SIMULATION OF SINGLE INPUT DISTRIBUTED PARAMETER SYSTEMS VIA VIRTUAL CONTROLLER DESIGN

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Abstract

For the analysis of dynamic systems and model-based controller design highly accurate models are required. They often consist of partial differential equations. Furthermore, in most cases an analytical solution of these equations cannot be found and a numerical method (e. g. finite element method etc.) has to be applied to obtain an adequate approximation. For the purpose of model-based controller design the resulting high dimensional state space models (systems of first order ordinary differential equations) are not convenient. Therefore, an order reduction scheme has to be applied, yielding a suitable low order description. In this report, an alternative way of numerical modeling and simulation of single input systems, described by linear partial differential equations, is presented. The exact solution is approximated using low order polynomials by means of the mean weighted residual method. These polynomials are defined only at a small part of the domain and the problem is divided into many different subproblems. The coupling of the subproblems is achieved by introducing constraints. The global solution is described by a differential-algebraic matrix equation (descriptor system). It can be solved by introducing a proportional state feedback, using pole assignment with output coupling. The concept will be evaluated by application on a beam equation.

Keywords: Pole Placement, Output Coupling, Order reduction, Distributed Parameter Systems, Partial Differential Equation.

Presenting Author's Biography

Jan Schlake received his Diploma in mathematics from the Technical University of Clausthal, Germany, in 2004. Currently he is working toward his Ph.D. degree in numerical modeling and model based control theory of distributed parameter systems at the electrical engineering department of the Technical University of Darmstadt, Germany.

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1 Introduction

In the last decades the possibilities to simulate complex systems have been improved continually. Today it is common to use CAD-Tools and to generate models of partial differential equations (PDE) before starting with real experiments. For increasingly complex geometries the standard methods to simulate these systems are based on finite element methods (FEM).
The problem to solve the global problem is divided into many subproblems, while the solution of these subproblems can be approximated by a linear combination of low order polynomials. Hence, by refining the mesh of the domain Ω, nearly any magnitude of accuracy of the approximated solution can be achieved, leading to large linear matrix equations [1, 2].

From this data many of the commercial FEM-programs offer the possibility to create highly accurate state space models

$$\begin{aligned} \mathbf{E}\,\dot{\mathbf{x}} &= \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\,\mathbf{x} \end{aligned} \tag{1}$$

of a previously chosen order N with constant matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{E}$. The matrix $\mathbf{E}$ has full rank N, so that the system can be simulated with standard (implicit) methods. These models can also be the basis for further investigations of the system dynamics with classical methods of control theory [3]. Using controller design schemes like pole placement these models can be used to achieve a desired dynamic behaviour. The advantage of this procedure is the precise separation of numerical modeling and analysis of system dynamics/controller design.

In contrast to simulation, in case of controller design only the output at some certain points (and not the whole solution) is of interest. For most controller design schemes the order of the state space system (1) has to be quite moderate. Therefore, the accuracy of the numerical solution has to be traded off in order to obtain a suitable order of the state space model (1). Problematic in this procedure is the fact that the control engineer has to decide whether the resulting state space model delivers a sufficient approximation of the original system. In fact, a lot of order reduction schemes exist that indicate how to chose the "right" order. Furthermore, a lot of research has been carried out to improve these methods, too (see [4] et al.).

Another problem is the fact that the states have no physical meaning (especially after applying order reduction schemes) and cannot be interpreted in any other way. The input/output behaviour of the state space system (1) only approximates the input/output behaviour of the original system. Therefore, in some cases it might happen that a controller based on such a model can lead to an unexpected behaviour of the closed loop system caused by the spill over effect e. g. [5].

In most instances, there is no need to have the same accuracy in every part $K \subset \Omega$ of the domain Ω for controller design. Some parts are more important for the input/ouput behaviour than others. This a priori knowledge can already be used during the generation phase of the numerical models and should result in a model of lower order. Formally, in FEM-notation, this is equivalent to the use of different elements in every part K.

In the following sections a simple method which combines the numerical modeling for simulation and the controller design is presented. A procedure equivalent to the freeing process for a body in mechanics leads to a description by a descriptor model or differential-algebraic equations (DAE) as in eq. (1) with a rank of the matrix $\mathbf{E}$ less than N.
The introduction of static boundary and coupling constraints allows a very flexible way of numerical modeling. For every part K a different numerical approximation method can be used. It can be shown that the controller design for DAE can be interpreted as a pole assignment problem with output coupling. Besides the assignment of all finite poles this procedure also allows for controllable systems governed by PDEs the arbitrary assignment of all infinite poles. Therefore, it gives the guarantee that all inconsistent states $\mathbf{x}$ can be obliterated.
It turns out that simulation is a special case of controller design where the poles and the static gain of the closed loop system and the descriptor system (1) are chosen identically.
Finally, the procedure will be demonstrated by the example of a onesided clamped beam.

2 Problem statement

The scope of the proposed simulation method is restricted to problems described by linear PDEs without mixed derivatives of time t and space z

$$D_t w + D_z w = u_\Omega \quad \text{on} \quad \Omega \tag{2}$$

$$B w = u_\Gamma \quad \text{on} \quad \Gamma\,, \tag{3}$$

where Ω is a subset of $\mathbb{R}^3$, $\Gamma = \partial\Omega$ the boundary of Ω,

$$\begin{aligned} D_t &= \sum_{k=0}^{k_1} \alpha_k \frac{\partial^k}{\partial t^k},\; D_z = \sum_{k=0}^{k_2} \beta_k(z) \frac{\partial^k}{\partial z^k}, \\ B &= \sum_{k=0}^{k_2-1} \gamma_k \frac{\partial^k}{\partial z^k} \end{aligned} \tag{4}$$

are linear differential operators, u_Γ and $u_\Omega := u(z)\bar{u}(t)$ are input quantities and $w \in W$ the solution at time $t \in \mathbb{R}^+$ and space $z \in \Omega \subset \mathbb{R}^3$. Since only single input systems are treated in this article only $\bar{u}$ or one component of u_Γ is unequal zero at any time t.
Because all operators (4) are linear the PDE (2) with inhomogeneous boundary conditions (3) can be transformed into a PDE with homogenous boundary conditions [6],

$$B w = 0 \quad \text{on} \quad \Gamma\,. \tag{5}$$

The approach proposed in the following sections is restricted to operators D_t with order $k_1 = 1$. The extension to higher order problems is unproblematic, but requires more formalism.

3 Simulation

3.1 Boundary Value Problem

The formulation (2) without mixed derivatives in t and z allows the application of line methods. Discretization with respect to space z by any method for boundary value problems (BVP) results in a matrix differential equation (1). Then, the finite approximation of the infinite dimensional solution w of the PDE (2) is represented by the state vector $\mathbf{x}$. Depending on the chosen method the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{E}$ and the vector $\mathbf{x}$ have different structure and meaning.

A common approach for a numerical solution of BVPs is the approximation of the solution w by finite linear combinations

$$\hat{w}(t,z) = \sum_{j=1}^{N} \bar{w}_j(t)\phi_j(z) \tag{6}$$

of pre-chosen basis functions $\phi_j(z)$. The quality of the approximation depends on the space

$$W_\mathrm{h} = \left\{ \hat{w} | \hat{w} = \sum_{j=1}^{N} \alpha_j \phi_j(z)\,; \alpha_j \in \mathbb{R}; B\phi_j = 0 \right\} .$$

The better W_h "approximates" the solution space W of functions which fulfil the homogenous boundary conditions (5) ($W_\mathrm{h} \subset W$), the closer comes the approximation $\hat{w} \in W_\mathrm{h}$ to the solution $w \in W$.

Multiplication of the PDE (2) with test functions $v \in V$ with support in Ω and the integration

$$D_t \underbrace{\int_\Omega vw\mathrm{d}z}_{=:m(v,w)} + \underbrace{\int_\Omega vD_z w\mathrm{d}z}_{=:a(v,w)} = \underbrace{\int_\Omega v u\mathrm{d}z}_{=:b(v)} \bar{u} \quad \forall v \tag{7}$$

gives a starting point for numerical simulation of eq. (2).

If the space of admissible test functions V and the space W in which the solution is sought are chosen as the corresponding Sobolev-spaces

$$W = V = H_0^{k_2-1}(\Omega) , \tag{8}$$

after application of Green's theorem eq. (7) turns into the well known weak formulation, whereas eq. (2) is known as the strong formulation.

Inserting the ansatz (6) into the weak formulation (7) and choosing N test functions $v_i \in V_\mathrm{h} \subset V$ results in the matrix differential equation

$$\mathbf{E}\,\dot{\mathbf{w}} = \mathbf{A}\,\mathbf{w} + \mathbf{B}\bar{u} \tag{9}$$

where $\mathbf{w} = (\bar{w}_j)_j$ is the state vector, $\bar{u}$ an scalar input, $\mathbf{B} = (b(v_i))_i$ the $N \times 1$ input matrix , $\mathbf{A} = (a(v_i,\phi_j))_{i,j}$ the $N \times N$ system matrix and $\mathbf{E} = (m(v_i,\phi_j))_{i,j}$ the $N \times N$ mass matrix. If the test functions v_i and basis functions ϕ_j respectively are linear independent the matrix $\mathbf{E}$ has full rank.
With the use of ansatz (6) an additional output equation

$$\mathbf{y} = \mathbf{C}\,\mathbf{w} , \tag{10}$$

with the $q \times N$ output matrix $\mathbf{C}$ can be defined to calculate the solution $\hat{w}$ at any q desired points z_k.

If the boundary values should be used for boundary control, the input function u_Ω has singularities in Ω and/or the state space model (9) has a bad condition number, it makes sense to use other spaces for V_h and W_h with less restrictions than in eq. (8). In this case it is not advantageous to use pre-chosen basis functions ϕ_j which fulfil the homogenous boundary conditions.

Instead of performing a transformation of the original PDE (2) into a PDE with homogenous boundary conditions (5), the linear combination $\hat{w}$ of basis functions ϕ_j should directly meet the boundary conditions in eq. (3)

$$B\hat{w} = u_\Gamma \quad \text{on} \quad \Gamma . \tag{11}$$

Therefore, the basis functions $\phi_j \in W_\mathrm{h}$ must be chosen in a more general way. In most cases, due to numerical reasons some kind of polynomial function sets (Chebyshev polynomials T_k [7], Legendre polynomials L_k [6] etc.) are the best choice.

In order to take eq. (11) into consideration "boundary-boarding" technics are applied [7]. The k_2 boundary conditions (11) will be inserted directly into the matrix equation (9) while only $r := N - k_2$ test functions v_i for the determination of the weak solution $\hat{w}$ in eq. (7) have to be used. This easy and appealing approach is strongly related to the introduction of fictive forces in mechanical engineering, thus resulting in a singular matrix $\mathbf{E}$ and in a set of differential-algebraic equations (DAE) or descriptor system (9) with $\mathrm{rank}(\mathbf{E}) = r < N$.

But even if it is possible to choose all basis functions ϕ_j in a way that all boundary conditions are matched, it could be useful to apply this concept [8]. The subdivision of the domain Ω into many small parts $\Omega_K \subset \Omega$ and the usage of different test functions $v_{K,i}(z)$ on every domain leads to different methods of discretization and DAEs

$$\mathbf{E}_K\,\dot{\mathbf{w}}_K = \mathbf{A}_K\,\mathbf{w}_K + \mathbf{B}_K\,\bar{u}_K ,$$

with $\mathbf{E}_K, \mathbf{A}_K \in \mathbb{R}^{N_K \times N_K}$ and $\mathbf{B}_K \in \mathbb{R}^{N_K}$ in each part Ω_K. Assembling these descriptor subsystems with help of static coupling equations

$$\lim_{„z \nearrow \partial K"} \frac{\partial^i \hat{w}}{\partial z^i} = \lim_{„z \searrow \partial K"} \frac{\partial^i \hat{w}}{\partial z^i} , \; i = 0, \ldots \tag{12}$$

into a global system (9) with an order of $N = \sum N_K$ is a highly flexible way of constructing a numerical model for linear PDEs.

Disadvantageous of the boundary boarding technics is that it results in a singular mass (or descriptor) matrix $\mathbf{E}$. This leads to an increased complexity in the simulation and the control of the matrix equation (9).

Due to the implementation of boundary conditions (11) and/or coupling equations (12), the descriptor system (9) has an index $\kappa \geq 1$. The index κ is a measure for the singularity of a descriptor system. For a more profound definition of the index and related topics see [9, 10].

Depending on the index κ the characteristic equation

$$\det\left[s\mathbf{E} - \mathbf{A}\right] = 0$$

of the descriptor system (9) has $r - m$ finite and $N - r + m$ infinite eigenvalues $\lambda_i \in \Lambda$, respectively.

There exist several numerical methods for the simulation of DAEs with an index $\kappa \geq 1$ (see [10, 11, 12] for example). Besides the input quantity $\bar{u}$ most of them require the calculation of one or more partial derivatives by time, too. This demand excludes the application of many test signals like step functions etc. in the analysis of the systems dynamic.

In the following sections 3.2 and 3.3 a controller design method for R-controllable descriptor systems with any index κ is introduced which allows the assignment of all eigenvalues Λ_R of the closed loop system. The method results simultaneously in a stable index reduction and a reduction of the system order. For a certain choice of the parameters of the controller, the proposed method can also be used for simulation of a DAE.

3.2 Descriptor System

With implementation of a proportional state feedback

$$\bar{u} = -\mathbf{R}\,\mathbf{w} + \mathbf{F}\,\breve{u} \tag{13}$$

for the descriptor system (9) with constant matrices $\mathbf{R} \in \mathbb{R}^{1\times N}$ and $\mathbf{F} \in \mathbb{R}^{1\times q}$ it is possible to reduce the index κ [12] and to assign r eigenvalues $\lambda_{\mathrm{R}i} \in \Lambda_\mathrm{R}$ to the closed loop system

$$\mathbf{E}\,\dot{\mathbf{w}} = (\mathbf{A} - \mathbf{B}\,\mathbf{R})\,\mathbf{w} + \mathbf{B}\,\mathbf{F}\,\breve{u}\,. \tag{14}$$

This fact is used in [13] for controller design of R-controllable linear multi input multi output (MIMO) descriptor systems (1). For simplification following calculations will be restricted without lack of generality to distinct eigenvalues $\lambda_{\mathrm{R}i} \in \Lambda_\mathrm{R}$.

Through numerical robust singular value decomposition (SVD [4]) of matrix $\mathbf{E}$, DAE. (9) can always be transformed into the differential-algebraic canonical form

$$\begin{aligned}\dot{\hat{\mathbf{x}}}_1 &= \mathbf{A}_{11}\hat{\mathbf{x}}_1 + \mathbf{A}_{12}\hat{\mathbf{x}}_2 + \mathbf{B}_1\bar{u}\\ 0 &= \mathbf{A}_{21}\hat{\mathbf{x}}_1 + \mathbf{A}_{22}\hat{\mathbf{x}}_2 + \mathbf{B}_2\bar{u}\\ \mathbf{y} &= \mathbf{C}_1\hat{\mathbf{x}}_1 + \mathbf{C}_2\hat{\mathbf{x}}_2\end{aligned} \tag{15}$$

with transformed descriptor variables

$$\hat{\mathbf{x}} = \left[\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2\right]^\mathrm{T}$$

of the DAE. The first equation describes the dynamic changes of the descriptor variables $\hat{\mathbf{x}}_1 \in \mathbb{R}^r$. $\hat{\mathbf{x}}_2 \in \mathbb{R}^{N-r}$ depends directly on $\bar{u}$ and $\hat{\mathbf{x}}_1$, according to the second equation (15). Thus, this equation ensures that the descriptor vector $\hat{\mathbf{x}}$ is consistent.

Introduction of a new "fictitious" input vector

$$\tilde{\mathbf{u}} := \left[\hat{\mathbf{x}}_2, \bar{u}\right]^\mathrm{T} \in \mathbb{R}^{N-r+1}$$

and new system matrices

$$\begin{aligned}\tilde{\mathbf{B}} &:= [\mathbf{A}_{12}, \mathbf{B}_1] \in \mathbb{R}^{r\times N-r+1}\,,\\ \tilde{\mathbf{C}} &:= \mathbf{A}_{21} \in \mathbb{R}^{N-r\times r}\,,\\ \tilde{\mathbf{D}} &:= [\mathbf{A}_{22}, \mathbf{B}_2] \in \mathbb{R}^{N-r\times N-r+1}\,,\\ \mathbf{D}_1 &:= [\mathbf{C}_2, 0] \in \mathbb{R}^{q\times N-r+1}\end{aligned}$$

yields the representation

$$\begin{aligned}\dot{\hat{\mathbf{x}}}_1 &= \mathbf{A}_{11}\,\hat{\mathbf{x}}_1 + \tilde{\mathbf{B}}\,\tilde{\mathbf{u}}\\ \mathbf{0} &= \tilde{\mathbf{C}}\,\hat{\mathbf{x}}_1 + \tilde{\mathbf{D}}\,\tilde{\mathbf{u}}\\ \mathbf{y} &= \mathbf{C}_1\,\hat{\mathbf{x}}_1 + \mathbf{D}_1\,\tilde{\mathbf{u}}\,.\end{aligned} \tag{16}$$

Compared to the original single input multi output (SIMO) DAE (9), this MIMO representation has a reduced set of $r < N$ state variables. Note that, for $\tilde{\mathbf{D}} = 0$ the second equation is reduced to conditions for the states $\hat{\mathbf{x}}_1$. Then, the system of equations (16) is the problem formulation of pole assignment with a coupling controller.

For $\tilde{\mathbf{D}} \neq \mathbf{0}$ and $\mathrm{rank}(\tilde{\mathbf{D}}) = d$ another application of the SVD for the matrix $\tilde{\mathbf{D}}$

$$\mathbf{S}\,\tilde{\mathbf{D}}\,\mathbf{T} = \begin{bmatrix}\boldsymbol{\Sigma}_\mathrm{d} & 0\\ 0 & 0\end{bmatrix}$$

results in a partition of the second equation of the transformed system (16)

$$\begin{aligned}\mathbf{0} &= \bar{\mathbf{C}}\,\hat{\mathbf{x}}_1 + \boldsymbol{\Sigma}_\mathrm{d}\,\hat{\mathbf{u}}_1\\ \mathbf{0} &= \hat{\mathbf{C}}_2\,\hat{\mathbf{x}}_1\end{aligned} \tag{17}$$

with a regular matrix $\boldsymbol{\Sigma}_\mathrm{d} \in \mathbb{R}^{d\times d}$ and the transformed input

$$\hat{\mathbf{u}} = \begin{bmatrix}\hat{\mathbf{u}}_1\\ \hat{\mathbf{u}}_2\end{bmatrix} = \mathbf{T}^\mathrm{T}\tilde{\mathbf{u}}\,.$$

Inserting the constraints (17) into eq. (16) results in the equivalent formulation

$$\begin{aligned}\dot{\hat{\mathbf{x}}}_1 &= \hat{\mathbf{A}}\,\hat{\mathbf{x}}_1 + \hat{\mathbf{B}}\,\hat{\mathbf{u}}_2\\ \mathbf{0} &= \hat{\mathbf{C}}_2\,\hat{\mathbf{x}}_1\\ \mathbf{y} &= \hat{\mathbf{C}}_1\,\hat{\mathbf{x}}_1 + \hat{\mathbf{D}}_2\,\hat{\mathbf{u}}_2\end{aligned} \tag{18}$$

of the original DAE (9). This problem formulation has $r < N$ states, $N - r + 1 - d$ inputs and q outputs. The matrix $\hat{\mathbf{C}}_2$ only exists for problems with an index greater than 1.
Notice that the transformation of a certain DAE (9) into the problem statement (18) has been performed only by application of the SVD and is therefore numerical

good-natured [4]. It follows directly that if the DAE (9) has r finite eigenvalues, all eigenvalues of the matrix $\hat{\mathbf{A}}$ should be elements of Λ. This is the case for problems with an index $\kappa = 1$. Obviously, for problems with an index greater than 1 the matrix $\hat{\mathbf{A}}$ has more finite eigenvalues than the DAE (9).

Because of the assumed R-controllability of the DAE (9) the state space representation (18) is controllable. Under this assumptions, a control law for coupling a subset of output variables can be found [13]. This method delivers a state feedback law

$$\hat{\mathbf{u}}_2 = -\hat{\mathbf{R}}\,\hat{\mathbf{x}}_1 + \hat{\mathbf{F}}\,\check{\mathbf{u}} \tag{19}$$

for eq. (18). The controller matrix $\hat{\mathbf{R}}$ and input filter matrix $\hat{\mathbf{F}}$ are chosen in a way that the $N - r - d$ constraints for the states $\hat{\mathbf{x}}_1$ given through the matrix $\hat{\mathbf{C}}_2$ are matched and the closed loop system has a desired input/output behaviour.

3.3 Coupling via Pole Assignment

Inserting control law (19) into eq. (18) leads to the equations

$$\begin{aligned} \dot{\hat{\mathbf{x}}}_1 &= [\hat{\mathbf{A}} - \hat{\mathbf{B}}\,\hat{\mathbf{R}}]\,\hat{\mathbf{x}}_1 + \hat{\mathbf{B}}\,\hat{\mathbf{F}}\,\check{\mathbf{u}} \\ \mathbf{y} &= [\hat{\mathbf{C}}_1 - \hat{\mathbf{D}}_2\,\hat{\mathbf{R}}]\,\hat{\mathbf{x}}_1 + \hat{\mathbf{D}}_2\,\hat{\mathbf{F}}\,\check{\mathbf{u}} \end{aligned} \tag{20}$$

of the closed loop system. Fig. 1 shows the corresponding block diagram. If the desired eigenvalues $\lambda_{\mathrm{R}i} \in \Lambda_{\mathrm{R}}$

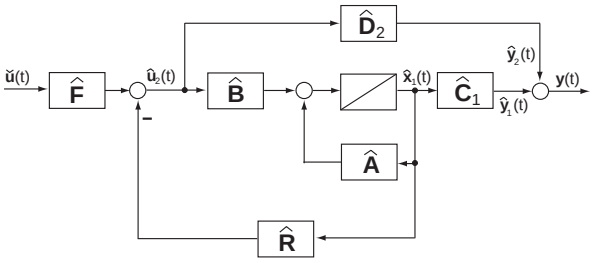


Fig. 1 Block diagram of the controller scheme

are the roots of the characteristic equation

$$\det(s\,\mathbf{I} - \hat{\mathbf{A}} + \hat{\mathbf{B}}\,\hat{\mathbf{R}}) \stackrel{!}{=} \Pi_{i=1}^{n}(s - \lambda_{\mathrm{R}i}) = 0$$

of the closed loop system (20), it is known that every controller matrix $\hat{\mathbf{R}}$ can be written in the form

$$\hat{\mathbf{R}} = \mathbf{P}\,\mathbf{V}_{\mathrm{R}}^{-1} = \mathbf{P}\,\mathbf{W}_{\mathrm{R}}\,, \tag{21}$$

where $\mathbf{V}_{\mathrm{R}}$ is the matrix of (right) eigenvectors $\mathbf{v}_{\mathrm{R}i}$ of the closed loop system, $\mathbf{W}_{\mathrm{R}}$ the matrix of (left) eigenvectors $\mathbf{w}_{\mathrm{R}i}$ of the closed loop system and $\mathbf{P}$ the matrix of parameter vectors $\mathbf{p}_i$. The eigenvectors $\mathbf{v}_{\mathrm{R}i}$ can be determined by the eigenvalues $\lambda_{\mathrm{R}i}$ and the parameter vectors $\mathbf{p}_i$ [14]

$$\mathbf{v}_{\mathrm{R}i} = -(\lambda_{\mathrm{R}i}\mathbf{I} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}}\mathbf{p}_i\,. \tag{22}$$

Thus, the parameter vectors $\mathbf{p}_i$, the filter matrix $\hat{\mathbf{F}}$ and the eigenvalues $\lambda_{\mathrm{R}i}$ are the degrees of freedom which have to be used for the fulfilment of the constraints in eq. (18).

This fact is used in [15] for output coupling in linear multivariable systems. The definition of the proportional controller equation (19) has to be carried out in two steps: change of the dynamic behaviour and adjustment of the static behaviour.

Without loss of generality, the dynamic effects can be described by the differential equation between the input $\check{\mathbf{u}}$ and the output $\hat{\mathbf{y}}_1$ (see fig. 1). Introduction of a new extended output

$$\hat{\mathbf{y}} := \begin{bmatrix}\hat{\mathbf{y}}_1 \\ \tilde{\mathbf{y}}_1\end{bmatrix} := \begin{bmatrix}\hat{\mathbf{C}}_1 \\ \hat{\mathbf{C}}_2\end{bmatrix}\hat{\mathbf{x}}_1 \tag{23}$$

and application of the Laplace-transform yields the description

$$\begin{aligned} \begin{bmatrix}\hat{\mathbf{y}}_1 \\ 0\end{bmatrix} &\stackrel{!}{=} \begin{bmatrix}\hat{\mathbf{C}}_1 \\ \hat{\mathbf{C}}_2\end{bmatrix}\left[s\mathbf{I} - \hat{\mathbf{A}} + \hat{\mathbf{B}}\hat{\mathbf{R}}\right]^{-1}\hat{\mathbf{B}}\left[\hat{\mathbf{F}}_1, \hat{\mathbf{F}}_2\right]\begin{bmatrix}\check{\mathbf{u}}_1(s) \\ \check{\mathbf{u}}_2(s)\end{bmatrix} \\ &=: \begin{bmatrix}G_{\mathrm{R}11}(s)\,G_{\mathrm{R}12}(s) \\ G_{\mathrm{R}21}(s)\,G_{\mathrm{R}22}(s)\end{bmatrix}\begin{bmatrix}\check{\mathbf{u}}_1(s) \\ \check{\mathbf{u}}_2(s)\end{bmatrix} \end{aligned} \tag{24}$$

in frequency domain.

Obviously, eq.(24) is fulfilled if

$$G_{\mathrm{R}21}(s) \stackrel{!}{=} 0 \tag{25}$$

and

$$\check{\mathbf{u}}_2(s) \equiv 0 \tag{26}$$

hold. Condition (25) results in the defining equation

$$\sum_{i=1}^{N}\frac{\tilde{\mathbf{C}}_2\,\mathbf{v}_{\mathrm{R}i}\,\mathbf{w}_{\mathrm{R}i}^{\mathrm{T}}\,\hat{\mathbf{B}}\,\hat{\mathbf{F}}_1}{s - \lambda_{\mathrm{R}i}} = \mathbf{0} \tag{27}$$

for the first part of the input filter matrix $\hat{\mathbf{F}}_1$ and the eigenvectors $\mathbf{v}_{\mathrm{R}i}$, $i = 1, \ldots, \hat{m}$. These eigenvectors $\mathbf{v}_{\mathrm{R}i}$ form the matrix $\mathbf{V}_{\mathrm{R}_1}$ which is the first part of the matrix of eigenvectors $\mathbf{V}_{\mathrm{R}}$.
The relation (27) is true for any complex variable s if

$$\tilde{\mathbf{C}}_2\mathbf{v}_{\mathrm{R}i} = 0\,, \quad i = 1, \ldots, \hat{m} \tag{28}$$

and

$$\mathbf{w}_{\mathrm{R}i}^{\mathrm{T}}\hat{\mathbf{B}}\hat{\mathbf{F}}_1 = 0\,, \quad i = \hat{m}+1, \ldots, r \tag{29}$$

hold for a specific index $\hat{m}$. Eq. (28) and (29) define coupling conditions for the output and the input, respectively. It can be shown that $\hat{m}$ equals the number of finite eigenvalues λ_i of the original DAE.

Inserting eq. (22) into the coupling conditions for the output (28) results in $\hat{m}$ homogeneous matrix equations

$$\underbrace{\hat{\mathbf{C}}_2(\lambda_{\mathrm{R}i}\mathbf{I} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}}}_{\tilde{G}_2(\lambda_{\mathrm{R}i})}\mathbf{p}_i = 0\,, \quad i = 1, \ldots, \hat{m} \tag{30}$$

to determine the parameter vectors $\mathbf{p}_i \not\equiv 0$. The matrix $\tilde{G}_2(\lambda_{\mathrm{R}i})$ has $N-r-d$ rows and $N-r-d+1$ columns. It can be shown that for most choices of eigenvalues $\lambda_{\mathrm{R}i}$ the kernel of $\tilde{G}_2(\lambda_{\mathrm{R}i})$ consists of exact one nontrivial vector. Therefore, the first $\hat{m}$ parameter vectors $\mathbf{p}_i$ are unequivocal except for their length. The other $r-\hat{m}$ parameter vectors can be chosen at random.

The coupling conditions for the input (29) result in a homogeneous matrix equation

$$\left[\hat{\mathbf{B}}\,,\,\mathbf{V}_{\mathrm{R}1}\right]\begin{bmatrix}\hat{\mathbf{F}}_1\\ *\end{bmatrix}=\mathbf{0} \tag{31}$$

to determine the first part of the input filter matrix $\hat{\mathbf{F}}_1$. $\mathbf{V}_{\mathrm{R}1}$ is the matrix of the first $\hat{m}$ eigenvectors (22) and $*$ can be ignored. If eq.(26) holds the second part $\hat{\mathbf{F}}_2$ can be chosen freely.

With this definition of the matrices $\hat{\mathbf{F}}$ and $\hat{\mathbf{R}}$ and condition (26), $G_{\mathrm{R}11}(s)$ describes the input/output behaviour of the coupled system (20). $G_{\mathrm{R}12}(s)$ and $G_{\mathrm{R}22}(s)$ only affect the output $\mathbf{y}$ if $\breve{\mathbf{u}}_2 \neq 0$, if the initial conditions are not consistent or if the system is disturbed by external forces.

Simulation

It follows directly that if the poles $\lambda_{\mathrm{R}i}, i=1,\ldots,\hat{m}$ of the subsystem $G_{\mathrm{R}11}(s)$ are equal to the finite eigenvalues λ_i of the original DAE (9) the dynamic behaviour of the closed loop system (20) and the original DAE (9) are the same under normal conditions.

Considering the static behaviour of the original DAE (9)

$$\mathbf{K}_{\mathrm{D}} = -\mathbf{C}\,\mathbf{A}^{-1}\mathbf{B}$$

and closed loop system (20)

$$\mathbf{K}_{\mathrm{C}} = -\left[\hat{\mathbf{C}}_1-\hat{\mathbf{D}}_2\hat{\mathbf{R}}\right]\left[\hat{\mathbf{A}}-\hat{\mathbf{B}}\hat{\mathbf{R}}\right]^{-1}\hat{\mathbf{B}}\hat{\mathbf{F}}+\hat{\mathbf{D}}_2\hat{\mathbf{F}}$$

another filter matrix $\breve{\mathbf{F}}$

$$\breve{\mathbf{u}} = \breve{\mathbf{F}}\bar{u} \tag{32}$$

must be used to adjust the static behaviour. If $q \neq 1$ the matrices $\mathbf{K}_{\mathrm{D}}$ and $\mathbf{K}_{\mathrm{C}}$ are not quadratic and therefore not invertible. In this case $\breve{\mathbf{F}}$ can only be used to minimize the error

$$\min_{\breve{\mathbf{F}}}(\mathbf{K}_{\mathrm{D}}-\mathbf{K}_{\mathrm{C}}\breve{\mathbf{F}}) \tag{33}$$

between the static reinforcements.

Remarks

This scheme could also be used for controller design for the original DAE (9). There exists a connection between the two control laws (13) and (19) (see [13]). For the simulation case this transformation is not necessary.

As shown before, the subsystem $G_{\mathrm{R}11}(s)$ describes the dynamic behaviour between the input $\bar{u}$ and the output $\mathbf{y}$. It has the $\hat{m}$ poles $\lambda_{\mathrm{R}1},\ldots,\lambda_{\mathrm{R}\hat{m}}$. The subsystem $G_{\mathrm{R}22}(s)$ describes how errors in the boundary conditions (11) or coupling conditions (12) affect the output $\mathbf{y}$. $G_{\mathrm{R}22}(s)$ has the $r-\hat{m}$ poles $\lambda_{\mathrm{R}\hat{m}+1},\ldots,\lambda_{\mathrm{R}r}$. In case of simulation these poles can be chosen arbitrarily. The transient time depends on the real parts of the chosen eigenvalues $\lambda_{\mathrm{R}i}$. The more negative they are, the faster the error tends to zero.

The PDEs in this paper are restricted to single input multi output systems. The restriction is useful to determine the controller matrix $\hat{\mathbf{R}}$. If the system has more than one input, the kernel of $\tilde{G}_2(\lambda_{\mathrm{R}i})$ in eq. (30) could have a basis consisting of at least two vectors. Then, the parameter vector which belongs to the eigenvalue $\lambda_{\mathrm{R}i}$ can be any linear combination of these basis vectors. Since the parameter vectors are strongly connected to the zeros of the closed loop system, the equality of the input/output behaviour of booth systems cannot be guaranteed for a random choice of the parameter vectors within the kernel of $\tilde{G}_2(\lambda_i)$. Hence, for the extension to multi input problems, additional conditions for the parameter vectors have to be taken into consideration.

The main aspect of this controller design scheme is the calculation of the matrix of eigenvectors $\mathbf{V}_{\mathrm{R}}$ of the closed loop system. According to [16] this matrix can be used to calculate empirical eigenfunctions $\tilde{\phi}_i$

$$\tilde{\phi} := \mathbf{V}_{\mathrm{R}}\,\hat{\mathbf{C}}_1\,. \tag{34}$$

$\tilde{\phi}$ is the matrix of empirical eigenfunctions $\tilde{\phi}_i$ evaluated at the output points. Since the output condition (28) defines $\hat{m}$ eigenvectors $v_{\mathrm{R}i}$, only the first $\hat{m}$ eigenfunctions $\tilde{\phi}_i$ approximate the eigenfunctions of the PDE (2). The other empirical eigenfunctions $\tilde{\phi}_i$ depend on the choice of the parameter vectors $\mathbf{p}_i$ and belong to the stabilization process. Therefore, they only affect the results at the transition points and at the boundary points.

4 Example: Beam-Equation

The undamped, onesided clamped beam can be described by the linear PDE

$$\ddot{w}(t,z)+\frac{\partial^4}{\partial z^4}w(t,z)=f(t,\delta(z=1.5)) \tag{35}$$

with the boundary conditions

$$w(t,0)=\frac{\partial w(t,0)}{\partial z}=\frac{\partial^2 w(t,2)}{\partial z^2}=\frac{\partial^3 w(t,2)}{\partial z^3}=0 \tag{36}$$

for $z\in[0;2]$ and $t\in\mathcal{R}^+$. The solution can be approximated by a linear combination of 5 ordinary polynomials

$$\phi_{j,K}(z_K)=z_K^{(j-1)}\,,\; j=1,\ldots,5 \tag{37}$$

in every part K. By subdivision of the domain in two equally spaced parts K and introduction of four transition conditions

$$\begin{aligned} w(t,1-) &= w(t,1+)\,, \\ w'(t,1-) &= w'(t,1+)\,, \\ w''(t,1-) &= w''(t,1+)\,, \\ w'''(t,1-) &= w'''(t,1+) \end{aligned} \tag{38}$$

the vector $\hat{\mathbf{w}}_K$ of coefficients is the solution of the descriptor system

$$\mathbf{E}_K \ddot{\mathbf{w}}_K(t) = \mathbf{A}_K \mathbf{w}_K(t) + \mathbf{B}_K \bar{u}_K(t)$$

obtained by collocation with

$$\mathbf{E}_{1/2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0.5 & 0.25 & 0.125 & 0.0625 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{A}_{1/2} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -24 \\ 0 & 0 & -2 & -6 & -12 \\ 0 & 0 & 0 & -6 & -24 \end{pmatrix},$$

$$\mathbf{B}_1^{\mathrm{T}} = (0,0,1,0,0)$$

and

$$\mathbf{B}_2^{\mathrm{T}} = (0,\ldots,0)$$

in parts of the domain.
Introducing a new state space vector $\tilde{\mathbf{w}}_K^{\mathrm{T}} = (\mathbf{w}_K^{\mathrm{T}}, \dot{\mathbf{w}}_K^{\mathrm{T}})$ and an output equation which provides the values of the numerical solution at 3 equidistant points in the two parts

$$\begin{aligned} z_{1,1} &= 0\,,\ z_{1,2} = 0.5 \text{ and } z_{1,3} = 1\,, \\ z_{2,1} &= 1\,,\ z_{2,2} = 1.5 \text{ and } z_{2,3} = 2 \end{aligned}$$

yield a descriptor system of order 20 and index 2. This system has four finite poles in $\Lambda_1 = [p_{1,2} = \pm 0.929\mathrm{i},\ p_{3,4} = \pm 8.7319\mathrm{i}]$ and 16 infinite poles.

Application of the transformations discussed in section 3.2 leads to a state space model (18) with 10 outputs, 9 fictitious inputs, 12 states and 8 coupling conditions for the states. Finally, construction of a proportional state feedback guarantees the fulfilment of eq. (38) and closed loop eigenvalues at Λ_1 and additional stabilization eigenvalues at $\Lambda_2 = [-1,-2,-3,-4,-5,-6,-7,-8]\cdot 10^3$.
In fig. 2 the step response of the controlled model is shown. In comparison with the results of a stable index reduction method by differentiation [10, 12] the absolute difference between both models is less than 10^{-7}, but the "controlled" system has only 12 states compared to the 20 of the system derived by differentiation. In fig. 3 the error in the transition conditions (38) (dashed line) and the boundary conditions (36) (solid line) of the closed loop system caused by an inconsistent initial

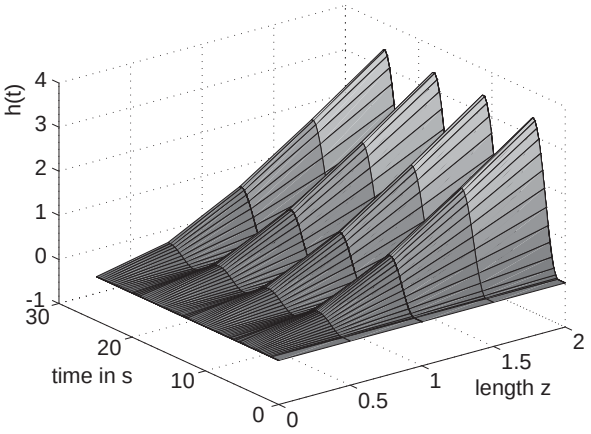


Fig. 2 Step response $h(t)$ of a onesided clamped beam

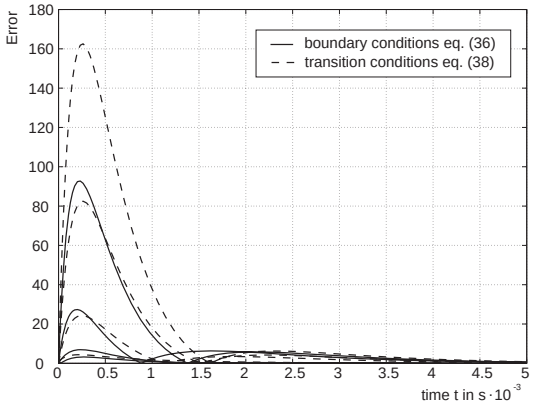
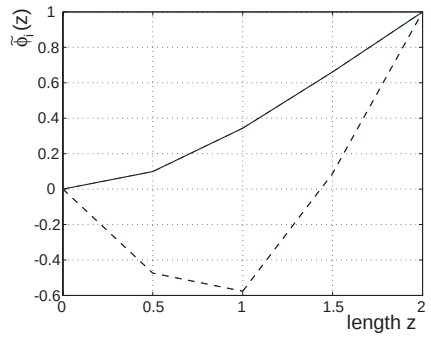


Fig. 3 Simulation of the error of the closed loop system output constraints with inconsistent initial conditions

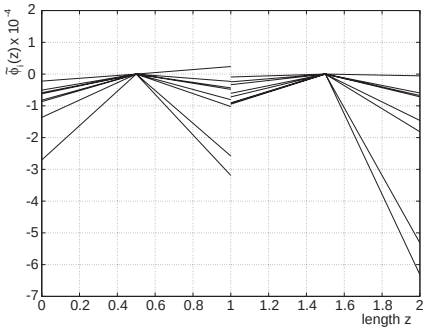
condition $w(0,0) = 1$ is shown. Due to the poles Λ_2 all errors vanish nearly complete after $5\cdot 10^{-3}$ s. The magnitude of these errors could be reduced among other things by choosing the poles Λ_2 in a more conservative way.

Multiplication of the matrix of eigenvectors $\mathbf{V}_{\mathrm{R}}$ with the output matrix $\hat{\mathbf{C}}_1$ reveals the first two empirical eigenfunctions shown in fig. 4(a). The solutions are normalized so that the maximum of booth functions is 1. Notice that the solutions are interpolated out of a few points. Fig. 4(b) shows the empirical eigenfunctions which belong to the stabilization eigenvalues. Obviously, they are nonzero at the transition point and at the boundary points and zero at interior points. Hence, they infect only the transition conditions and boundary conditions.

Comparison with the exact solution shows that the empirical solutions give a good impression of the systems behaviour [17]. All results in this section can be improved by increasing the number of basis functions N_K in every subdomain or by increasing the number of subdivisions.



(a)



(b)

Fig. 4 Emprical Eigenfunctions.

5 Conclusions

A method for simulation of linear single input partial differential equations without mixed derivatives was presented. Based on line methods the problem can be transformed into a matrix differential equation. For the discretization of the space the approach of approximating the solution by a finite linear combination of basis functions was used. Instead of using a special function set, the proposed scheme allows any set of linearly independent functions. The boundary conditions of the partial differential equation are taken into account by "boundary boarding". With the introduction of static transition equations the proposed scheme allows the subdivision of the domain into many subdomains. In every subdomain different methods for determination of the numerical solution of boundary value problem can be used.

The complex simulation of the resulting differential-algebraic equations can be neglected by transformation into an output coupling problem. By application of a proportional state feedback controller the consistency conditions for the state vector can be fulfilled. For a special choice of the parameters of the control law, the input/ouput behaviour of the closed loop system can be adjusted to the input/ouput behaviour of the differential-algebraic equation system.
Moreover, the proposed method allows the optimization of the systems dynamics. Due to the combination of a virtual controller design and simulation, "real" control laws can be deduced easily.

So far, the proposed scheme is restricted to single input systems governed by partial differential equations. Further research will be carried out to extend the method to multi input systems.

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