(n)} = \sum_{k=0}^{\infty} G_{(n-1)}^k B_k$$

the equation (5) can be solved numerically by the iterations method. The zero matrix is suitable as the zero iteration $G_{(0)}$. Then the iterative procedure is the non-decreasing matrix sequence converge to the solution of the equation (5) i.e. to the matrix G.

The rest unknown vectors $\vec{p}_r$, $r = \overline{0, R+1}$, are found from the equations (1), (2). Then

$$\vec{p}_0 = \sum_{r=1}^{R} \vec{p}_r B_{r0} + \vec{p}_{R+1} \tilde{B}_0,$$

$$\vec{p}_r = \sum_{n=r-1}^{R} \vec{p}_n B_{n,r-1} + \vec{p}_{R+1} \tilde{B}_r, \;\; r = \overline{1, R+1},$$

where

$$\tilde{B}_r = \sum_{n=R+1}^{\infty} G^{n-R-1} B_{nr}, \;\; r = \overline{0, R+1}.$$

Herewith the normalization condition (3) takes the form

$$\left(\sum_{r=0}^{R} \vec{p}_r + \vec{p}_{R+1}(1-G)^{-1} \right) \vec{1} = 1.$$

In conclusion of this section let's cite the formulas for calculation of string vector $\vec{p}_r^*$, $r \geq 0$, whose coordinates p_{rn}^*, where $n = \overline{1, IJ_r}$ under $r = \overline{0, R-1}$ and $n = \overline{1, IJ}$ under $r \geq R$, are the steady-state probabilities in arbitrary moments (per time) that there are r customers in the system, and the servicing and generation phases are i and j respectively. Either as previously it is marked here that $n = (\vec{e}_i' \otimes \vec{e}_j'')\vec{r}_r$, $\vec{r}_r = (1, 2, \ldots, IJ_r)^T$ is the string vector of size IJ_r or IJ, and $\vec{e}_i'$ and $\vec{e}_j''$ the string vectors of sizes I and J_r or J, whose the i-th and the j-th coordinates are ones respectively, and the rest coordinates are the zeros. These formulas have the next form:

$$\vec{p}_r^* = \frac{1}{\bar{a}} \sum_{n=r}^{\infty} \vec{p}_n \int_0^{\infty} A^{(d)}(x) \otimes B_{nr}(x)\, dx, \;\; r = \overline{0, R},$$

$$\vec{p}_r^* = \frac{1}{\bar{a}} \sum_{n=r}^{\infty} \vec{p}_n \int_0^{\infty} A^{(d)}(x) \otimes B_{n-r}(x)\, dx, \;\; r \geq R+1,$$

where $A^{(d)}(x)$ is a diagonal matrix with the elements $1 - \tilde{a}_i(x)$ on the major diagonal.

4 The Steady-State Distribution of Waiting Time

Primarily let's define the steady-state probability π_{kil} of the (positive) customer "killing". It should be mentioned that that this probability does not depend on what queued customer will be "killed" by the negative arriving customer. Since the customer is "killed" in the case only if the negative customer arrives and catches $k \geq R+1$ customers in the system then on changing the state of the embedded Markov chain under the condition that $r \geq R+1$ customers had been in the system at the previous moment when the embedded Markov chain varied its state the "killing" probability is

$$\sum_{k=R+1}^{r} \int_0^{\infty} A_0'(x) \otimes B_{r-k}(x)\, dx\, \vec{1} =$$

$$\int_0^{\infty} [A_0'(x)\vec{1}] \otimes [\vec{1} - \tilde{B}_{r-R}(x)\vec{1}]\, dx, \;\; r \geq R+1.$$

Hence with provision for (4) we have

$$\pi_{\text{kil}} = \sum_{r=R+1}^{\infty} \vec{p}_r \int_0^{\infty} [A_0'(x)\vec{1}] \otimes [\vec{1} - \tilde{B}_{r-R}(x)\vec{1}]\, dx =$$

$$\vec{p}_{R+1} \sum_{r=0}^{\infty} G^r \int_0^{\infty} [A_0'(x)\vec{1}] \otimes [\vec{1} - \tilde{B}_{r-R}(x)\vec{1}]\, dx.$$

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Now let's consider some abstract auxiliary queueing system with an infinite buffer and semi-Markovian control. This system consists of the same semi-Markovian input flow of positive and negative customers like for the source system but herewith the positive customers do not service and merely agglomerate in the system and each negative customer "kills" exactly one positive customer.

Via $q_{ij}(x),\ i,j=\overline{1,I}$, let's denote the probability that the busy period (BP) of auxiliary queueing system will less than x and the customers generation process passes to the j-th phase after completion of BP provided BP begins under the i-th phase immediately after a positive customer arrives in the system. Denoting the matrix with elements $q_{ij}(x)$ via $Q(x)$ we have

$$Q'(x)=A'_0(x)+\int\limits_0^x A'_1(y)\,dy\int\limits_0^{x-y}Q'(z)Q'(x-y-z)\,dz. \qquad (6)$$

It is necessary to note that if the system load $\rho=\lambda_{\mathrm{pos}}/\lambda_{\mathrm{neg}}$ is more than one in the auxiliary queueing system then BP of this system is unowned casual value, i.e. all coordinates of the vector $Q(x)\vec{1}$ is strictly less one.

We mark also that using the next notation in terms of Laplace–Stieltjes transform

$$\hat{Q}(s)=\int\limits_0^\infty e^{-sx}Q'(x)\,dx,$$

$$\hat{A}_1(s)=\int\limits_0^\infty e^{-sx}A'_1(x)\,dx,$$

$$\hat{A}_0(s)=\int\limits_0^\infty e^{-sx}A'_0(x)\,dx$$

the equation (6) has the form

$$\hat{Q}(s)=\hat{A}_0(s)+\hat{A}_1(s)\hat{Q}^2(s).$$

Let's turn to the source queueing system. The steady-state probability $\vec{p}_r^{\,+}$ that there will r (positive) customers in the system after a (positive) customer will arrive is of the form

$$\vec{p}_r^{\,+}=\frac{1}{\vec{\pi}_{\mathrm{SM}}A_1\vec{1}}\sum_{n=r-1}^{\infty}\vec{p}_n\int\limits_0^\infty A'_1(x)\otimes B_{n,r-1}(x)\,dx,\quad r=\overline{1,R+1},$$

$$\vec{p}_r^{\,+}=\frac{1}{\vec{\pi}_{\mathrm{SM}}A_1\vec{1}}\sum_{n=r-1}^{\infty}\vec{p}_n\int\limits_0^\infty A'_1(x)\otimes B_{r-n-1}(x)\,dx,\quad r\ge R+2.$$

We find now the steady-state distribution $F(x)$ of the sojourn time in the system for "killed" customer. If there are $r\ge R+1$ customers in the system after the arriving of positive customer then the density of the distribution of the sojourn time for this customer till the moment of the "killing" is of the form

$$\sum_{k=R+1}^{r}[Q'(x)\otimes B_{r-k}(x)]\,\vec{1}=[Q'(x)\vec{1}]\otimes[\vec{1}-\tilde{B}_{r-R}(x)\vec{1}],\quad r\ge R+1.$$

Thence we get that the steady-state density of distribution $F'(x)$ of the sojourn time in the system for "killed" customer is given by

$$F'(x)=\frac{1}{\pi_{\mathrm{kil}}}\sum_{r=R+1}^{\infty}\vec{p}_r^{\,+}[Q'(x)\vec{1}]\otimes[\vec{1}-\tilde{B}_{r-R}(x)\vec{1}].$$

Let's turn to definition of the steady-state distribution $W(x)$ of waiting time for a "non-killed" customer. If there are $r\ge R+1$ customers in the system after the arriving of positive customer then the density of the distribution of the sojourn time for this customer is of the form

$$[(E-Q(x))\,\vec{1}]\otimes[\tilde{B}'_{r-R}(x)M_1\,\vec{1}]=[\vec{1}-Q(x)\vec{1}]\otimes[B_{r-R-1}(x)M_1\,\vec{1}],\quad r\ge R+1.$$

Thus $W(x)$ is given by

$$W(x)=1-\frac{1}{1-\pi_{\mathrm{kil}}}\sum_{r=R+1}^{\infty}\vec{p}_r^{\,+}\int\limits_x^\infty[\vec{1}-Q(y)\vec{1}]\otimes[B_{r-R-1}(y)M_1\,\vec{1}]\,dy,\quad x>0.$$

In particular the steady-state probability π_{imm} that the arriving "non-killed" customer begins to service immediately is of the form

$$\pi_{\mathrm{imm}}=1-\frac{1}{1-\pi_{\mathrm{kil}}}\sum_{r=R+1}^{\infty}\vec{p}_r^{\,+}\times\int\limits_0^\infty[\vec{1}-Q(x)\vec{1}]\otimes[B_{r-R-1}(x)M_1\,\vec{1}]\,dx.$$

5 Queueing System with Finite Buffer

Let's consider now the same queueing system like earlier, but with a finite buffer of S-capacity. A customer arriving in the filling system (all servers and the whole buffer are busy) is lost.

We shall not examine the cases of buffer lack and buffer with one place that are given formulas with some differences from the other cases shall intend farther that $S>1$.

The steady-state distribution exists in the system with finite buffer under any (finite) load ρ.

Introducing either as previously an embedded Markov chain formed by numbers of (positive) customers immediately after the moments when (any) customer arrives in the system and by phases of customers generation and servicing (now this chain has already a finite set of states) we get the next EES to calculate the steady-state probabilities of embedded Markov chain:

$$\vec{p}_0 = \sum_{r=0}^{R+S} \vec{p}_r B_{r0},$$

$$\vec{p}_r = \sum_{n=r-1}^{R+S} \vec{p}_n B_{nr}, \;\; r = \overline{1, R+1},$$

$$\vec{p}_r = \sum_{n=r-1}^{R+S} \vec{p}_n B_{r-n-1}, \;\; r = \overline{R+2, R+S},$$

with normalization condition

$$\sum_{r=0}^{R+S} \vec{p}_r \vec{1} = 1,$$

where the matrices $B_{kl}, \;\; k,l = \overline{0, R+S}$, are defined by the same formulas like ones were used earlier with the exclusion of the matrix $B_{R+S,R+S}$ having the form

$$B_{R+S,R+S} = B_1 + B_0.$$

The steady-state probabilities of states at arbitrary moments are given by the formulas

$$\vec{p}_r^* = \frac{1}{a} \sum_{n=r}^{R+S} \vec{p}_n \int_0^\infty A^{(d)}(x) \otimes B_{nr}(x)\, dx, \;\; r = \overline{0, R},$$

$$\vec{p}_r^* = \frac{1}{a} \sum_{n=r}^{R+S} \vec{p}_n \int_0^\infty A^{(d)}(x) \otimes$$
$$B_{n-r}(x)\, dx, \;\; r = \overline{R+1, R+S}.$$

The steady-state probabilities π_{kil} of the "killing" of customer and π_{los} of the customer lost due to buffer overflow are defined by the next formulas

$$\pi_{\text{kil}} = \sum_{r=R+1}^{R+S} \vec{p}_r \int_0^\infty [A_0'(x)\, \vec{1}] \otimes [\vec{1} - \tilde{B}_{r-R}(x)\, \vec{1}]\, dx,$$

$$\pi_{\text{los}} = \vec{p}_{R+S} \int_0^\infty [A_1'(x)\, \vec{1}] \otimes [B_0(x)\, \vec{1}]\, dx.$$

The steady-state probability $\vec{p}_r^{\,+}$ meaning that there are r positive customers in the system after an arrival of a (positive) customer is (with provision for phases of customers generation and servicing at this moment)

$$\vec{p}_r^{\,+} = \frac{1}{\vec{\pi}_{\text{SM}} A_1 \vec{1}} \sum_{n=r-1}^{R+S} \vec{p}_n \int_0^\infty A_1'(x) \otimes$$
$$B_{n,r-1}(x)\, dx, \;\; r = \overline{1, R+1},$$

$$\vec{p}_r^{\,+} = \frac{1}{\vec{\pi}_{\text{SM}} A_1 \vec{1}} \sum_{n=r-1}^{R+S} \vec{p}_n \int_0^\infty A_1'(x) \otimes$$
$$B_{r-n-1}(x)\, dx, \;\; r = \overline{R+2, R+S-1},$$

$$\vec{p}_{R+S}^{\,+} = \frac{1}{\vec{\pi}_{\text{SM}} A_1 \vec{1}} \Bigg(\vec{p}_{R+S-1} \int_0^\infty A_1'(x) \otimes B_0(x)\, dx +$$
$$\vec{p}_{R+S} \int_0^\infty A_1'(x) \otimes [B_0(x) + B_1(x)]\, dx \Bigg).$$

Let $\vec{F}_{rn}(x), \;\; r = \overline{1, S}, \;\; n = \overline{0, S-r}$, is probability that a customer will be "killed" moreover till the moment x provided at the initial moment the customer is the r-th in the queue and there are n customers after this one in the queue yet (a record in the form of a column vector takes into account the phases of customer generation and servicing at the initial moment). Then

$$\vec{F}_{r0}'(x) = [A_0'(x)\, \vec{1}] \otimes [\vec{1} - \tilde{B}_r(x)\, \vec{1}] +$$
$$\int_0^x \sum_{n=0}^{r-1} [A_1'(y) \otimes B_n(y)] \vec{F}_{r-n,1}'(x-y)\, dy, \;\; r = \overline{1, S-1},$$

$$\vec{F}_{rn}'(x) = \int_0^x \sum_{k=0}^{r-1} [A_0'(y) \otimes B_k(y)] \vec{F}_{r-k,n-1}'(x-y)\, dy +$$
$$\int_0^x \sum_{k=0}^{r-1} [A_1'(y) \otimes B_k(y)] \vec{F}_{r-k,n+1}'(x-y)\, dy,$$
$$r = \overline{1, S-2}, \;\; n = \overline{1, S-r-1},$$

$$\vec{F}_{S0}'(x) = [A_0'(y)\, \vec{1}] \otimes [\vec{1} - \tilde{B}_S(x)\, \vec{1}] +$$
$$\int_0^x \sum_{n=1}^{S-1} [A_1'(y) \otimes B_n(y)] \vec{F}_{S-n,1}'(x-y)\, dy +$$
$$\int_0^x [A_1'(y) \otimes B_0(y)] \vec{F}_{S0}'(x-y)\, dy,$$

$$\vec{F}_{r,S-r}'(x) =$$
$$\int_0^x \sum_{k=0}^{r-1} [A_0'(y) \otimes B_k(y)] \vec{F}_{r-k,S-r-1}'(x-y)\, dy +$$
$$\int_0^x \sum_{k=1}^{r-1} [A_1'(y) \otimes B_k(y)] \vec{F}_{r-k,S-r+1}'(x-y)\, dy +$$
$$\int_0^x [A_1'(y) \otimes B_0(y)] \vec{F}_{r,S-r}'(x-y)\, dy, \;\; r = \overline{1, S-1}.$$

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The steady-state density of distribution $F'(x)$ of sojourn time for "killed" customer is given by the relation

$$F'(x) = \frac{1}{\pi_{\text{kil}}}\left(\sum_{r=1}^{S-1}\vec{p}^{\,+}_{R+r}\vec{F}'_{r0}(x) + \left[\vec{p}_{R+S-1}\int_0^\infty A'_1(y)\otimes B_0(y)\,dy + \vec{p}_{R+S}\int_0^\infty A'_1(y)\otimes B_1(y)\,dy\right]\vec{F}'_{S0}(x)\right).$$

Let's denote via $\vec{W}_{rn}(x)$, $r=\overline{1,S}$, $n=\overline{0,S-r}$, the probability that a customer will be served (i.e. a customer will not be "killed" by a negative customer and will not be lost on account of the buffer overflow) and will wait to start the service during the time period which is less than x provided at the initial moment this customer occupies the r-th place in the queue and n customers stay behind it in the queue yet (with provision for phases of customers generation and servicing at the initial moment). Then the next relations hold good

$$\vec{W}'_{r0}(x) = [\vec{1}-A(x)\vec{1}]\otimes[B_{r-1}(x)M_1\,\vec{1}] + \int_0^x\sum_{n=0}^{r-1}[A'_1(y)\otimes B_n(y)]\vec{W}'_{r-n,1}(x-y)\,dy,\ \ r=\overline{1,S-1},$$

$$\vec{W}'_{rn}(x) = [\vec{1}-A(x)\vec{1}]\otimes[B_{r-1}(x)M_1\,\vec{1}] + \int_0^x\sum_{k=0}^{r-1}[A'_0(y)\otimes B_k(y)]\vec{W}'_{r-k,n-1}(x-y)\,dy + \int_0^x\sum_{k=0}^{r-1}[A'_1(y)\otimes B_k(y)]\vec{W}'_{r-k,n+1}(x-y)\,dy,$$

$$r=\overline{1,S-1},\ \ n=\overline{1,S-r-1},$$

$$\vec{W}'_{S0}(x) = [\vec{1}-A(x)\vec{1}]\otimes[B_{S-1}(x)M_1\,\vec{1}] + \int_0^x\sum_{n=1}^{S-1}[A'_1(y)\otimes B_n(y)]\vec{W}'_{S-n,1}(x-y)\,dy + \int_0^x[A'_1(y)\otimes B_0(y)]\vec{W}'_{S0}(x-y)\,dy,$$

$$\vec{W}'_{r,S-r}(x) = [\vec{1}-A(x)\vec{1}]\otimes[B_{r-1}(x)M_1\,\vec{1}] + \int_0^x\sum_{k=0}^{r-1}[A'_0(y)\otimes B_k(y)]\vec{W}'_{r-k,S-r-1}(x-y)\,dy + \int_0^x\sum_{k=1}^{r-1}[A'_1(y)\otimes B_k(y)]\vec{W}'_{r-k,S-r+1}(x-y)\,dy + \int_0^x[A'_1(y)\otimes B_0(y)]\vec{W}'_{r,S-r}(x-y)\,dy,\ \ r=\overline{1,S-1}.$$

The steady-state distribution $W(x)$ of waiting time for served customer is given by formula

$$W(x) = 1-\int_x^\infty\frac{1}{1-\pi_{\text{kil}}-\pi_{\text{los}}}\left(\sum_{r=R+1}^{S-1}\vec{p}^{\,+}_r\vec{W}'_{r0}(y) + \left[\vec{p}_{R+S-1}\int_0^\infty A'_1(z)\otimes B_0(z)\,dz + \vec{p}_{R+S}\int_0^\infty A'_1(z)\otimes B_1(z)\,dz\right]\vec{W}'_{S0}(y)\right)dy.$$

In particular the steady-state probability π_{imm} that an arriving (served) customer begins to service at once has the form

$$\pi_{\text{imm}} = 1-\int_0^\infty\frac{1}{1-\pi_{\text{kil}}-\pi_{\text{los}}}\left(\sum_{r=R+1}^{S-1}\vec{p}^{\,+}_r\vec{W}'_{r0}(y) + \left[\vec{p}_{R+S-1}\int_0^\infty A'_1(z)\otimes B_0(z)\,dz + \vec{p}_{R+S}\int_0^\infty A'_1(z)\otimes B_1(z)\,dz\right]\vec{W}'_{S0}(y)\right)dy.$$

6 Queueing System with Phase Distribution for Input Flow of Positive and Negative Customers

Now we show how the queueing system with phase distribution for the input flow of positive and negative customers which was considered in [10] can be led to the system researched here.

Let the new queueing system differs from the system examined above only that the arrival input flow (the generation process) of negative and positive customers is Markovian, moreover a matrix whose elements are the intensities of arrivals of positive customers is marked through Λ_1, a matrix whose elements are the intensities of arrivals of negative customers is marked through Λ_0 and a matrix whose elements are the intensities of states changes of generation process without an arrival of customer is marked through Λ. Then in order to lead the new system to the system researched earlier it is necessary to suppose that

$$A'_0(x) = e^{\Lambda x}\Lambda_0,\quad A'_1(x) = e^{\Lambda x}\Lambda_1. \tag{7}$$

The other formulas stays without any changes.

Let the general Markovian input flow of positive and negative customers is superposition of two independent Markovian flows (the flow of positive and the flow of negative customers) and at the same time the flow of positive customers is defined by the matrices Λ_{pos} and $\tilde{\Lambda}_{\text{pos}}$ of order I_1 with elements that are intensities of phase changes for the generation process of positive customers with and without an arrival of customer respectively, but the flow of negative customers is defined by the matrices Λ_{neg} and $\tilde{\Lambda}_{\text{neg}}$ of order I_2 whose elements are intensities of phase changes for the generation process of negative customers with and without an arrival of customer respectively. To lead this system to the queueing system which is considered above it should be assumed in this case (see [10]) that

$$\Lambda_1 = \Lambda_{\text{pos}} \otimes E_{I_2},$$

$$\Lambda_0 = E_{I_1} \otimes \Lambda_{\text{neg}},$$

$$\Lambda = \tilde{\Lambda}_{\text{pos}} \oplus \tilde{\Lambda}_{\text{neg}} = \tilde{\Lambda}_{\text{pos}} \otimes E_{I_2} + E_{I_1} \otimes \tilde{\Lambda}_{\text{neg}},$$

and then to apply the formula (7) mentioned above.

It should be noted that the matrix algorithm for the queueing system with phase distribution of input flow of positive and negative customers which is obtained in [10] is greatly more effective than this algorithm proposed here since that one rids of the rather bulky calculation of matrices that are required under the analysis of the systems by means of embedded Markov chain.

7 Queueing System with Phase Distribution for Service of Customers

Let us consider a multi-server (R-server) queuing system with the SEMI-Markovian input flow of positive and negative customers, phase distribution (PH-distribution) of the customer servicing time, and the aforementioned order of action of the negative customers. We assume that the PH-distribution $H(x)$ of the customer servicing time by each server is characterized by an irreducible PH-representation (h, H) of the order K (see, for example, [11]).

Let us see how this system can be reduced to the aforementioned general queuing system. To this end, one can define the initial parameters of the queuing system in the following way. Let us assume that $J_r = K^r$, $J = K^R$, $H^* = \vec{h} \otimes \vec{1}$. We denote by $H^{(d)}$ the diagonal matrix with the diagonal elements $h_{ii}^{(d)} = \sum_{j=1}^{K} h_{ij}$. The matrices of the elements M_{1r}, M_{0r}, M_1 and M_0 are defined as follows

$$M_{1r} = [H^{(d)}\vec{1}] \otimes E_K \otimes \ldots \otimes E_K +$$
$$E_K \otimes [H^{(d)}\vec{1}] \otimes \ldots \otimes E_K + \ldots +$$
$$E_K \otimes E_K \otimes \ldots \otimes [H^{(d)}\vec{1}], \;\; r = \overline{1, R},$$

where the number of addends and the number of multipliers in each addend are equal to r,

$$M_1 = [H^{(d)}H^*] \otimes E_K \otimes \ldots \otimes E_K +$$
$$E_K \otimes [H^{(d)}H^*] \otimes \ldots \otimes E_K + \ldots +$$
$$E_K \otimes E_K \otimes \ldots \otimes [H^{(d)}H^*],$$

where the number of addends and the number of multipliers are equal to R,

$$M_{0r} = H \otimes E_K \otimes \ldots \otimes E_K +$$
$$E_K \otimes H \otimes \ldots \otimes E_K + \ldots +$$
$$E_K \otimes E_K \otimes \ldots \otimes H, \;\; r = \overline{0, R-1},$$

where the number of addends and the number of multipliers are equal to r,

$$M_0 = H \otimes E_K \otimes \ldots \otimes E_K +$$
$$E_K \otimes H \otimes \ldots \otimes E_K + \ldots + E_K \otimes E_K \otimes \ldots \otimes H,$$

where the number of addends and the number of multipliers are equal to R,

$$\Omega_r = \frac{1}{r}(H^* \otimes E_K \otimes \ldots \otimes E_K +$$
$$E_K \otimes H^* \otimes \ldots \otimes E_K + \ldots +$$
$$E_K \otimes E_K \otimes \ldots \otimes H^*), \;\; r = \overline{0, R-1},$$

where the number of addends and the number of multipliers are equal to $r+1$.

The total time of customer sojourn in a phase-type system is consists of the waiting time and the servicing time proper. Therefore, the stationary distribution $V(t)$ of this time for the serviced customer obeys the formula

$$V(x) = \int_0^x W(x-y)\, dH(y).$$

We note that the above numeration of the states leads to an extremely great number of states J even for a small number of servers R. A much more economic method of numeration can be found in [12].

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9 References

[1] E. Gelenbe. Random neural networks with positive and negative signals and product form solution. *Neural Computation*, 1:502–510, 1989.

[2] E. Gelenbe. Reseaux neuronaux aleatoires stables. *Comptes-Rendus Acad. Sci. II*, 310:177–180, 1990.

[3] E. Gelenbe and G. Pujolle. *Introduction to queueing networks*. John Wiley, New York, 1998.

[4] J.R. Artalejo. G-networks: A versatile approach for work removal in queueing networks. *Eur. J. Oper. Res.*, 126:233–249, 2000.

[5] P.P. Bocharov and V.M. Vishnevskii. G-networks: development of the theory of multiplicative networks. *Automation and Remote Control*, 64:714–739, 2003.

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[6] F. Baskett, K.M. Chandy, R.R. Muntz et al. Open, closed and mixed networks of queues with different classes of customers. *J. ACM*, 22:248–260, 1975.

[7] Y. Shin. Multiserver retrial queue with negative customers and disasters. In B. Choi, editor, *Proc. of the Fifth Internat. Workshop on Retrial Queues*, pages 53–60, Seoul, 2005. Korea University.

[8] V. Anisimov and J. Artalejo. Analysis of Markov multiserver retrial queues with negative arrivals. *Queueing Systems*, 39:157–182, 2001.

[9] V.V. Chaplygin. Stationary characteristics of queueing system $G/MSP/n/r$ with flow of negative customers. *Information processes*, 5(1):1–19, 2005 (in Russian).

[10] P.P. Bocharov, C. D'Apice, R. Manzo and A.V. Pechinkin. Analysis of the multi-server Markov queuing system with unlimited buffer and negative customers. *Automation and Remote Control*, 68(1):85–94, 2007.

[11] P.P. Bocharov, C. D'Apice, A.V. Pechinkin and S. Salerno. *Queueing Theory*. VSP, Utrecht, Boston, 2004.

[12] A.V. Pechinkin and V.V. Chaplygin. Stationary Characteristics of the $SM/MSP/n/r$ Queuing System. *Automation and Remote Control*, 65(9):1429-1443, 2004.

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