MODELLING UNCERTAINTY IN JOINTS USING COMPONENT MODE SYNTHESIS AND A STOCHASTIC REDUCED BASIS METHOD

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Abstract

There are often uncertainties in the properties of joints, which subsequently produce uncertainties in the dynamic response of built-up structures. Line joints, such as glued or continuously welded joints, have spatially distributed uncertainty and can be modelled by a discretised random field. Simulation techniques such as the commonly used direct Monte Carlo simulation (MCS) can be applied to approximate the output statistics to an arbitrary degree of accuracy, provided that a sufficient number of samples is used. Unfortunately, the computational cost of this technique can be prohibitive for large-scale models. This paper addresses how spatially correlated uncertainties in joints might be included straightforwardly in a mechanical finite element model, with particular reference to approaches based on fixed interface (Craig-Bampton) component mode synthesis and a stochastic reduced basis method. The methods are reviewed and an efficient way of implementation based on an exact matrix identity is proposed with a significantly lower computational cost. Unlike perturbation-based methods, good accuracy can be achieved even when the coefficients of variation of the input random variables are large. A numerical example of two line-coupled plates is investigated to benchmark the accuracy and calculation time. For the problem considered, the proposed formulation is an efficient and effective implementation of a stochastic reduced basis method. It is seen that the method can be up to orders of magnitude faster than direct Monte Carlo simulation, while providing results of comparable accuracy. Furthermore, the proposed implementation is more efficient the fewer joints are affected by uncertainty.

Keywords: Uncertainty in joints, Random field, Component mode synthesis, Stochastic reduced basis method, Stochastic Krylov subspace.

Presenting Author's Biography

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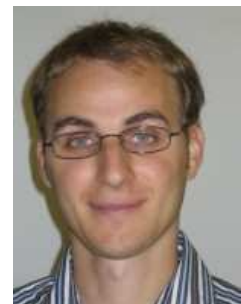
1 Introduction

When constructing numerical models of real-life engineering systems it is often assumed that the system under consideration is deterministic. In practice, however, some degree of uncertainty in the system properties and operating environment is inevitable, e.g. boundary conditions, material properties, etc. In such situations, an individual deterministic realisation of the system properties and its environment might be undesirable, since this could lead to misleading response predictions. A commonly used approach in engineering design is to introduce safety factors to indirectly account for parameter uncertainty. However, this approach typically leads to highly conservative designs and may not be appropriate for new lightweight materials and novel design concepts.

With continuous growth in computing power and recent development of sophisticated numerical techniques, reliable numerical simulations of systems with uncertainty can be obtained by quantifying input uncertainties to the model equations and propagating them to the system response numerically. In contrast to deterministic analysis, which only provides response predictions at a single point in the ensemble corresponding to the nominal values of the system parameters, this probabilistic approach provides a range of response values, which can be valuable in the design process.

Simulation techniques commonly used, such as direct Monte Carlo simulation (MCS), can be applied to approximate the output statistics to an arbitrary degree of accuracy, provided that a sufficient number of samples is used. Unfortunately, the computational cost of direct simulation techniques can be prohibitive for large-scale models. There are three main approaches to reduce the computational cost of large models: (1) advanced Monte Carlo methods, e.g. importance sampling, line sampling, etc., (2) perturbation-based methods and (3) stochastic finite element methods. The latter techniques offer computationally efficient alternatives to MCS and have been widely applied to approximate the first two statistical moments of the system response. Unlike perturbation-based methods, stochastic finite element methods, e.g. the stochastic reduced basis method (SRBM), can achieve good accuracy even when the coefficients of variation of the random variables are large.

There are typically uncertainties in the properties of joints in mechanical structures e.g. bolts, rivets, spot welds, glue, etc., which subsequently produce uncertainties in the dynamic response of built-up structures. This paper addresses how such uncertainties might be included straightforwardly in a finite element model of a structure, with particular reference to approaches based on fixed interface (Craig-Bampton) component mode synthesis (CMS), see [1, 2]. The spatially distributed uncertainty in the joints is modelled by a discretised random field [3]. As proposed here, the efficiency of the SRBM [4] can be enhanced further by a combination of CMS and matrix algebra. The former approach truncates the number of degrees of freedom for each component separately, while the latter reduces the global stochastic algebraic equation to a much smaller size, the size being the number of joint degrees of freedom, instead of retaining all of the system degrees of freedom.

In the next sections CMS is briefly reviewed with focus on the fixed interface (Craig-Bampton) approach, subsequently the modelling of joints with spatial correlation in terms of a random field is described. The calculation of system response statistics by direct MCS with CMS and SRBM with CMS is described. Finally, a numerical example is given.

2 Component mode synthesis

CMS is a technique which can reduce the size of the system matrices of built-up structures, in order to reduce the computational cost of the dynamic response calculation. The systems under considerations are assumed to be built-up from a number of connected subsystems. In this paper, the subsystems are modelled with respect to their local modes. At this component level the substructure is represented by an undamped FE model that can be written in the form

$$\mathbf{M}^{(s)}\ddot{\mathbf{x}}^{(s)} + \mathbf{K}^{(s)}\mathbf{x}^{(s)} = \mathbf{f}^{(s)} \tag{1}$$

where $\mathbf{M}^{(s)}$ and $\mathbf{K}^{(s)}$ are the $n \times n$ mass and stiffness matrices, and $\mathbf{x}^{(s)}$ and $\mathbf{f}^{(s)}$ the $n \times 1$ vectors of physical displacement coordinates and external forces acting on the substructure/component s. The superscript (s) will be omitted for the rest of this section. The displacement coordinates $\mathbf{x}$ are partitioned into n_i interior (i) and n_b boundary (b) coordinates, where $n = n_\mathrm{i} + n_\mathrm{b}$, resulting in

$$\begin{bmatrix} \mathbf{M}_\mathrm{ii} & \mathbf{M}_\mathrm{ib} \\ \mathbf{M}_\mathrm{bi} & \mathbf{M}_\mathrm{bb} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{x}}_\mathrm{i} \\ \ddot{\mathbf{x}}_\mathrm{b} \end{bmatrix}$$
$$+ \begin{bmatrix} \mathbf{K}_\mathrm{ii} & \mathbf{K}_\mathrm{ib} \\ \mathbf{K}_\mathrm{bi} & \mathbf{K}_\mathrm{bb} \end{bmatrix} \begin{bmatrix} \mathbf{x}_\mathrm{i} \\ \mathbf{x}_\mathrm{b} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_\mathrm{i} \\ \mathbf{f}_\mathrm{b} \end{bmatrix} \tag{2}$$

Different CMS methods exist for reducing the size of the system matrices, e.g. [1, 2, 5]. Here the Craig-Bampton [2] method, outlined in the subsequent section, is followed.

2.1 Craig-Bampton procedure

This method uses the normal modes of the components with the boundary coordinates fixed in combination with static constraint modes. The fixed interface normal modes of a single component are the natural modes of the component with all boundary coordinates fixed, given by the right-eigenvalue problem

$$\left(\mathbf{K}_\mathrm{ii} - \omega_l^2 \mathbf{M}_\mathrm{ii}\right) \boldsymbol{\phi}_l = \mathbf{0} \tag{3}$$

Here ω_l are the n_i eigenfrequencies of the fixed component and the corresponding eigenvectors $\boldsymbol{\phi}_l$ are mass-normalised so that

$$\boldsymbol{\phi}_i^T \mathbf{M}_\mathrm{ii} \boldsymbol{\phi}_i = \mathbf{I}_{ii}, \tag{4}$$

$$\boldsymbol{\phi}_i^T \mathbf{K}_\mathrm{ii} \boldsymbol{\phi}_i = \mathrm{diag}\left(\omega_1^2, \omega_2^2, \ldots, \omega_{n_i}^2\right) \tag{5}$$

where $\mathbf{I}_{ii}$ is the identity matrix of size $n_i \times n_i$. The modal matrix with respect to all component modal coordinates $\mathbf{x} = [\mathbf{x}_i^T, \ \mathbf{x}_b^T]^T$ is composed from these Ritz vectors as

$$\mathbf{\Phi} = \begin{bmatrix} \boldsymbol{\phi}_1 & \boldsymbol{\phi}_2 & \cdots & \boldsymbol{\phi}_{n_i} \\ & \mathbf{0} & & \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_i \\ \mathbf{0} \end{bmatrix} \qquad (6)$$

This set of modes is complemented by a set of static constraint modes. Both sets are linearly independent. The static constraint modes chosen are the static displacements of a component due to unit displacement of a single boundary coordinate while keeping all other boundary coordinates fixed. The matrix of constraint modes is defined by

$$\mathbf{\Psi} = \begin{bmatrix} -\mathbf{K}_{ii}^{-1}\mathbf{K}_{ib} \\ \mathbf{I}_{bb} \end{bmatrix} = \begin{bmatrix} \mathbf{\Psi}_i \\ \mathbf{I}_{bb} \end{bmatrix} \qquad (7)$$

where $\mathbf{I}_{bb}$ is the unity matrix of size $n_b \times n_b$ and the upper part corresponds to a Guyan reduction ([6]). The matrices in Eq. 6 and Eq. 7 define a transformation of the physical coordinates $\mathbf{x}$ to component modal coordinates $\mathbf{q}$ defined by

$$\begin{bmatrix} \mathbf{x}_i \\ \mathbf{x}_b \end{bmatrix} = \mathbf{B} \begin{bmatrix} \mathbf{q}_i \\ \mathbf{q}_b \end{bmatrix}, \qquad \mathbf{B} = [\mathbf{\Phi} \quad \mathbf{\Psi}] \qquad (8)$$

A reduction in the size of the component matrices is achieved by retaining only some of the fixed interface normal modes, i.e. those with the lowest eigenfrequencies. If only the first k modes are kept the transformation in Eq. 8 is approximated by

$$\mathbf{B}_k = [\mathbf{\Phi}_k \quad \mathbf{\Psi}] \qquad (9)$$

where $\mathbf{\Phi}_k$ is the matrix of kept mode shapes. Applying this transformation, the component matrices reduces to

$$\mathbf{M}_k = \mathbf{B}_k^T \mathbf{M} \mathbf{B}_k, \ \mathbf{K}_k = \mathbf{B}_k^T \mathbf{K} \mathbf{B}_k, \ \mathbf{f}_k = \mathbf{B}_k^T \mathbf{f}. \quad (10)$$

The equations of motion in terms of physical coordinates in Eq. 2 is thus transformed to the modal system

$$\begin{bmatrix} \mathbf{I}_{kk} & \mathbf{M}_{kc} \\ \mathbf{M}_{kc}^T & \mathbf{M}_{cc} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_i \\ \ddot{\mathbf{q}}_b \end{bmatrix}$$
$$+ \begin{bmatrix} \mathbf{\Lambda}_{kk} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{cc} \end{bmatrix} \begin{bmatrix} \mathbf{q}_i \\ \mathbf{q}_b \end{bmatrix} = \mathbf{B}_k^T \begin{bmatrix} \mathbf{f}_i \\ \mathbf{f}_b \end{bmatrix} \qquad (11)$$

where $\mathbf{\Lambda}_{kk}$ is a diagonal matrix of the eigenvalues ω_k^2. The sub-matrices in Eq. 11 are expressed explicitly as

$$\mathbf{M}_{kc} = \mathbf{\Phi}_k^T \left(\mathbf{M}_{ib} + \mathbf{M}_{ii}\mathbf{\Psi}_i \right),$$
$$\mathbf{M}_{cc} = \mathbf{M}_{ii} + \mathbf{M}_{ib}^T\mathbf{\Psi}_i + \mathbf{\Psi}_i^T\mathbf{M}_{ib} + \mathbf{\Psi}_i^T\mathbf{M}_{ii}\mathbf{\Psi}_i,$$
$$\mathbf{K}_{cc} = \mathbf{K}_{bb} + \mathbf{K}_{ib}^T\mathbf{\Psi}_i. \qquad (12)$$

2.2 Synthesis of components.

After the reduction at the component level, the components are assembled into global system matrices of reduced size. First, the system matrices of the components are arranged in global system matrices

$$\mathbf{M}_d = \mathrm{diag}\left(\mathbf{M}^{(1)}, \cdots, \mathbf{M}^{(s)} \right),$$
$$\mathbf{K}_d = \mathrm{diag}\left(\mathbf{K}^{(1)}, \cdots, \mathbf{K}^{(s)} \right), \qquad (13)$$
$$\mathbf{f}_d = [\mathbf{f}^{(1),T}, \cdots, \mathbf{f}^{(s),T}]^T,$$

with respect to the vector of all component coordinates

$$\mathbf{q} = [\mathbf{q}_i^{(1),T}, \ \mathbf{q}_b^{(1),T}, \cdots, \mathbf{q}_i^{(s),T}, \ \mathbf{q}_b^{(s),T}]^T, \qquad (14)$$

where s is the number of components and $\mathbf{M}^{(i)}$, $\mathbf{K}^{(i)}$ are as defined in Eq. 11. The components are coupled by enforcing displacement continuity along the interface coordinates of two components s and p,

$$\mathbf{x}_b^{(s)} = \mathbf{x}_b^{(p)} \ \text{or} \ \text{equivalently} \ \mathbf{q}_b^{(s)} = \mathbf{q}_b^{(p)}, \qquad (15)$$

leading to linearly dependent global matrices in Eq. 13. A transformation between the linearly dependent component modal coordinates $\mathbf{q}$ and the linearly independent set of global component coordinates $\mathbf{p}$ is introduced,

$$\mathbf{q} = \mathbf{S}\mathbf{p} = \begin{bmatrix} \mathbf{I}_{ii} & \mathbf{0}_{ib} \\ & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{p}_i \\ \mathbf{p}_b \end{bmatrix}, \qquad (16)$$

where $\mathbf{D}$ is defined by the relations in Eq. 15. Since only interface coordinates are related in Eq. 15, the interior coordinates are not affected by this transformation. Finally, the global system matrices with respect to the global component coordinates $\mathbf{p}$ are

$$\mathbf{M} = \mathbf{S}^T\mathbf{M}_d\mathbf{S}, \ \mathbf{K} = \mathbf{S}^T\mathbf{K}_d\mathbf{S}, \ \mathbf{f} = \mathbf{S}^T\mathbf{f}_d, \qquad (17)$$

leading to the global system

$$\begin{bmatrix} \mathbf{I}_{kk} & \mathbf{M}_{kc} \\ \mathbf{M}_{kc}^T & \mathbf{M}_{cc} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{p}}_i \\ \ddot{\mathbf{p}}_b \end{bmatrix}$$
$$+ \begin{bmatrix} \mathbf{\Lambda}_{kk} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{cc} \end{bmatrix} \begin{bmatrix} \mathbf{p}_i \\ \mathbf{p}_b \end{bmatrix} = \begin{bmatrix} \mathbf{f}_i \\ \mathbf{f}_b \end{bmatrix} \qquad (18)$$

3 Modelling uncertainty in joints

Uncertainty and variability are unavoidable. In this paper the effects of uncertainties. are considered for the particular case of uncertainties in line joints, e.g. continuously welded or glued joints.

A spatial correlation of a joint property is introduced and modelled as a random field defined by a correlation function $R(r; a, \sigma)$, where σ^2 is the variance, r is the distance between two points in the joint and a the correlation parameter. In the numerical example the specific correlation function

$$R(r; a, \sigma) = \sigma^2 \exp\{-\frac{|r|}{a}\} \qquad (19)$$

is used. Assuming an exponential dependency means that adjacent values of the uncertain parameter do not

differ, on average, as much as values that are further apart. The continuous correlation function is discretised at finite element coordinates yielding a covariance matrix. For small values of a the discretised covariance matrix is close to a diagonal matrix leading to weak correlation in space. For large values of a this matrix becomes almost fully occupied and the correlation is strong.

For a given covariance matrix the Karhunen-Loève expansion (or equivalently a Polynomial Chaos of first order, see [7]) can be applied to model a spatially correlated physical property d as

$$d\{\vartheta\} = \langle d\{\vartheta\} \rangle + \sum_{r=1}^{m} \sqrt{\lambda_r} \chi_r \xi_r\{\vartheta\} \qquad (20)$$

where ϑ is a parameter for the random subspace, $\langle \cdot \rangle$ denotes the mean value, ξ_r are Gaussian variables with mean 0 and standard deviation 1, and λ_r and χ_r are the eigenvalues and eigenvectors of the covariance matrix. In the numerical example below, only the Young's modulus Y and, consequently, the stiffness matrix of the joint is considered to be uncertain. However, it is straightforward to adapt the procedure to an arbitrary joint with uncertain stiffness and/or mass matrix. If the continuous joint is treated as a single component in the CMS-model, its stiffness matrix can be written as

$$\mathbf{K}^{(j)}\{\vartheta\} = \left\langle \mathbf{K}^{(j)}\{\vartheta\} \right\rangle + \sum_{r=1}^{m} \mathbf{K}_r^{(j)} \xi_r\{\vartheta\}, \qquad (21)$$

where $\left\langle \mathbf{K}^{(j)}\{\vartheta\} \right\rangle$ and $\mathbf{K}_r^{(j)}$ are deterministic component matrices.

3.1 Monte Carlo Simulation

MCS is a widely used numerical method for calculating the statistics of a system's response for nondeterministic system properties. For numerical calculation only a fraction of all possible property samples is considered at which the system model is evaluated repeatedly. According to Eq. 21, in this study each sample consists of a set of m random numbers $\xi_r\{\vartheta\}$. The accuracy of the method depends highly on the number of samples considered. If the number of samples is large enough convergence towards the correct statistics is guaranteed, however, with an increasing number of property samples the simulation rapidly becomes time consuming. Instead of recalculating the global system matrices for each property sample, the concept of CMS is applied which has a twofold advantage. Firstly, only the components with nondeterministic parameters need to be evaluated at each sample, while the deterministic components are calculated only once. Secondly, the dimension of the global system matrices is reduced considerably.

The Ritz vectors in Eq. 9 are recalculated for the nondeterministic joint components by applying Eq. 21 while the deterministic components are calculated only once. The reduced component matrices are derived for all components according to Eq. 11. Finally, the system

is assembled as in Eq. 18 for each sample leading to

$$\left(\langle \mathbf{K}\{\vartheta\} \rangle - \omega^2 \langle \mathbf{M} \rangle + \sum_{r=1}^{m} \mathbf{K}_r \xi_r\{\vartheta\} \right) \mathbf{u}\{\vartheta; \omega\} = \mathbf{f}.$$
$$(22)$$

For a dynamic system response $\mathbf{u}\{\vartheta; \omega\}$, the term in parentheses needs to be inverted at every frequency of interest and for each sample. Instead of direct inversion an acceleration scheme is introduced. Solving the global eigenproblem of the fully assembled system

$$\mathbf{K}\{\vartheta\} \mathbf{v}_j = \omega_j^2 \langle \mathbf{M} \rangle \mathbf{v}_j, \qquad (23)$$

these system matrices can be diagonalised to

$$\mathbf{\Lambda}\{\vartheta\} = \mathbf{V}^T\{\vartheta\} \mathbf{K}\{\vartheta\} \mathbf{V}\{\vartheta\},$$
$$\mathbf{I} = \mathbf{V}^T\{\vartheta\} \mathbf{M} \mathbf{V}\{\vartheta\}, \qquad (24)$$

where $\mathbf{V}$ consists of mass-normalised eigenvectors $\mathbf{v}_j$. The solution to Eq. 22 thus becomes

$$\mathbf{u}\{\vartheta; \omega\} = \mathbf{V}\{\vartheta\} \left(\mathbf{\Lambda}\{\vartheta\} - \omega^2 \mathbf{I} \right)^{-1} \mathbf{V}^T\{\vartheta\} \mathbf{f}. \quad (25)$$

Now only diagonal matrices need to be inverted, which is much faster (for the example system in this paper this corresponds to a reduction of about 33%) than inverting the matrix in Eq. 22.

3.2 Stochastic Reduced Basis Method

Recently, a stochastic reduced basis method (SRBM) was developed in [4] based on [8] for solving systems of linear random algebraic equations in space and the random dimension as in Eq. 22. In contrast to the classical approach in [7], a set of basis vectors spanning a preconditioned stochastic Krylov subspace is employed to approximate the system response. Subsequent application of the Galerkin scheme leads to a reduced-order deterministic system of equations with a significantly lower computational cost.

Introducing the abbreviation of the deterministic baseline system

$$\mathbf{A}\{\omega\} = \langle \mathbf{K}\{\vartheta\} \rangle - \omega^2 \langle \mathbf{M} \rangle, \qquad (26)$$

and premultiplying Eq. 22 by $\mathbf{A}^{-1}\{\omega\}$ gives the preconditioned, nonsingular algebraic equation

$$\left(\mathbf{I} + \mathbf{A}^{-1}\{\omega\} \sum_{r=1}^{m} \mathbf{K}_r \xi_r\{\vartheta\} \right) \mathbf{u}\{\vartheta; \omega\}$$
$$= \mathbf{A}^{-1}\{\omega\} \mathbf{f}. \qquad (27)$$

Instead of inverting the term in parentheses as in the previous section, the solution for the system response $\mathbf{u}\{\vartheta, \omega\}$ is approximated in the preconditioned Krylov subspace, see [4], by

$$\mathbf{u}\{\vartheta; \omega\} \approx \sum_{j=0}^{p} \alpha_j\{\omega\} \mathbf{u}_j\{\vartheta; \omega\} = \mathbf{U}\boldsymbol{\alpha} \qquad (28)$$

where α_j are deterministic coefficients and $\mathbf{u}_j$ are recursive stochastic basis vectors which forms the columns of $\mathbf{U}$ and are defined as

$$\mathbf{u}_0\{\omega\} = \mathbf{A}\{\omega\}^{-1}\mathbf{f},$$

$$\mathbf{u}_{j+1}\{\vartheta;\omega\} = \mathbf{A}\{\omega\}^{-1}\sum_{r=1}^{m}\mathbf{K}_r\xi_r\{\vartheta\}\mathbf{u}_j\{\vartheta;\omega\}. \quad (29)$$

The inverse of the system matrix Eq. 26 is calculated by applying an acceleration scheme similar to Eq. 25, but now the eigenproblem of the deterministic baseline system

$$\langle\mathbf{K}\{\vartheta\}\rangle\,\mathbf{w}_j = \omega_j^2\,\langle\mathbf{M}\rangle\,\mathbf{w}_j \quad (30)$$

is solved, leading to the diagonal matrices

$$\mathbf{\Lambda} = \mathbf{W}^T\mathbf{K}\mathbf{W}, \quad \mathbf{I} = \mathbf{W}^T\mathbf{M}\mathbf{W}. \quad (31)$$

Finally, the inverse of the baseline system is given by

$$\mathbf{A}\{\omega\}^{-1} = \mathbf{W}\left(\mathbf{\Lambda} - \omega^2\mathbf{I}\right)^{-1}\mathbf{W}^T. \quad (32)$$

In contrast to Eq. 25, Eq. 32 needs to be calculated only once at each frequency.

To determine the deterministic coefficients α_j in Eq. 28 a Bubnov-Galerkin projection scheme [4] is applied

$$\left\langle\mathbf{U}^T\left(\mathbf{A}\{\omega\} + \sum_{r=1}^{m}\mathbf{K}_r\xi_r\{\vartheta\}\right)\mathbf{U}\right\rangle\alpha\{\omega\}$$
$$= \langle\mathbf{U}^T\mathbf{f}\rangle \quad (33)$$

This deterministic algebraic equation is of size p, the number of chosen basis vectors in Eq. 28. The projection scheme guarantees convergence with increasing value of p. For mechanical systems, in general, only two or three basis vectors are needed for accurate approximation to the preconditioned Eq. 27. Note that so far only the undamped system has been considered. For the proportionally damped case (Rayleigh damping) the system matrices become complex and the transpose of $\mathbf{U}$ is replaced by the Hermitian transpose.

With the stochastic basis vectors in Eq. 29 and the deterministic coefficients from Eq. 33 the first two moments of the system response can be determined. The mean of the system response results in

$$\langle\mathbf{u}\{\vartheta;\omega\}\rangle = \sum_{j=1}^{p}\alpha_j\,\langle\mathbf{u}_j\{\vartheta;\omega\}\rangle \quad (34)$$

and the covariance matrix in

$$\langle\mathbf{u}\{\vartheta;\omega\}\mathbf{u}^*\{\vartheta;\omega\}\rangle$$
$$= \sum_{j,k=1}^{p}\alpha_j\alpha_k^*\,\langle\mathbf{u}_j\{\vartheta;\omega\}\mathbf{u}_k^*\{\vartheta;\omega\}\rangle. \quad (35)$$

Since the basis vectors in Eq. 29 are computed recursively the number of multiplications increases rapidly with increasing values of p. However, the basis vectors need never be calculated explicitly because only their moments are of interest in Eqs. 34 and 35.

3.3 Matrix identity

The calculation procedures can be optimised further by performing matrix algebra, as proposed in the following. The general form of the Sherman-Morrison-Woodbury matrix identity [9] is

$$(\mathbf{A} + \mathbf{PCQ})^{-1} = \mathbf{A}^{-1}$$
$$- \mathbf{A}^{-1}\mathbf{P}\left(\mathbf{C}^{-1} + \mathbf{QA}^{-1}\mathbf{P}\right)^{-1}\mathbf{QA}^{-1}. \quad (36)$$

Herein $\mathbf{A}$ and $\mathbf{C}$ are square matrices while $\mathbf{P}$ and $\mathbf{Q}$ are of rectangular shape. This identity is exact. If the matrix $\mathbf{C}$ has a much smaller dimension than $\mathbf{A}$, this is much more efficient than inverting $\mathbf{A} + \mathbf{UCV}$ directly. By postmultiplying Eq. 36 by $\mathbf{A}\{\omega\}$ the inverse of the term in parentheses in Eq. 27 can be written as

$$\left(\mathbf{I} + \mathbf{A}^{-1}\{\omega\}\mathbf{H}^T\mathbf{C}\{\vartheta\}\mathbf{H}\right)^{-1} = \mathbf{I} \quad (37)$$
$$- \mathbf{A}^{-1}\{\vartheta\}\mathbf{H}^T\left(\mathbf{C}^{-1}\{\vartheta\} + \mathbf{HA}^{-1}\{\omega\}\mathbf{H}^T\right)^{-1}\mathbf{H},$$

where

$$\sum_{r=1}^{p}\mathbf{K}_r\xi_r\{\vartheta\}$$
$$= \mathbf{H}^T\mathbf{C}\{\vartheta\}\mathbf{H} = \sum_{r=1}^{p}\mathbf{H}^T\mathbf{K}_r^W\mathbf{H}\xi_r\{\vartheta\}. \quad (38)$$

Here the dimension of $\mathbf{K}_r^W$ is the same as the number of joint coordinates while that of $\mathbf{K}_r$ equals the number global coordinates. Usually the number of global system coordinates is much larger than the number of joint coordinates, so that the size of $\mathbf{K}_r^W$ is much smaller than the size of $\mathbf{K}_r$. The advantage of the identity is that the inverse of the large system matrix $\mathbf{A}$ is already available from a deterministic eigenanalysis of the system. On the right hand side of Eq. (37) the inverse of $\mathbf{A}$ is still needed, but is transformed by $\mathbf{H}$ to a fraction of its original size before inversion. Based on Eq. 37, new recursive stochastic basis vectors $\mathbf{v}_j$ are defined by

$$\mathbf{v}_0\{\vartheta;\omega\} = \sum_r\mathbf{K}_r^W\xi_r\mathbf{V}^T\mathbf{u}_0\{\omega\}, \quad (39)$$

$$\mathbf{v}_{j+1}\{\vartheta;\omega\} = \sum_r\mathbf{K}_r^W\xi_r\{\vartheta\}\mathbf{V}^T\mathbf{A}\{\omega\}^{-1}\mathbf{V}\mathbf{v}_j\{\vartheta;\omega\}$$

which leads to smaller matrix operations compared to the originally assembled system in Eq. 29. This method is referred here to as the enhanced stochastic reduced basis method (ESRBM).

4 Comparison of calculation methods

A numerical example is presented to illustrate the approach. The example is shown in Fig. 1. Two plates are clamped on one edge and joined to each other on the opposite edge. The parameters of the deterministic plates and the nondeterministic joint are listed in Tab. 1. The system is discretised using a mesh of 10×5 and 8×5 thin isotropic plate elements [10] and the joints are modelled by equidistant elastic elements. The stiffness

Tab. 1 Physical and mesh parameters

	plate 1	plate 2	joint	units
ρ	2700	2700	1350	kg/m^3
Y	$7 \cdot 10^7$	$7 \cdot 10^7$	35	kN/m^2
ν	0.3	0.3	–	–
L_x	$0.5\sqrt{2}$	0.5	0.006	m
L_y	0.5	0.5	0.5	m
h	3	3	3	mm
x-elements	10	8	1	–
y-elements	5	5	5	–
	$r = L_y/5$		$\sigma = 20\%$	

of the line-coupling is uncertain and the Young's modulus is expressed by a random field with an assumed spatial correlation.

A typical frequency response between the points P1 and P2 in Fig. 1 is shown in Fig. 2a. The baseline system is plotted together with the envelopes of the system responses calculated by MCS. The frequency range of interest covers the first 12 modes of the structure with equally distributed frequency points.

Firstly, the calculation times of one sample using direct MCS are compared for the full finite element model (FEM) and the reduced model (CMS). For CMS, the first 15 modes of each plate are kept. The relative calculation times are summarised in Tab. 2, emphasising the great advantage of applying CMS to reduce the number of interior coordinates of each deterministic component while the constraint coordinates remain.

Secondly, the direct MCS with CMS is compared in terms of accuracy and computational efficiency with the projection schemes SRBM with CMS and the proposed ESRBM with CMS, both employing the preconditioned stochastic Krylov subspace. Since no approximation is made from SRBM to ESRBM both methods lead to exactly the same results. The mean values and covariance of the frequency responses calculated by MCS and SRBM are shown in Fig. 2b and c, respectively. The accuracy of SRBM depends on the number of considered basis vectors in Eq. 29 or Eq. 39. The higher the value of p the closer the results of SRBM are to those of MCS. However, for the minimum number $p = 2$ as chosen in this study, the accuracy of the first two statistical moments is good and the characteristics of the distributions are reproduced very well by SRBM. The overall calculation times for all three methods described in section 3 using CMS are summarised in Tab. 3 emphasising the computational efficiency of SRBM and ESRBM. Note that the calculation time of 100% corresponds to the calculation time using CMS, which is itself 3.1% the time of the full FEM solution in Tab. 2.

On one hand, for a MCS the system response are calculated repeatedly over the whole frequency range and the statistics are obtained by postprocessing. Alternatively, for SRBM these statistics are calculated directly at a fixed frequency and, consequently, the statistics at each frequency are calculated independently. Hence, the calculation time for MCS strongly depends on the number of samples considered while the calculation time for SRBM is not affected. So increasing the number of samples for better accuracy of MCS even enhances the efficiency of SRBM according to Tab. 3. The main difference between MCS and SRBM is the type of sampling. While MCS uses samples with respect to the stochastic subspace ϑ, SRBM uses samples with respect to the frequency ω.

5 Conclusions

For the problem considered, the proposed formulation is an efficient and effective implementation of a stochastic reduced basis method. It is seen that ES-RBM can be up to orders of magnitude faster than MCS, while providing results of comparable accuracy. Furthermore, the proposed implementation is more efficient the fewer joints are affected by uncertainty.

6 Acknowledgement

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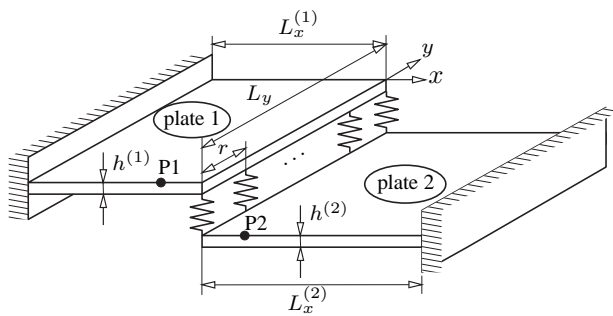
Fig. 1 Plates coupled by a joint with spatially distributed uncertainty

Tab. 2 Comparison of calculation times for baseline system at 1000 frequency points

	matrix size n	constr. DOF	relative time
FEM	324	36	100%
CMS	56	36	3.1%

Tab. 3 Comparison of calculation times using CMS

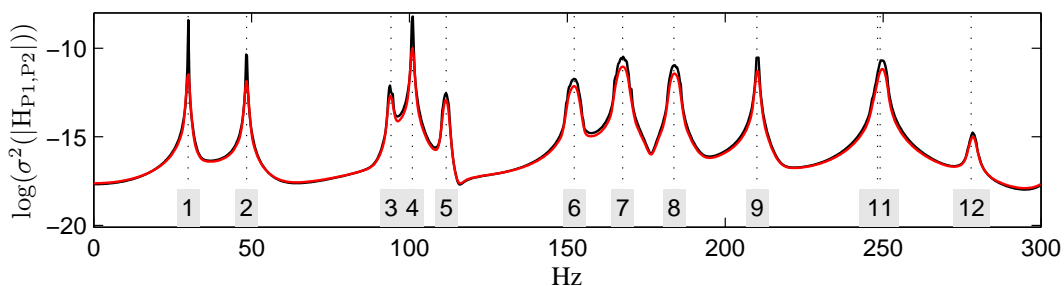| method	relative time
MCS with 1500 samples	100%
SRBM using 2 basis vectors	24.6%
ESRBM using 2 basis vectors	16.8%
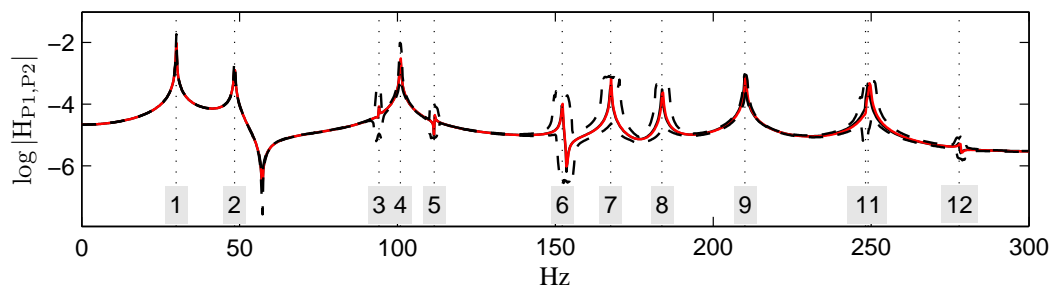
(a) Reference transfer mobility for system at mean value (1 sample): full model (black) and CMS (red), and envelopes of MCS with CMS (1500 samples) (dashed lines)
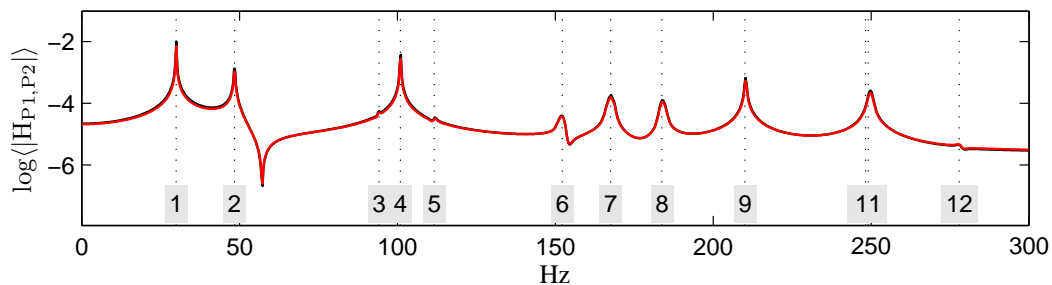
(b) Mean transfer mobility for MCS (black) and SRBM (red).

(c) First moment of transfer mobility for MCS (black) and SRBM (red).

Fig. 2 Comparison of calculation procedures MCS with CMS and SRBM with CMS