ON ELASTIC-RIGID COUPLING IN MODELLING OF MOTION OF ELASTIC LINKAGES BY FEM

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Abstract

In the paper the discussion on elastic-rigid coupling in finite element analysis of flexible mechanisms is presented. The planar case is consider and the Euler-Bernoulli beam finite elements are used. It is commonly assumed that the shape function of elastic motion can represent rigid body motion. In this paper, in opposite to this assumption, the model of a shape function specially dedicated for the rigid-body motion is presented and its influence on elastic motion appears in equations of motion - the inertia matrix standing by rigid body acceleration depends on both shape function of beam and rigid elements. The assumed shape function for rigid motion is also useful for analysis of high-speed mechanisms when the mutual dependence between the rigid-body motion and elastic deformation of links is needed. In this case the displacement vector contains both rigid body and elastic displacements and is substituted as generalized coordinate to Gibbs-Appel equations of motion. The rigid body components are separated from elastic ones in the displacement vector. Thus the dimension of the displacement vector of a finite element is larger compare to the previous studies on this field but equations of motion are obtained by solving one set of system equations. The numerical calculations are conducted in order to determine the influence of the assumed shape function for rigid body motion on the vibration of links in the case of two examples: a four-bar linkage representing the closed-loop mechanisms, and the open-loop example of two link planar manipulator. The results of numerical simulation show that for transient analysis and for some specific conditions (e.g. starting range, open-loop mechanisms) the influence of assumed shape functions on vibration response can be quite significant.

Keywords: Flexible mechanism, Rigid body motion, Elastic-rigid coupling.

Presenting Author's biography

Michał Hać. Received the M.S. and Ph.D. in Mechanics and Machine Structures from the Faculty of Automotive and Construction Machinery Engineering, Warsaw University of Technology (WUT), Poland in 1978 and 1986, respectively. Since 2001 he is a Professor of WUT in the Institute of Machine Design Fundamentals. His research contribution include the usage of the finite element method in machine design. His research interests are directed towards dynamics of flexible mechanism and analysis of multibody systems. He is a member of Polish Society of Computer Simulation.

1. Introduction

In recent years considerable attention has been given to the analysis of flexible mechanisms. The need of taken into account flexibility of linkages appears due to much higher speed operation and on the other hand the greater restrictions on weight and power requirements of mechanism members. In consequence for high speed operation, both rigid body and elastic effects should be included in the mechanism design process in order to reduce dynamic reactions and allow the linkage to perform its prescribed kinematic functions.

Many papers that has been published in recent years has involved the application of special-purpose finite element methods. Usually in these studies the response of a mechanism was obtained by using superposition theory. In this method firstly rigid-body motion is determined and then the elastic displacements are solved for as the unknowns of the system. In the derivation of equations of motion it is made an assumption that the shape function for rigid-body motion is the same as for elastic motion [1,2,3,4,5]. In consequence the system inertia matrix for elastic displacement appears also together with the rigid-body acceleration on the right hand side of equations of motions.

In the present paper a model of a shape function for the planar rigid-body motion proposed in [6,7] is used. This causes the difference in dynamic equations of motion that the inertia matrix standing by rigid body acceleration depends on both shape functions of beam and rigid elements.

The superposition procedure, however, does not yield accurate results when high speed systems are considered, since it does not provide for the mutual dynamic coupling of both the rigid and elastic motions. Analysis procedure developed by Song and Haug [8] introduced the "one pass" method which models both the large motions and small elastic displacements of the links simultaneously. However, the interconnections of the bodies are described by a large set of constraint equations formulated for each type of joints. This procedure increases the dimension of the problem considerably and yields to the set of equations composed of both differential and algebraic equations, which are not easy to solve.

The aim of this paper is also to present the theoretical background for analysis of moving elastic linkages in the case of mutual dependence between rigid and elastic motions. The presented shape function for rigid body motion makes possible to formulate the displacement vector containing both rigid body and elastic displacements. This vector is than substituted as generalized coordinate to Gibbs-Appel equations of motion. The rigid body components are separated from the elastic ones in the displacement vector. Thus the dimension of the displacement vector of a finite element is larger compare to the previous studies on this field ([2,9]) but equations of motion are obtained by solving one set of system equations. Moreover, any correction terms are not used due to the finite element modelling of large displacement motion.

2. Equations of motion by using the superposition theory

The coupling between the non-linear rigid body motions and the linear small elastic deformation stands as the main problem in the solution of the dynamics of flexible mechanisms. In the superposition theory [1,4,5] the elastic displacements are separated from the rigid body displacements (which are previously calculated) and solved as the unknowns of the system. In this way the rigid-body motion is not influenced by the elastic motion.

2.1 Finite element formulation

Figure 1 presents a general planar beam element in three frames of reference. Frame *XY* is fixed to the ground and serves as the global coordinate system. The origin of the *xy* frame is located at a node point of a finite element and it remains parallel to the *XY* system. The element oriented frame $\xi\eta$ is a rotating reference frame and its ξ-axis is parallel to the undeformed center line of the element. The frames *xy* and $\xi\eta$ are updated continually as the element moves.

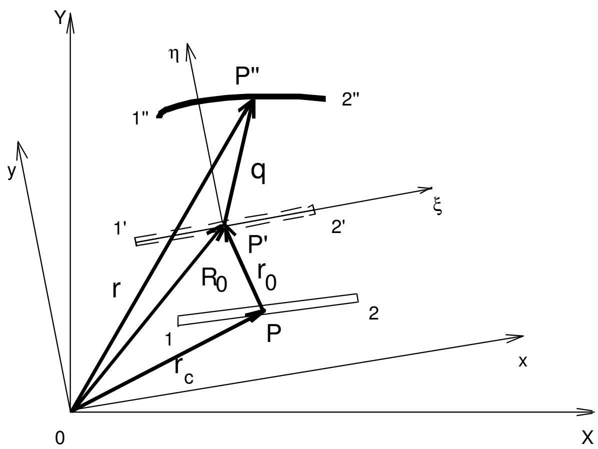

Fig. 1. Displacements of a finite element: in global and local coordinates

Let us assume that during the motion of the mechanism a finite element has changed its position determined by endpoints 1 and 2 to the points 1' and 2' (dotted) due to the rigid-body motion and to the points 1" and 2" due to the elastic deformation. The components of nodal displacement vectors 11' and 22' can be expressed in the relative coordinate system *xy* by the following vector

$$\{s_0\}^T = [u_{01}, w_{01}, u_{02}, w_{02}] \tag{1}$$

or in the local coordinate system by

$$\{\delta_0\}^T = [p_{01}, v_{01}, p_{02}, v_{02}] \tag{2}$$

where $u_{01}, w_{02}, u_{01}, w_{02}$ are displacements of nodes 1 and 2 in x and y direction respectively, $p_{01}, p_{02}, v_{01}, v_{02}$ - displacements of nodes 1 and 2 in ξ and η direction respectively.

The displacement vector $\mathbf{r}_0$ of a general point belonging to the finite element can be expressed in terms of the displacements of the element's endpoints using the formula

$$r_0 \equiv \{r_0\} = [r_{0x}, r_{0y}]^T = [N_{e0}]\{\delta_0\} = [N_{e0}][T_0]\{s_0\} \tag{3}$$

where $[N_{e0}]$ is the shape function for the rigid body motion which can be expressed as [6]

$$[N_{e0}] = \begin{bmatrix} 0.5 & 0 & 0.5 & 0 \\ 0 & 1-\varsigma/L & 0 & \varsigma/L \end{bmatrix} \tag{4}$$

where $0 \le \varsigma \le L$, L is the length of a finite element, and $[T_0]$ is a transformation matrix ($a = \cos\alpha$, $b = \sin\alpha$, α is the angle between $\xi\eta$ and global coordinate systems):

$$[T_0] = \begin{bmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -b & a \end{bmatrix} \tag{5}$$

Now let us consider the elastic motion which describes the vibration of links. The components of nodal displacement vectors 1'1'' and 2'2'' due to elastic deformation can be expressed in the xy and $\xi\eta$ systems by the vectors $\{s\}$ and $\{\delta\}$ respectively

$$\{s\}^T = [u_1, w_1, \Theta_1, u_2, w_2, \Theta_2] \tag{6}$$

where: u_1, w_1, u_2, w_2 are displacements of nodes 1 and 2 in x and y direction, respectively; Θ_1, Θ_2 are angular deformations of nodes 1 and 2,

$$\{\delta\}^T = [p_1, v_1, \Theta_1, p_2, v_2, \Theta_2] \tag{7}$$

where p_1, p_2, v_1, v_2 – displacements of nodes 1 and 2 in ξ and η direction, respectively.

The vector $\{\delta\}$ is transformed into $\{s\}$ by

$$\{\delta\} = [T]\{s\} \tag{8}$$

where $[T]$ is a transformation matrix

$$[T] = \begin{bmatrix} a & b & 0 & 0 & 0 & 0 \\ -b & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & -b & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

Introducing the matrix notation essential in the finite element formulation the displacement vector **q** can be expressed in terms of the displacements of the element's endpoints:

$$\mathbf{q} \equiv \{q\} = [N_e]\{\delta\} = [N_e][T]\{s\} \tag{10}$$

where the shape function $[N_e]$ for elastic displacement is as follows

$$[N_e] = \begin{bmatrix} 1-\varsigma/L & 0 & 0 & \varsigma/L & 0 & 0 \\ 0 & N_1 & N_2 & 0 & N_3 & N_4 \end{bmatrix} \tag{11}$$

where $0 \le \varsigma \le L$, N_1, N_2 N_3 N_4 are functions for Euler-Bernoulli beam type finite element:

$$\begin{aligned} N_1 &= 1 - 3(\varsigma/L)^2 + 2(\varsigma/L)^3 \\ N_2 &= \varsigma - 2(\varsigma^2/L) + \varsigma^3/L^2 \\ N_3 &= 3(\varsigma/L)^2 - 2(\varsigma/L)^3 \\ N_4 &= -\varsigma^2/L + \varsigma^3/L^2 \end{aligned} \tag{12}$$

2.2 Equations of motion

In most of the studies in the flexible multibody formulation it is assumed that the element shape function adopted for elastic motion can be used to describe an arbitrary large rigid body translation [2,3,4,5,9]. Thus the shape function for elastic displacement $[N_e]$ (11) can be used to determine rigid body displacement, i.e.

$$r_0 = [N_e]\{\delta'_0\} \tag{13}$$

where vector $\{\delta'_0\}$ is similar to given by Eq. (7) but the displacements are due to rigid body motion.

This assumption leads to equations of motion in which the inertia matrix standing by rigid body acceleration is the same as the system inertia matrix. In simplified form it can be presented as follows

$$\begin{aligned} &[M]\{\ddot{x}\} + ([C]+[C_g])\{\dot{x}\} + ([K]+[K_c])\{x\} = \\ &\{Q_{ex}\} - [M]\{\ddot{x}_0\} \end{aligned} \tag{14}$$

where $[M]$ is the global inertia matrix, $[K]$ is the stiffness matrix, $[K_c]$ is the centrifugal matrix, $[C]$ is the damping matrix, $[C_g]$ is the gyroscopic matrix, $\{Q_{ex}\}$ is a vector of generalized forces, $\{\ddot{x}\}$, $\{\dot{x}\}$ and $\{x\}$ represent acceleration, velocity and displacement vectors (in nodal points), and $\{\ddot{x}_0\}$ is the rigid-body acceleration.

Assuming that the shape function for the rigid-body motion is given by Eq. (4) the equations of motion in XY frame in simplified form are stated as

$$\begin{aligned} &[M]\{\ddot{x}\} + ([C]+[C_g])\{\dot{x}\} + ([K]+[K_c])\{x\} = \\ &\{Q_{ex}\} - [M_0]\{\ddot{x}_0\} \end{aligned} \tag{15}$$

where $[M_0]$ is the coefficient matrix defined for element level formulation (denoted by $[M_{e0}]$) for constant cross-sectional area A and constant material density ρ by the following notation [6]

$$\{\dot{s}\}^T [M_{e0}]\{\dot{s}_0\} = \frac{1}{2}\rho A \int_0^L (\{\dot{s}_0\}^T [T_0]^T [N_{e0}]^T [N_e][T]\{\dot{s}\} + \{\dot{s}\}^T [T]^T [N_e]^T [N_{e0}][T_0]\{\dot{s}_0\}) d\varsigma \quad (16)$$

By comparing Eq. (14) and (15) it can be seen that assuming the shape function for rigid-body motion (4) the inertia matrix standing by rigid-body acceleration at right hand side of equations of motion differs considerably in the two cases considered.

The solution $\{x\}$ of matrix Eq. (14) and (15) is obtained by using Newmark method for integrating the equations. In numerical examples the structural damping appearing in equations of motion are not taken into account.

3. Equations of motion for mutual elastic-rigid coupling

In order to take into account the mutual dependence between rigid body motion and elastic deformation of links both the large motions and small elastic displacements of the links must be calculated simultaneously. The equations of motion are derived by using the Gibbs-Appel equations which simplify derivation procedure and minimize the work done for preparing equations of motion (as compared to Lagrangian formulation). Thus the displacement vector contains both rigid body and elastic displacements and is substituted as generalize coordinates to the Gibbs-Appel equations of motion. In this way the obtained equations of motion are more accurate due to the mutual dependence between the rigid body motion and elastic deformation of links. The rigid-body mechanism motion can be unknown and is not required to compute before solving the system equations.

As compared to other investigators, the rigid body components are separated from elastic ones in the displacement vector. The dimension of the displacement vector is larger than in the few existing previous studies on this field concerning one-pass methods ([2,7]), but equations of motion are obtained by solving one set of system equations. The shape function for this case is composed of the shape function for rigid and beam elements.

Let us denote the total displacement of the general point P by $\mathbf{q_c}$ (not marked in Fig. 1) due to rigid body ($\mathbf{r_0}$) and elastic deformation ($\mathbf{q}$). Thus the displacement from P (initial position of the general point) to P" can be determined by

$$\mathbf{q}_c = \mathbf{r_0} + \mathbf{q} \quad (17)$$

where $\mathbf{q}$ is an elastic deformation vector in the local coordinate system $\xi\eta$.

In order to take into account the mutual coupling between rigid-body motion (large-displacement motion) and elastic deformation (small-displacement motion) the nodal displacement vector consists of both rigid-body and elastic motions. The components of nodal displacement vectors 11" and 22" can be expressed in the global coordinate system XY by the following vector

$$\{s\}^T = [u_1, w_1, u_1', w_1', \Theta_1, u_2, w_2, u_2', w_2', \Theta_2] \quad (18)$$

where u_1, w_1, u_2, w_2 are displacements of element nodes due to rigid body motion in X and Y direction, respectively; u'_1, w'_1, Θ_1, u'_2, w'_2, Θ_2 are displacements (u',w') and angular deformation (Θ) due to elastic motion.

Taking into account that xy system is parallel to $\xi\eta$ coordinate system, the displacements due to the large-displacement motion are the same in both coordinate systems. Thus, in the local coordinate system $\xi\eta$ the global deformation vector is as follows

$$\{\delta\}^T = [p_1, v_1, p_1', v_1', \Theta_1, p_2, v_2, p_2', v_2', \Theta_2] \quad (19)$$

where $p_1, p'_1, p_2, p'_2, v_1, v'_1, v_2, v'_2$ - displacements of nodes 1 and 2 in ξ and η direction; Θ_1, Θ_2 - angular deformation of nodes 1 and 2.

The vector $\{\delta\}$ is transformed into $\{s\}$ by

$$\{\delta\} = [T]\{s\} \quad (20)$$

where $[T]$ is a transformation matrix.

The displacement vector can be divided into two parts: 1) connected with rigid-body motion ($\{\delta_1\}$ and 2) connected with elastic displacements ($\{\delta_2\}$). Vectors $\{\delta_1\}$, and $\{\delta_2\}$ contain appropriate displacements for rigid, and elastic motions, respectively, and are as follows:

$$\{\delta_1\}^T = [p_1, v_1, p_2, v_2]; \ \{\delta_2\}^T = [p_1', v_1', \Theta_1, p_2', v_2', \Theta_2] \quad (21)$$

The vectors representing rigid-body and elastic displacements can be transformed into global displacement vector $\{\delta\}$ by

$$\underset{4x1}{\{\delta_1\}} = \underset{4x10}{[T_1]}\underset{10x1}{\{\delta\}}; \qquad \underset{6x1}{\{\delta_2\}} = \underset{6x10}{[T_2]}\underset{10x1}{\{\delta\}} \quad (22)$$

where $[T_1]$ and $[T_2]$ are Boolean type transformation matrices.

The displacement vector $\mathbf{q_c}$ of a general point belonging to the finite element can be expressed in terms of the displacements of the element's endpoints by using the formula

$$q_c = [N]\{\delta\} = [N][T]\{s\} \tag{23}$$

where the shape function is given by

$$[N] = \begin{bmatrix} 0.5 & 0 & 1-\varsigma/L & 0 & 0 & 0.5 & 0 & \varsigma/L & 0 & 0 \\ 0 & \varsigma/L & 0 & N_1 & N_2 & 0 & \varsigma/L & 0 & N_3 & N_4 \end{bmatrix} \tag{24}$$

The equations of motion are formulated by applying the Gibbs-Appel equations [10]. Both rigid body and elastic displacements are treated as unknowns. The Gibbs-Appel equations of motion are written for each element as:

$$\frac{\partial G_e}{\partial\{\ddot{\delta}\}} + \frac{\partial U_e}{\partial\{\delta\}} = \{Q\} \tag{25}$$

where $\{\delta\}$ represents the generalized nodal degree of freedom, G_e is the Gibbs function ("energy of acceleration"), U_e is the potential energy, and $\{Q\}$ are the generalized forces acting on the element.

In order to obtain an "energy of acceleration" of the mechanism element the acceleration of the general point is needed. The velocity and acceleration of the point P'' are given from Eq. (17) by

$$\begin{aligned} \dot{\mathbf{q}}_c &= \dot{r}_0 + \overline{\omega} \times \mathbf{q} + \dot{\mathbf{q}} = \dot{r}_0 + \omega[\Omega]\mathbf{q} + \dot{\mathbf{q}} \\ \ddot{\mathbf{q}}_c &= \ddot{r}_0 + 2\omega[\Omega]\dot{\mathbf{q}} - \omega^2\mathbf{q} + \ddot{\mathbf{q}} + \dot{\omega}[\Omega]\mathbf{q} \end{aligned} \tag{26}$$

where $\overline{\omega} = \mathrm{col}[0,0,\omega]$ is the absolute angular velocity of the xy frame and $[\Omega]$ is the operator matrix of size 2x2: $[\Omega]=[0\ -1;\ 1\ 0]$.

The time derivatives of vectors $\mathbf{r}_0$ and $\mathbf{q}$ are obtained from Eq. (17,21,22,23) by taking into account that shape functions do not depend on time. The Gibbs function is expressed as follows

$$\begin{aligned} G_e &= \frac{1}{2}\int_0^L \ddot{\mathbf{q}}_c{}^T \ddot{\mathbf{q}}_c dm = \\ & \frac{1}{2}\int_0^L \Big(\Big(\{\ddot{\delta}\}^T [T_1]^T [N_{e0}]^T + 2\omega\{\dot{\delta}\}^T [T_2]^T [N_e]^T [\Omega]^T \\ & -\omega^2\{\delta\}^T [T_2]^T [N_e]^T + \{\ddot{\delta}\}^T [T_2]^T [N_e]^T \dot{\omega}\{\delta\}^T [T_2]^T [N_e]^T [\Omega]^T\Big) \\ & \Big([N_{e0}][T_1]\{\ddot{\delta}\} + 2\omega[\Omega][N_e][T_2]\{\dot{\delta}\} - \omega^2[N_e][T_2]\{\delta\} + \\ & [N_e][T_2]\{\ddot{\delta}\} + \dot{\omega}[\Omega][N_e][T_2]\{\delta\}\Big)\Big) dm \end{aligned} \tag{27}$$

For further analysis the uniform cross-sectional area A and uniform mass density ρ of an finite element are assumed.

The strain energy U_e of the element is associated with the stiffness matrix for a beam finite element. The rigid body components connected with longitudinal displacements of links (in accordance with the assumed shape function $[N_{e0}]$) must appear in addition in the stiffness matrix for rigid-body motion. Thus the element stiffness matrices are associated with the displacement vector $\{\delta_3\}$ which consists of both elastic displacements and longitudinal displacements in large-displacement motion:

$$\{\delta_3\}^T = [p_1, p_1', v_1', \Theta_1, p_2, p_2', v_2', \Theta_2] \tag{28}$$

and is as follows

$$[k] = \frac{E}{L^3}\begin{bmatrix} AL^2 & 0 & 0 & 0 & -AL^2 & 0 & 0 & 0 \\ 0 & AL^2 & 0 & 0 & 0 & -AL^2 & 0 & 0 \\ 0 & 0 & 12I & 6IL & 0 & 0 & -12I & 6IL \\ 0 & 0 & 6IL & 4IL^2 & 0 & 0 & -6IL & 2IL^2 \\ -AL^2 & 0 & 0 & 0 & AL^2 & 0 & 0 & 0 \\ 0 & -AL^2 & 0 & 0 & 0 & AL^2 & 0 & 0 \\ 0 & 0 & -12I & -6IL & 0 & 0 & 12I & -6IL \\ 0 & 0 & 6IL & 2IL^2 & 0 & 0 & -6IL & 4IL^2 \end{bmatrix} \tag{29}$$

where E is the Young's modulus, and I is the cross-sectional moment of inertia.

The strain energy U_e of the element is expressed as follows

$$U_e = \frac{1}{2}\{\delta_3\}^T [k]\{\delta_3\} = \frac{1}{2}\{\delta\}^T [T_3]^T [k][T_3]\{\delta\} \tag{30}$$

where $[T_3]$ is a Boolean type transformation matrix between vectors $\{\delta\}$ and $\{\delta_3\}$.

Combining all the element equations, taking into account the boundary conditions and introducing damping matrix $[C]$, the equations of motion for the system are stated as

$$[M]\{\ddot{x}\} + ([C]+[C_g])\{\dot{x}\} + ([K]+[K_c])\{x\} = \{F\} \tag{31}$$

where $\{F\}$ represents generalized forces, $\{\ddot{x}\}$, $\{\dot{x}\}$, and $\{x\}$ represent acceleration, velocity, and displacement vectors (in nodal points). Vector $\{x\}$ is an assembled form of element vector $\{s\}$ with regard taken to boundary conditions. The global matrices $[M]$, $[C_g]$, $[K]$ and $[K_c]$ are assembled forms of the appropriate element matrices and their names are mentioned in description of Eq. (14). The linear viscous damper is associated with the elastic deformation represented by vector $\{\delta_2\}$. Therefore the damping matrix $[C]$ is a linear combination of the mass and stiffness matrices built on the basis of the standard beam type elements obtained from the shape function $[N_e]$. To obtain the proper dimension of the damping matrix, these matrices are premultiplied by transformation matrix $[T_2]$.

4. Illustrative examples

In order to determine the influence of the assumed rigid-body motion shape function on vibration of links two numerical examples are considered: the first one is an example of closed-chain mechanism - a four-bar linkage, and the second one represents the open-chain mechanism of two-link planar manipulator. Also it is interesting to compare the presented method with the commonly used assumption of applying element type (in our case beam finite element) shape function to rigid-body motion. Thus the calculations are conducted for two cases:

case I – by solving equations of motion (15) which have been formulated for the shape function for the rigid-body motion given by Eq. (4);

case II - by solving commonly used equations of motion of type (14).

In numerical examples gravity and damping effects are not taken into account.

4.1 Four-bar linkage

All three moving members of the linkage are assumed to be deformable. A total of six elements were employed – each link is represented by two finite elements. The corresponding placement of nodes is shown in Fig. 2.

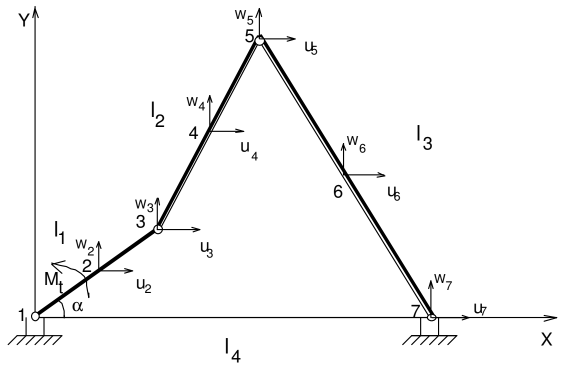

Fig. 2 Four-bar linkage with nodes

The geometrical data taken for numerical simulation is given in Tab. 1. This particular mechanism (in the case when the crank circulates with constant angular velocity) was studied extensively by Turcic and Midha [4].

Tab. 1 Four-bar linkage parameters

Parameters	Crank	Coupler	Follower
Length, m	0.10795	0.2794	0.27051
Cross-section, m^2	1.07×10^{-4}	0.406×10^{-4}	0.406×10^{-4}
Bending m of iner., m^4	1.62×10^{-10}	8.67×10^{-12}	8.67×10^{-12}
Distance between ground pivots: 0.254 m.			
Lumped mass at node 3: 0.0450 kg			
Lumped mass at node 5: 0.0375 kg			
Modulus of elasticity: 0.71×10^{11} N/m^2			
Mass density: 2710 kg/m^3			

The input torque M_t is applied to the crank of the mechanism and is assumed to be as follows

$$M_t = M_{t0} + M_{t1} \sin \alpha \qquad (32)$$

where α is the crank angle. The data given for numerical simulation are: M_{t0} = 1 Nm, M_{t1} = 0.5 Nm. The initial conditions (crank angle and crank angular velocity) are α = 0 rad, ω = 0 rad/sec. Simulation time is from 0 to 0.25 sec and assures the crank to do a full rotation.

The results of computer simulation show that in the case of considered four-bar linkage the differences in equations of motion do not significantly influence the results. The midspan bending strains for the coupler (Fig. 3) and for the follower (Fig. 4) are very alike for the two cases considered and the relative difference in results does not exceed 3%. Thus it would be hardly visible in the figures and additional charts are provided in order to show differences in midspan strains: in Fig. 5 for the coupler, and in Fig. 6 for the follower.

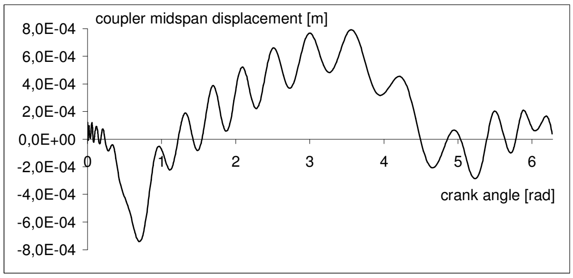

Fig. 3. Midspan bending strains for coupler with crank angle

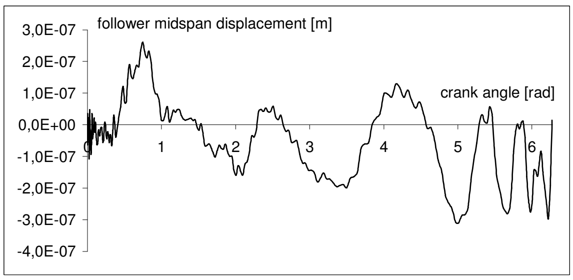

Fig. 4. Midspan bending strains for follower with crank angle

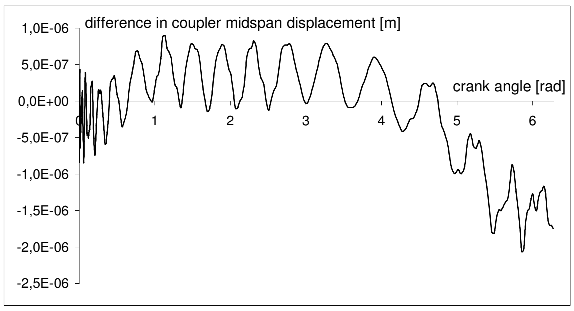

Fig. 5. Difference between two cases in midspan coupler strains with crank angle

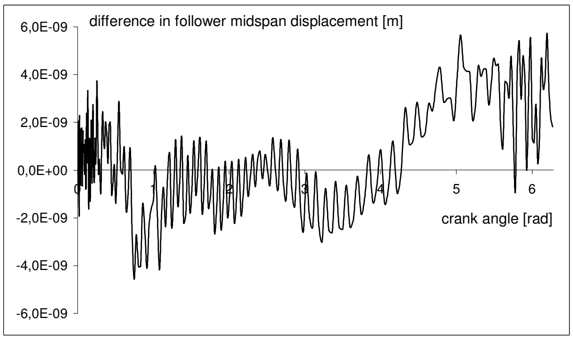

Fig. 6. Difference between two cases in midspan coupler strains with crank angle

The difference in equations of motion (15) and (14) responsible for case I and II, respectively, is only in the inertia force vector standing by the right hand sides of these equations. For Eq. (15) this vector is equal to $F_{c0}=[M_0]\{\ddot{x}_0\}$, and for Eq. (14) $F_c=[M]\{\ddot{x}_0\}$. It was interesting to see if the courses of these extortion forces are also very similar. In Fig. 7 the inertia force with the crank angle is presented - in both cases the results are very close and the differences are not visible in the figure. Thus in next Fig. 8 the difference between these forces (i.e. $F_c - F_{c0}$) is presented – its value is approximately 1% of the value of component forces.

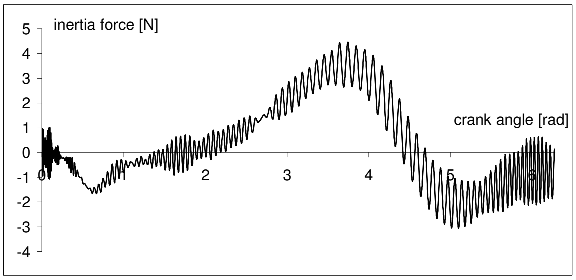

Fig. 7. Inertia force with the crank angle

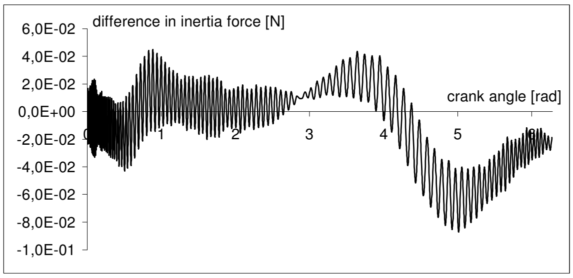

Fig. 8. Difference in inertia force between two cases with crank angle

4.2 Two link planar manipulator

The second example concerns the open-loop mechanism of a two-link planar manipulator (Fig. 9). The mechanical basic characteristics of this manipulator are the same as in [11] and are given in Tab. 2. In addition there are taken into account the mass at the joint 3 (m_3 = 0.5 kg), moment of inertia at the hub I_3 = 0.04 kgm^2, and nominal payload m_4 = 0.3 kg.

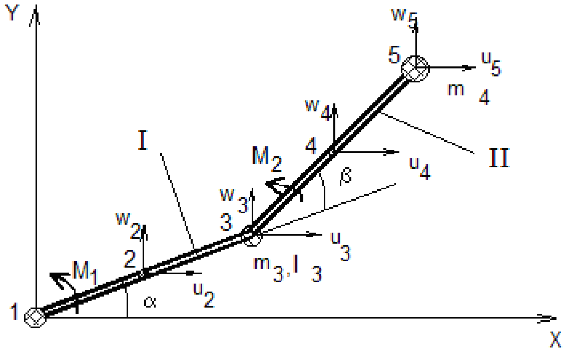

Fig. 9 Two link planar manipulator

The vibration of links arises due to the motion of the manipulator excited by torques applied to manipulator links. It was assumed that the values of torques are as follows: $M_1=-M_2=10$ Nm. The vibration analysis is conducted by solving equations of motion (15) and (14) which respond to the two cases considered. The initial conditions are: α = 0 [rad], β = 0.5 [rad], zero angular velocities, time of simulation is equal to 0.25 sec.

Tab. 2 Two-link manipulator parameters

Parameters	Link I	Link II
Length, m.	l_1= 1	l_2= 1
Cross-section area, m^2	F_1= 4×10^{-4}	F_2= 4×10^{-4}
Moment of inertia, m^4	I_1= 16/3×10^{-8}	I_2= 16/3×10^{-8}
Modulus of elasticity, N/m^2	$E = 2.1×10^{11}$	
Mass density, kg/m^3	ρ= 7800	

The horizontal and vertical displacements of manipulator's tip are presented in Fig. 10 and Fig. 11, respectively. Now the difference in results for the two cases considered is considerable and can be clearly seen. Especially the displacements in Y direction for starting range differ significantly but it should be noted that the calculations have been provided for transient analysis with sudden applied torques to both links of the manipulator.

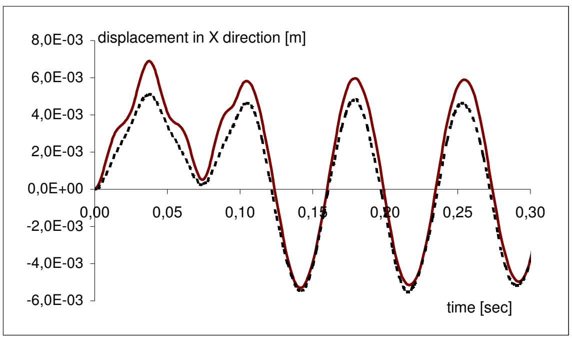

Fig. 10. Dispalcement of manipulator's tip in X direction for cases: I (solid line) and II (dashed line) with time

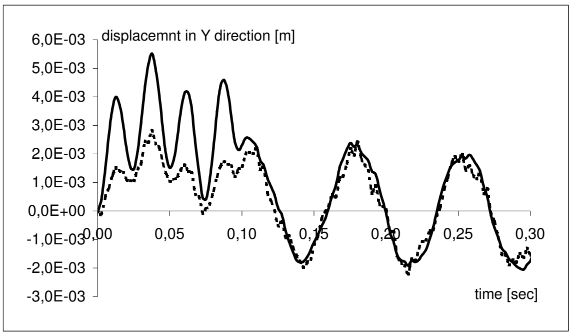

Fig. 11. Displacement of manipulator's tip in Y direction for case I (solid line) and II (dashed line) with time

In order to analyze the influence of payload on displacements of manipulator's tip the numerical simulation with no payload were conducted. The results in vertical direction are presented in Fig. 12. From the figure it can be seen that for given data due to smaller resistance (inertial) forces – no payload – the increase of vibration response is observed.

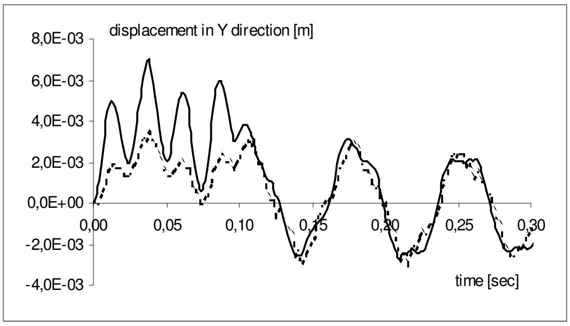

Fig. 12. Displacement of manipulator's tip (no payload) in Y direction for case I (solid line) and II (dashed line) with time

5. Conclusions

A finite element model for analysing the kineto-elastodynamic transient characteristics of a flexible mechanism is developed in the present paper. In multi-body dynamics there is a direct relationship between the selection of mode shapes in assumed coordinate system and the vibration response of constrained deformable bodies. By assuming the introduced shape function for rigid-body motion the vibration equations of motion are derived and the comparative analysis with existing commonly used approach is presented. The difference in equations of motion is in the coupling between elastic and rigid body modes – the inertia force vector connecting with rigid-body acceleration is different.

The method presented here is illustrated for the case of planar open- and closed-loop linkages. The results of numerical simulation show that in the case of considered four-bar linkage the differences in equations of motion do not considerably influence the results. Opposite observation is in the case of open-loop mechanism – two link planar manipulator – the differences in results between two discussed approaches are significant. However in this case the severe excitation conditions were assumed – the transient response was examined after stepped torques applied to links. Apart from starting range the solutions coincide which leads to the conclusion that for stable conditions of work (e.g. mechanisms circulating at constant speed, steady-state response is analysed) the two approaches presented in this paper could give quite converged results.

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