MATHEMATICAL MODELS OF THE AUJESZKY DISEASE IN THE CUNEO PROVINCE FARMS, ITALY.

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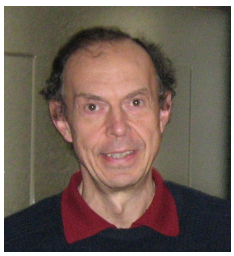
Abstract

This work has been carried out in strict collaboration with the veterinarians studying the Aujeszky disease (A.D.), [1]. This disease is caused by the Herpevirus 1 suis (ADV or SHV-1). Field data concerning the blood samples collected according to the law (D.M. April 1st, 1997) in 1997-2004 for the serologic exam for A.D. of breeding animals of the Cuneo province have been examined. For each breeding farm birth rates have been determined. Also, mortality rates have been subdivided into those connected with Aujeszky disease and those which were not. The Villafalletto and Vottignasco towns where the farms are located has a swine density exceeding 3200 units per Km^2 and a total number of 90000 units. These facts allow to take the area as a single giant raising farm. On the basis of these assumptions we have formulated and analyzed mathematical models for the description of the disease evolution, to determine strategies for its desirable if at all possible control, and also indirectly for simulating the human intervention, which if inappropriate may adversely affect the disease spread. In this context, the effects of biosafety measures, together with the vaccination policy according to the regulations in effect, are also considered. The ultimate goal would be the realisation of a disease eradication plan with nonprohibitive costs. We started from a well-known and accepted epidemics models for realistic situations, [2], in which also the total population is not constant, [3, 4], contrary to the basic assumption of the classical epidemiological model, [5]. We then modified these models to take into account the possible fluxes between susceptible, "vaccinated" and infected animals, incorporating the lack of biosafety measures and possibly mimicking also the farmer's behavior who does not fully comply with correct vaccination policies. We discuss the outcomes of our analysis in terms of possible policies to contain the epidemics.

Keywords: epidemics, Aujeszky disease, biosafety, vaccination.

Presenting Author's Biography



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1 Introduction

The Aujeszky disease (A.D.) is caused by the Herpevirus 1 suis (ADV or SHV-1), affecting several wild and domestic species, but in particular hogs. It is not lethal, but it causes several disorders in the affected animals and ultimately it constitutes an economical burden for the farmer. Once contracted the disease, the infected animal cannot recover from it. Almost every country in Europe is affected. An eradication policy based on vaccination has been attempted in Italy in the past ten years, with mixed results.

Here we formulate and analyze a mathematical model possibly aimed at defining the realisation of a disease eradication plan at nonprohibitive costs. To determine strategies for the desirable if at all possible disease control, we begin by modeling the description of the evolution of the latter. A partial result has already been obtained, [6]. Noteworthy in the previous study is the modeling of the absence of biosafety measures, which may allow disease infiltration into a non affected farm by external vectors and not by direct contact between infected and susceptible animals. In this investigation we extend the previous model by taking into account in the simulation the human intervention, which if inappropriate adversely affects the disease spread.

2 Methods

This study is an attempt to investigate the situation using mathematical methods, in strict collaboration with the veterinarians studying the disease on the field, [1]. We have considered breeding farms in the area of the towns Villafalletto and Vottignasco in the Cuneo province in Piedmont, NW Italy. This part of the region is considered as a single giant epidemiological unit, since the swine density exceeds 3200 units per Km^2 with a total number of 90000 units.

Blood sample data collected according to the law (D.M. April 1st, 1997) in the period 1997-2004 for the serologic exam for A.D. constitute the basis for our analysis. From these and for each breeding farm birth rates have been determined together with natural and disease-related mortalities. These informations have been used in the model to give to the relevant parameters reasonably accurate numeric values, to validate the subsequent analysis and simulations.

From the mathematical point of view, in [6] we started from a well-known and accepted epidemics model for realistic situations, [2]. But contrary to the assumptions of the classical epidemiological model, [5], we allowed the total population to reproduce, as done in more recent models for disease spread, [3, 4]. The model studied in [6] has been here modified to include one more important feature. Based on the fact that disease prevalence went down after the first three years of law implementation, to come up again in the years 2000-2004, it is indeed argued whether the vaccination is in the end at all useful. This may be due to an intrinsic weakness of the vaccine, or to the bad implementation on the part of the farmers. The vaccine should indeed be administered three times in the lifetime of the animal, the first two times in the first months of life, the third one when the animal is about one year and a half old. Since by that time the farmer is in general ready to sell it, it may happen that the farmer avoids to administer the third vaccination, maybe thinking it is then unnecessary and to save on its costs. Whatever the cause, in any case the distinguishing feature of this model is here represented by the introduction of the class of animals on which the vaccine is ineffective, allowing possible fluxes between susceptible, "vaccinated" and infected animals. We discuss the outcomes of our analysis in terms of possible policies to contain the epidemics.

3 The model

In formulating the model we then take into account the following basic variables. First of all we consider the susceptible animals, $S(t)$. The latter is then subdivided into two further classes, the class of "formally" vaccinated animals, $V(t)$ and the one of susceptible or unvaccinated animals, $U(t)$. Then there is the infected class, $I(t)$. We allow transitions among these classes assuming that the vaccine does not always have a full effect, or is not implemented correctly. The transitions are described in the following equations. Notice that the disease is unrecoverable, so that once infected, an animal carries it for life, no transition back from the class I to either U or V is allowed.

We also assume that all individuals reproduce and newborns at birth are susceptible, due to some immunization gotten from the mother, there is no possibility of vertical transmission of the disease. Of course they will lose this immunity as they grow older. They then all belong to class U or V at birth, with respective birth rates ρ_U and ρ_V.

We consider then the following model

$$\begin{aligned} \dot{U} &= \rho_U N - \mu_S U - \beta_U \frac{UI}{N} - \tau U + \alpha V - \sigma U \\ \dot{I} &= \beta_U \frac{UI}{N} + \beta_V \frac{VI}{N} + \tau U + \tau V - \mu_I I \\ \dot{V} &= \rho_V N - \mu_S V - \beta_V \frac{VI}{N} - \tau V - \alpha V + \sigma U \end{aligned} \tag{1}$$

in addition to

$$N = U + V + I. \tag{2}$$

The first equation says that all newborns coming from parents of whichever class are born sound, i.e. they are susceptible to the disease. The class of susceptibles is subject to natural mortality μ_S. Some of its members migrate to the class V of the vaccinated at rate σ, but either for ineffectiveness of the vaccine or faults in its implementation we assume they can migrate back from class V into the susceptibles at rate α. This is clearly modeled by the last two terms of the first equation (1). The disease affects them via the incidence β, which expresses contagion of a susceptible upon direct contact with an infected animal, while the parameter τ expresses the fact that the susceptible can get infected also by other means, by vectors carried into its environment

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by external factors. Thus in a sense this parameter models the biohazards, which should be tackled by suitable biosafety measures in the farm.

The second equation gives the dynamics of infected individuals. They enter class I via direct contact at the rate β specified above, or via an external vector at rate τ; in both cases they can enter class I either from class U or from class V if the vaccine is ineffective. Finally they are subject this time to disease-related mortality μ_I.

The third equation (1) describes the evolution of the vaccinated class. The first term again represents the newborns, then there is the natural mortality term, the disease incidence which may affect also these animals, if the vaccine is not so effective, again the infection caused by external factors, and finally the vaccination at rate σ and the loss of immunization at rate α.

The available parameter values from veterinarians field measuments are $N = 90000$ total hogs population in the epidemiological unit, $\mu_S = 0.084$ represents the average natural mortality, $\mu_I = 0.087$ is the average disease-related mortality, $\rho = 0.107$ represents the mean birth rate in the whole epidemiological unit. Since the field measurements provide only a lump natality rate ρ, we assume the birth rates as follows

$$\rho_U = \rho\frac{U}{N}, \quad \rho_V = \rho\frac{V}{N}. \tag{3}$$

The remaining parameter description is as mentioned above, with β_U and β_V denoting the horizontal disease incidences respectively for the classes of susceptibles and vaccinated, τ the absence of biosafety measures, α the loss of immunity due to failure in the vaccine or in its administration, therefore it is a migration rate into the class of susceptibles from the class of vaccinated, and σ the vaccination rate, expressing a migration from U into V.

We now introduce three new variables, given by the subpopulations fractions u,i e v, namely

$$u = \frac{U}{N}, \quad i = \frac{I}{N}, \quad v = \frac{V}{N}. \tag{4}$$

From (2) it then follows

$$u + v + i = 1. \tag{5}$$

Notice that the position (4) entails for instance that upon differentiation

$$\dot{u} = \frac{\dot{U}}{N} - \frac{U}{N}\frac{\dot{N}}{N} = \frac{\dot{U}}{N} - \frac{U}{N}\left[\frac{\dot{U}}{N} + \frac{\dot{V}}{N} + \frac{\dot{I}}{N}\right], \tag{6}$$

and similarly for the other fractions. Thus substituting from (1) we obtain

$$\begin{aligned}\dot{u} = \mu_S u^2 + ui(\mu_I - \beta_U) + \rho_U + \mu_S uv + \alpha v \\ -u(\mu_S + \tau + \sigma + \rho_U + \rho_V), \\ \dot{v} = \mu_S v^2 - v(\mu_S + \tau + \alpha + \rho_U + \rho_V) \\ +vi(\mu_I - \beta_V) + \rho_V + \sigma u + \mu_S vu, \\ \dot{i} = \mu_I i^2 + ui(\mu_S + \beta_U) + vi(\beta_V + \mu_S) \\ -i(\mu_I + \rho_V + \rho_U) + \tau u + \tau v.\end{aligned} \tag{7}$$

Then by using the assumption on the birth rates (3) into (7) we have

$$\begin{aligned}\dot{u} = \mu_S u^2 - u(\mu_S + \tau + \sigma) + \mu_S vu \\ +ui(\mu_I - \beta_U + \rho) + \alpha v, \\ \dot{v} = \mu_S v^2 + vi(\rho + \mu_I - \beta_V) + \mu_S uv \\ +\sigma u - v(\mu_S + \tau + \alpha), \\ \dot{i} = i^2(\mu_I + \rho) - i(\mu_I + \rho + \tau) \\ +ui(\beta_U + \mu_S) + vi(\beta_V + \mu_S) + \tau.\end{aligned} \tag{8}$$

Finally on eliminating the variable i from (5) we have the reduced model description via the equations

$$\begin{aligned}\dot{u} = (\mu_S - \mu_I - \rho + \beta_U)u^2 \\ +uv(\mu_S - \mu_I - \rho + \beta_U) \\ -u(\mu_S + \tau + \sigma - \rho - \mu_I + \beta_U) + \alpha v, \\ \dot{v} = v^2(\mu_S - \rho - \mu_I + \beta_V) + \sigma u \\ +uv(\mu_S - \rho - \mu_I + \beta_V) \\ +v(\rho - \mu_S - \tau - \alpha + \mu_I - \beta_V)\end{aligned} \tag{9}$$

In view of the constraint (5), we seek the solutions of (9) in the unit simplex $\Omega \equiv \{(u, v) \in \mathbf{R}^2 : 0 \le u, v \le 1\}$.

4 Analysis

Let us now seek the system equilibria. By equating to zero the right hand sides of (9) we find the following equations

$$\begin{aligned}(\mu_S - \mu_I - \rho + \beta_U)u^2 + \alpha v \\ +uv(\mu_S - \mu_I - \rho + \beta_U) \\ -u(\mu_S + \tau + \sigma - \rho - \mu_I + \beta_U) = 0\end{aligned} \tag{10}$$

$$\begin{aligned}v^2(\mu_S - \rho - \mu_I + \beta_V) + \sigma u \\ +uv(\mu_S - \rho - \mu_I + \beta_V) \\ +v(\rho - \mu_S - \tau - \alpha + \mu_I - \beta_V) = 0.\end{aligned} \tag{11}$$

These represent conic sections. To study the first conic, we consider its invariants

$$\Delta_1 \equiv \begin{vmatrix} (\Delta_1)_{1,1} & (\Delta_1)_{1,2} & (\Delta_1)_{1,3} \\ (\Delta_1)_{2,1} & 0 & (\Delta_1)_{2,3} \\ (\Delta_1)_{3,1} & (\Delta_1)_{3,2} & 0 \end{vmatrix}$$

with elements given by

$$\begin{aligned}(\Delta_1)_{1,1} = \mu_S - \rho + \beta_U - \mu_I \\ (\Delta_1)_{1,2} = (\Delta_1)_{2,1} = \frac{\mu_S - \rho + \beta_U - \mu_I}{2} \\ (\Delta_1)_{1,3} = (\Delta_1)_{3,1} = \frac{\rho - \mu_S - \tau - \beta_U + \mu_I - \sigma}{2} \\ (\Delta_1)_{2,3} = (\Delta_1)_{3,2} = \frac{\alpha}{2}\end{aligned}$$

and

$$\delta_1 \equiv \begin{vmatrix} (\Delta_1)_{1,1} & (\Delta_1)_{1,2} \\ (\Delta_1)_{2,1} & 0 \end{vmatrix}$$

Evaluating the determinants, then we find

$$\Delta_1 = \frac{\alpha}{4}[-\alpha(\mu_S - \rho + \beta_U - \mu_I) + (\mu_S - \rho - \mu_I + \beta_U)(\rho - \mu_S - \tau - \sigma + \mu_I + \beta_U)] = \frac{\alpha}{4}(\mu_S - \rho - \mu_I + \beta_U) \times (\rho - \mu_S - \tau - \sigma + \mu_I + \beta_U - \alpha)$$

and

$$\delta_1 = -\frac{\alpha}{4}(\mu_S - \rho + \beta_U - \mu_I)^2.$$

Since $\delta_1 < 0$ it is therefore a hyperbola. Its center is the point (u^*, v^*) where

$$u^* \equiv \frac{1}{\delta_1}\begin{vmatrix} (\Delta_1)_{1,2} & (\Delta_1)_{1,3} \\ (\Delta_1)_{2,2} & (\Delta_1)_{2,3} \end{vmatrix} = -\frac{\alpha}{\mu_I - \mu_S - \beta_U + \rho},$$

$$v^* \equiv \frac{1}{\delta_1}\begin{vmatrix} (\Delta_1)_{1,3} & (\Delta_1)_{1,1} \\ (\Delta_1)_{2,3} & (\Delta_1)_{2,1} \end{vmatrix} = \frac{\mu_S + \tau + \sigma - \mu_I - \rho + \beta_U + 2\alpha}{\mu_S - \rho - \mu_I + \beta_U}$$

The asymptotes of this hyperbola are given by

$$(\mu_S - \rho + \beta_U - \mu_I)u^2 + (\mu_S - \rho - \mu_I + \beta_U)uv + (\rho - \mu_S - \tau - \sigma + \mu_I - \beta_U)u + \alpha v - \frac{\Delta_1}{\delta_1} = 0, \quad (12)$$

where

$$\frac{\Delta_1}{\delta_1} = \frac{\alpha^2 - \alpha(\rho - \mu_S - \sigma + \mu_I - \beta_U)}{\mu_S - \rho - \mu_I + \beta_U}.$$

To find them explicitly, upon division of (12) by $\mu_S - \rho - \mu_I + \beta_U$, we find

$$T(u,v) = u^2 + \frac{\rho - \mu_S - \tau - \sigma + \mu_I - \beta_U}{\mu_S - \rho - \mu_I + \beta_U}u + vu + \frac{\alpha}{\mu_S - \rho - \mu_I + \beta_U}v - \frac{\alpha^2 - \alpha(\rho - \tau - \mu_S - \sigma + \mu_I - \beta_U)}{(\mu_S - \rho - \mu_I + \beta_U)^2} = 0.$$

Let us assume $T(u,v)$ to be the product of two linear functions with undetermined coefficients, so that

$$T(u,v) = (\widetilde{A}u + \widetilde{B}v + \widetilde{C})(\widetilde{D}u + \widetilde{E}v + \widetilde{F}) = 0.$$

Upon equating coefficients of like powers, we find that the following equations must be satisfied by $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, $\widetilde{D}$, $\widetilde{E}$, $\widetilde{F}$,

$$u^2: \quad \widetilde{A}\widetilde{D} = 1, \quad (13)$$

$$uv: \quad \widetilde{A}\widetilde{E} + \widetilde{B}\widetilde{D} = 1,$$

$$v^2: \quad \widetilde{B}\widetilde{E} = 0,$$

$$u: \quad \widetilde{A}\widetilde{F} + \widetilde{D}\widetilde{C} = \frac{(\rho - \tau - \mu_S - \sigma + \mu_I - \beta_U)}{(\mu_S - \rho - \mu_I + \beta_U)}$$

$$v: \quad \widetilde{B}\widetilde{F} + \widetilde{C}\widetilde{E} = \frac{\alpha}{(\mu_S - \rho - \mu_I + \beta_U)}$$

$$1: \quad \widetilde{F}\widetilde{C} = \frac{\alpha^2 - \alpha(\rho - \tau - \mu_S - \sigma + \mu_I - \beta_U)}{(\mu_S - \rho - \mu_I + \beta_U)^2}.$$

Without loss of generality, taking for instance $\widetilde{E} = 0$, to satisfy the second above equation, we have then the straight lines in the form

$$u = -\frac{\widetilde{F}}{\widetilde{D}}, \quad v = -\frac{\widetilde{A}}{\widetilde{B}}u - \frac{\widetilde{C}}{\widetilde{B}}.$$

We find now their coefficients as follows. From the fifth and the second of (13), since $\widetilde{E} = 0$ we have

$$\frac{\widetilde{F}}{\widetilde{D}} \equiv \frac{\widetilde{F}\widetilde{B}}{\widetilde{D}\widetilde{B}} = -\frac{\alpha}{\mu_S + \beta_U - \mu_I - \rho}.$$

so that the first asymptote is

$$u = -\frac{\alpha}{\mu_S + \beta_U - \mu_I - \rho}. \quad (14)$$

Then the first two equations of (13) give

$$\frac{\widetilde{A}}{\widetilde{B}} \equiv \frac{\widetilde{A}\widetilde{D}}{\widetilde{B}\widetilde{D}} = 1$$

and the last two in turn yield

$$\frac{\widetilde{C}}{\widetilde{B}} \equiv \frac{\widetilde{F}\widetilde{C}}{\widetilde{B}\widetilde{F}} = \frac{\alpha - (\rho - \mu_S - \sigma - \tau + \mu_I - \beta_U)}{(\mu_S - \rho - \mu_I + \beta_U)}.$$

Thus the second asymptote is

$$v = -u + \frac{\alpha - (\rho - \mu_S - \sigma - \tau + \mu_I - \beta_U)}{(\mu_S - \rho - \mu_I + \beta_U)}. \quad (15)$$

The intersections with the coordinate axes of the hyperbola (10) are the origin and the points $v = 0$ and the roots of the quadratic

$$u^2(\mu_S - \rho + \beta_U - \mu_I) + u(\rho - \mu_S - \tau - \sigma + \mu_I - \beta_U) = 0$$

which are explicitly

$$u = 0, \quad u = -\frac{(\rho - \mu_S - \tau - \sigma + \mu_I - \beta_U)}{(\mu_S - \rho + \beta_U - \mu_I)}.$$

We study now the conic (11). Its invariants can be determined as follows.

$$\Delta_2 \equiv \begin{vmatrix} 0 & (\Delta_2)_{1,2} & (\Delta_2)_{1,3} \\ (\Delta_2)_{2,1} & (\Delta_2)_{2,2} & (\Delta_2)_{2,3} \\ (\Delta_2)_{3,1} & (\Delta_2)_{3,2} & 0 \end{vmatrix}$$

has the elements

$$(\Delta_2)_{1,2} = (\Delta_2)_{2,1} = \frac{\mu_S - \rho - \mu_I + \beta_V}{2},$$

$$(\Delta_2)_{1,3} = (\Delta_2)_{3,1} = \frac{\sigma}{2},$$

$$(\Delta_2)_{2,2} = \mu_S - \rho - \mu_I + \beta_V,$$

$$(\Delta_2)_{2,3} = (\Delta_2)_{3,2} = \frac{\rho + \mu_I - \mu_S - \tau - \alpha - \beta_V}{2},$$

from which upon evaluation, we find

$$\Delta_2 = \frac{\sigma}{4}(\mu_S - \rho - \mu_I + \beta_V) \times(\rho - \mu_S - \tau - \alpha + \mu_I - \beta_V - \sigma).$$

The second invariant is

$$\delta_2 \equiv \begin{vmatrix} 0 & (\Delta_2)_{1,2} \\ (\Delta_2)_{2,1} & (\Delta_2)_{2,2} \end{vmatrix} = -\frac{(\mu_S - \rho + \beta_V - \mu_I)^2}{4}.$$

Again $\delta_2 < 0$ shows that also (11) is a hyperbola. To find its center we use once again the invariant method

$$u_2 \equiv \frac{1}{\delta_2}\begin{vmatrix} (\Delta_2)_{1,2} & (\Delta_2)_{1,3} \\ (\Delta_2)_{2,2} & (\Delta_2)_{2,3} \end{vmatrix} = -\frac{-(\rho - \mu_S - \tau - \alpha + \mu_I - \beta_V) + 2\sigma}{\mu_S - \rho - \mu_I + \beta_V}$$

and

$$v_2 \equiv \frac{1}{\delta_2}\begin{vmatrix} (\Delta_2)_{1,3} & 0 \\ (\Delta_2)_{2,3} & (\Delta_2)_{2,1} \end{vmatrix} = -\frac{\sigma}{\mu_S - \rho - \mu_I + \beta_V}.$$

The asymptotes are found from

$$(\mu_S - \rho - \mu_I + \beta_V)v^2 + (\mu_S - \rho - \mu_I + \beta_V)uv + (\rho - \mu_S - \tau - \alpha + \mu_I - \beta_V)v + \sigma u - \frac{\Delta_2}{\delta_2} = 0$$

where

$$\frac{\Delta_2}{\delta_2} = -\frac{\sigma(\rho - \mu_S - \tau - \alpha + \mu_I - \beta_V) + \sigma^2}{(\mu_S - \rho - \mu_I + \beta_V)}.$$

Substitution into the above equation thus yields

$$(\mu_S - \rho - \mu_I + \beta_V)v^2 + (\mu_S - \rho - \mu_I + \beta_V)uv + (\rho - \mu_S - \tau - \alpha + \mu_I - \beta_V)v + \sigma u + \frac{\sigma(\rho - \mu_S - \tau - \alpha + \mu_I - \beta_V) + \sigma^2}{(\mu_S - \rho - \mu_I + \beta_V)} = 0.$$

To explicitly determine the asymptotes, upon division by $\mu_S - \rho - \mu_I - \beta_V$ we have

$$Q(u,v) = v^2 + \frac{(\rho - \mu_S - \tau - \alpha + \mu_I - \beta_V)}{(\mu_S - \rho - \mu_I + \beta_V)}v + uv + \frac{\sigma}{(\mu_S - \rho - \mu_I + \beta_V)}u + \frac{\sigma(\rho - \mu_S - \tau - \alpha + \mu_I - \beta_V) + \sigma^2}{(\mu_S - \rho - \mu_I + \beta_V)^2} = 0.$$

Again let us take $Q(u,v)$ in factored form

$$Q(u,v) \equiv (Gu + Hv + I)(Lu + Mv + N) = 0.$$

Equating like powers of the variables, we thus find

$$\begin{aligned} u^2: &\quad GL = 0, \\ uv: &\quad GM + HL = 1, \\ u: &\quad GN + IL = \frac{\sigma}{A}, \\ v: &\quad HN + IM = \frac{B}{A}, \\ v^2: &\quad HM = 1, \\ 1: &\quad IN = \frac{\sigma E}{A^2}. \end{aligned} \tag{16}$$

In these equations we have used the following shorthands

$$\begin{aligned} C = \mu_S - \rho - \mu_I = -0.110 < 0, \\ A = C + \beta_V, \\ D = C + \beta_U, \\ \Gamma = C + \tau, \\ B = \Gamma + \alpha + \beta_V = A + \alpha, \\ E = B - \sigma, \\ F = \Gamma + \sigma + \beta_U = D + \sigma. \end{aligned} \tag{17}$$

In this way $Q(u,v)$ can be rewritten as

$$Q(u,v) = v^2 + \frac{B}{A}v + uv + \frac{\sigma}{A}u + \frac{\sigma E}{A^2}$$

Let us take $G = 0$, without loss of generality. We have then the asymptotes

$$v = -\frac{I}{H}, \quad v = -\frac{L}{M}u - \frac{N}{M}.$$

The second and third equations (16) give

$$\frac{I}{H} = \frac{IL}{HL} = \frac{\sigma}{A}.$$

From the second and fifth equation (16) we have

$$\frac{L}{M} = \frac{HL}{HM} = 1.$$

Also using the second, the sixth, the fifth and the third equation (16) we have

$$\frac{N}{M} = \frac{HL}{HM}\frac{IN}{IL} = \frac{E}{A}.$$

The asymptotes have then the equations

$$v = -u + \frac{E}{A}, \quad v = -\frac{\sigma}{A}. \tag{18}$$

5 Discussion

Let us now introduce new notations, to simplify the subsequent discussion. Using (17) the conics can be rewritten as

$$Du^2 - Fu + Duv + \alpha v = 0 \tag{19}$$

$$Av^2 - Bv + Auv + \sigma u = 0 \tag{20}$$

The center of (19) becomes

$$u_0 = -\frac{\alpha}{D}, \quad v_0 = 1 + \frac{\tau + \sigma + 2\alpha}{D}.$$

Set

$$u_* = 1 + \frac{\tau + \sigma}{D}, \quad v_0 = u_* - 2u_0,$$

its axes intersections are then the origin and $(u_*, 0)$. Its asymptotes are finally

$$u = u_0, \quad v = -u + 1 + \frac{\tau + \sigma - \alpha}{C + \beta_U} = -u + u_* + u_0.$$

Studying the intersection with the straight line $u+v = 1$ we find

$$\bar{u} = \frac{\alpha}{\tau + \alpha + \sigma} < 1, \quad \bar{v} = \frac{\tau + \sigma}{\tau + \alpha + \sigma} < 1.$$

Set now

$$\tilde{v}_* = \frac{\tau + \alpha + 2\sigma}{A}.$$

The center of (20) becomes

$$\tilde{u}_0 = 1 + \frac{\tau + \alpha + 2\sigma}{A} = \tilde{v}_* - 2\tilde{v}_0, \quad \tilde{v}_0 = -\frac{\sigma}{A},$$

its axes intersections are then the origin and $(0, \tilde{v}_*)$. Its asymptotes are finally

$$v = \tilde{v}_0, \quad v = -u - \tilde{v}_* + \tilde{v}_0.$$

Studying the intersection with $u + v = 1$ we have

$$\tilde{v} = \frac{\sigma}{\tau + \alpha + \sigma}, \quad \tilde{u} = \frac{\tau + \alpha}{\tau + \alpha + \sigma}.$$

To study the flow in the unit simplex Ω we need to determine the mutual positions of the hyperbolae (19) and (20). To do this, we can discriminate between their slopes at the origin, on top of using the informations summarized above. In particular we find that the slope at the origin of (19) is larger than the one of (20) if the following inequalities are satisfied

$$\begin{aligned} C + \beta_V + \tau + \alpha &> 0 \\ C + \beta_U + \tau + \sigma &> \frac{\alpha\sigma}{C + \beta_V + \tau + \alpha} > 0, \end{aligned} \tag{21}$$

or

$$\begin{aligned} B &< 0 \\ F &> \frac{\alpha\sigma}{B}, \\ \frac{\alpha\sigma}{B} &< 0. \end{aligned} \tag{22}$$

Conversely the slope at the origin of (20) is larger than the one of (19) if the following inequalities are satisfied

$$\begin{aligned} B &< 0 \\ F &< \frac{\alpha\sigma}{B} < 0, \end{aligned} \tag{23}$$

or

$$\begin{aligned} B &> 0 \\ F &< \frac{\alpha\sigma}{B}, \\ \frac{\alpha\sigma}{B} &> 0. \end{aligned} \tag{24}$$

On the border $u + v = 1$ of the unit simplex Ω in the uv phase plane the flow is directed upwards (increasing v) if

$$u < u^\dagger \equiv \frac{1}{2}\frac{\tau - 2\alpha}{\tau + \sigma - \alpha}. \tag{25}$$

Notice that $u^\dagger < \frac{1}{2}$ if and only if $\tau + \sigma > \alpha$. The point $(u^\dagger, 1 - u^\dagger)$ on the line $u + v = 1$ represents thus a saddle. Above it the flow goes upwards, below it goes downwards.

We need finally to determine the flow inside the unit simplex in the uv phase plane. To this end the informations on the two conics (19) and (20) need to be merged. There are several pictures that can be drawn corresponding to several cases of possible intersections among the two curves and positions of the other relevant points on the coordinate axes.

6 Conclusions

We summarize the ultimate behavior of the system here below, identifying when possible its ω-limit points. There are some cases in which the system trajectories naturally evolve toward the line $u+v = 1$ which corresponds to the disease-free environment, since it means $i = 0$ in the three dimensional phase space uvi, recalling (5). These are the equilibria we should strive for.

Some instances in which they are found are as follows.

For $D > 0$, $A > 0$, $\tilde{u}_0 > 0$, $\tilde{v}_0 > 1$ and $u_* > 1$ in cases (21) or (22) an internal saddle point arises, the origin is a stable equilibrium, implying from $u = v = 0$ that $i = 1$ i.e. the epidemics spreads to the whole population. But then there is also a stable equilibrium on the line $i = 0$ so that it is enough that trajectories lie in its basin of attraction for the disease to vanish.

An endemic stable equilibrium is found instead for the case $C + \beta_V < 0$, $C + \beta_U > 0$ with $\tilde{u} > \bar{u}$. But in the same situation instead with $\tilde{u} < \bar{u}$ the stable equilibrium moves on the line $u + v = 1$.

Other situations leading to the same final outcome are $A < 0$, $D > 0$, $\tilde{v}_0 > 0$, $0 < \tilde{v}_* < \tilde{v}_0$, $\tilde{u} < \bar{u}$.

Also $A < 0$, $D > 0$, $\tilde{v}_0 < 0$, $\tilde{v}_* < 0 < \tilde{v}_0$, $\tilde{u} < \bar{u}$.

Again $A < 0$, $D > 0$, $\tilde{v}_0 > 0$, $1 < \tilde{v}_*$, $0 > \tilde{v}_0$, $\tilde{u} < \bar{u}$.

One more $D < 0$, $A < 0$, $\tilde{u} < \bar{u}$.

But also $D < 0$, $A < 0$, $\tilde{u} > \bar{u}$.

Again $D > 0$, $A > 0$, $1 < \tilde{v}_*$, $\tilde{u} < \bar{u}$.

And $D < 0$, $A < 0$, $\tilde{u}_0 > 0$, $1 < \tilde{u}_*$, $\tilde{u} < \bar{u}$.

Finally for $D < 0$, $A > 0$, $\tilde{v}_0 > 0$, $\tilde{u} < \bar{u}$ and for $D < 0$, $A < 0$, $\tilde{u}_0 < 0$, $\tilde{v}_* < 0 < \tilde{v}_0$, $\tilde{u} > \bar{u}$.

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In all possible cases, the above result show that it would be possible to eradicate the epidemics by acting appropriately on the relevant parameters of the model, so as to satisfy the conditions leading to stable disease-free equilibria. Moreover there is also the possibility of choosing which parameters to act upon, so that the above inequalities are satisfied. This allows some freedom for the policy maker in the choice of the most appropriate means of fighting the epidemics. In particular there would be the possibility of better enforcing the vaccination program, so as to augment σ and at the same time decrease α, or rather to act on preventive measures, such as to counteract the biohazards which are prone to spread the disease horizontally. This can be implemented by taking suitable biosafety restrictions, so as to diminish the disease incidence β_U and β_V, and also to reduce the possibility of importing the disease through external vectors, thus obtaining a smaller τ.

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