ON SIMULATION OF A BIVARIATE UNIFORM BINOMIAL PROCESS AND ITS USE TO SCAN STATISTICS

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Abstract

Scan statistics studies maximal clusters of random points on an interval (or a *map*) $I \subset \mathcal{R}^n, n \geq 1$, determined by a *scan window* which is moving on the whole I.This paper considers the problem of simulating a bivariate uniform binomial process (BUBP) on I, with the purpose of estimating the *critical value* of a *scan test.* The first section defines and studies BUBP in analogy with the bivariate uniform Poisson process (BUPP) and introduces the bivariate scan statistics on a rectangle. The second section presents details on simulating a BUBP, based on known algorithms for simulating binomial and normal variates. The third section gives the implementation of algorithms and compares estimated distributions of the scan statistics for BUBP and BUPP, concluding that these distribitions are approximately the same for a large I. Finally is presented a practical application of the simulated bivariate scan statistics to a problem of healthcare.

Keywords:Simulation of discreete stochastic processes, Scan statistics,Binomial process.

Presenting Author's Biography.

Ion C. Văduva. Professor of the University of Bucharest (see the address), Chair of Informatics. Fields of interest: Computer Simulation; Stochastic Modeling; Monte Carlo Methods; Multivariate Statistics; Software Reliability; other related fields. Published more than 100 scientific papers in international journals or proceedings; published also 21 books and lecture notes (in Romanian). Is a member of AMS and IASC.

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1 Basic notions

The aim of this paper is to study a bivariate scan statistic and to approximate (by simulation) the *critical value* of a scan test.
In [8] is considered a scan statistics for a bivariate uniform Poisson process (BUPP) of the intensity λ. Here, in a similar manner, we will consider a discrete bivariate uniform binomial process (BUBP) of the parameters p, λ,
$I, p \in (0,1), \lambda \in R$, $I \subset R^2$, defined as

Definition 1 *Let I be the bivariate interval $I = [0,T] \times [0,W], 0 < T, W < \infty$ and $n = [\lambda \times mes(I))]$, $mes(I) = T \times W, ([x] - integer\ part)$ a positive integer. Let p be a given probability, $0 < p < 1$. Let $X_1, X_2, \ldots, X_N$ be a set of random points, uniformly distributed on I where N is an integer random variable having a binomial distribution of parameters (n,p) (denoted $Bin(n,p)$). The set of points $X_1, X_2, \ldots, X_N$ is called a trajectory of the bivariate uniform binomial process of parameters(p, λ, I) on I (denoted $BUBP(p, \lambda, I)$) if:*

1). Points $X_1, X_2, \ldots, X_N$ are stochastically independent;

2). For any bivariate disjoint intervals $B_i = [\alpha_{1i}, \beta_{1i}] \times [\alpha_{2i}, \beta_{2i}], 1 \le i \le k, \alpha_{mi}, \beta_{mi} \in R, m = 1,2$ and every finite k, the number of points N_i falling in B_i is distributed as $Bin(n_i, p)$, $n_i = [\lambda(\beta_{1i} - \alpha_{1i})(\beta_{2i} - \alpha_{2i})]$, and N_1, $N_2, \ldots, N_k$ are independent random variables. The constant λ will be also called the intensity of the process.

This binomial process has a *property of stability* similar to a Poisson process, namely

Theorem 1 *If $B_1, B_2, .., B_k$ are disjoint subsets in the interval $I = [0,T] \times [0,W]$ and $\{X_t\}$ is $BUBP(p, \lambda, I), t \in \mathcal{N}$ then the processes $\{X_i(t)\}$, $t \in \mathcal{N}$, are $BUBP(p, \lambda, B_i)$ and X_i is independent of X_j, $i \ne j$. Particularly, if $B \subset I$ then X_k is $BUBP(p, \lambda, B)$, $k = [\lambda mes(B)]$. On the other hand if $I = B_1 \cup B_2 \cup \ldots \cup B_m$, $B_i \cap B_j = \emptyset, i \ne j$ and $\{X_t\}$ is $BUBP(p, \lambda, B_i)$ then $X_{B_1} + X_{B_2} + \ldots + X_{B_m}$ is a $BUBP(p, \lambda, I)$.*

Proof. The proof can be easily done using the characteristic function of the binomial distribution. Thus, for the binomial distribution (X *is* $Bin(n,p)$) the characteristic function is

$$\varphi(t) = E[e^{itX}] = (p + qe^{it})^n,\ t \in R, \quad (1)$$

$q = 1 - p$.

Let us consider the random variables $X_{B_i}, 1 \le i \le m$ which are independent (points defining X_{B_i} beeing uniformly distributed on B_i), and X_{B_i} *is* $Bin([\lambda mes(B_i)], p)$. Then the random variable $Y = X_{B_1} + X_{B_2} + \ldots + X_{B_m}$ has the characteristic function of $Bin(n,p)$ distribution, i.e.

$$\varphi_{Y(t)} = \prod_{i=1}^{m} \varphi_{X_{B_i}}(t) = \prod_{i=1}^{m} (p + qe^{it})^{n_i} =$$

$$(p + qe^{it})^n, n = \sum_{i=1}^{m} n_i, n_i = [\lambda mes(B_i)].$$

The last formula gives the end of the proof.

Definition 2 *Let $I = [0,T] \times [0,W]$ be a two-dimensional interval and $u, v > 0$ two positive numbers such as $0 < u < T < \infty$, $0 < v < W < \infty$. (The numbers u, v define a two-dimensional scan window with dimensions u and v). Assume that in the interval I there are N points $\{X_1, X_2, \ldots, X_N\}$ which are uniformly distributed on I. Denote $\nu_{t,s} = \nu_{t,s}(u,v)$ = the number of points which fall in the scanning window $[t, t+u] \times [s, s+v]$. Then the bivariate scan statistic is*

$$S = S((u,v), T, W) =$$

$$= \max_{0 \le t \le T-u,\ 0 \le s \le W-v} \nu_{t,s}. \quad (2)$$

The points $X_i, 1 \le i \le N$ could be a trajectiry of a BUBP or BUPP. The probability of interest is:

$$P(S((u,v), T, W) \ge k) = P_k((u,v), T, W). \quad (3)$$

The probability distribution (3) is hard to calculate. Therefore a simulation procedure is the simplest way to estimate it. In [2] is introduced a method for estimating the probability distribution (3) using the simulation of conditional scan statistics and relation between this and actual scan statistics.
We estimate the probability distribution (3) using the simulation of scan statistic described by the following algorithm [8]:

Algorithm SIMSCAN

- Input W, T, u, v, N, m;

1. for $j = 1$ **to** m **do**

begin

- generate $X_1, X_2, ..., X_N$, N points uniformly distributed on $I = [0,T] \times [0,W]$;
- Determine $S_{(u,v)}$, take $K_j = S_{(u,v)}$;

end;

(*In the Section 3 we describe the implementation of the algorithm SIMSCAN which determines $S_{(u,v)}$, denoted there by n_w*).

2. Determine the empirical distribution of the sample $K_1, ..., K_m$ as follows:

- Determine the order statistics $K_{(1)} < K_{(2)} < ... < K_{(r)}$, $r < m$;
- Determine the frequencies f_i, $1 \leq i \leq r$, f_i = number of sampling values $K's$ equal to $K_{(i)}$, $1 \leq i \leq r$, $\sum_{i=1}^{r} f_i = m$;
- Determine the relative frequencies (i.e. sampling probabilities) $\pi_i = \frac{f_i}{m}$. **Stop**

(*In fact, step 2 builds-up a histogram of the scan statistics*).

If m is large enough, then the sampling distribution converges to (3) (according to the consistency property of the estimates π_i).
Given a probability α, $0 < \alpha < 1$ one can determine the critical test value k_α of the scan statistics defined as

$$P(S_{(u,v)} > k_\alpha) = \alpha. \tag{4}$$

The number N of random points uniformly distributed on $I = [0,T] \times [0,W]$ is assumed to have a binomial distribution and therefore $X_1, X_2, ..., X_N$ is a trajectory of a bivariate uniform binomial process.

In the next section we discuss the simulation of such a trajectory. Some methods for simulating binomial and normal distributions are used.

2 Algorithms for the simulation of a bivariate uniform binomial process

The Definition 1 leads to the following algorithm for simulating a *trajectory* of N points of the bivariate uniform binomial process of the intensity λ:

Algorithm SIMBIN2

- $i = 0$,input $\lambda, W, T, p, 0 < p < 1$;
- calculate $n = [\lambda \times W \times T]$;
- Generate N a sampling value of $Bin(n,p)$;

repeat

- Generate $U\, uniform\, [0,T]$ and $V\, uniform\, [0,W]$;
  (This can be done as follows:
  -Generate U_1 uniform $(0,1)$;
  Take $U = U_1 T$;
  -Generate U_2 uniform $(0,1)$;
  Take $V = U_2 W$;)
- $i := i + 1$,; take $X_i = (U,V)$;

until $i = N$.**Stop**

The algorithm produces the trajectory $X_1, X_2, ..., X_N$ of the $BUBP(p, \lambda, I)$. Now, using the algorithm SIMSCAN for the bivariate case, we can determine an empirical distribution of the scan statistics (i.e. a histogram) and then, estimate the critical value k_α.

The simulation of the random variable X which is binomially distributed with the parameters $p, n, 0 < p < 1$, $n \in \mathcal{N}^+$, can be done in various ways (see [8]). For large n,we use the fact that $X := Bin(n,p) \approx N(np, \sqrt{npq})$, $q = 1 - p$ i.e. X is normally distributed. Therefore, the algorithm to simulate X is the following [7]

Algorithm BINCL

- Input n, p; Calculate $m = np, \sigma = \sqrt{npq}$;
- Generate $Z\, normal\, N(0,1)$;
- Calculate $X = m + Z\sigma$;
- Take $N = round(X)$. **Stop**

The function $round(x)$ means the closest positive integer to x.

Simulation of a normal deviate $Z := N(0,1)$ can be done in several ways; two methods will be presented in short in the following. The first one is based on *Central Limit Theorem* (CLT) [5,6,7].

Algorithm CLNORM (simulates Z *normal* $N(0,1)$ based on CLT)

- $Z = 0$;

• **for** $i := 1$ **to** 12 **do begin**
Generate U uniform $(0,1)$;
$Z := Z + U$;
end; Stop

Another algorithm combines a *rejection* (*enveloping*) and a *discrete composition* method [6,7]. It looks as follows:

Algorithm RJNORM

repeat
• Generate U_1 $as\,uniform(0,1)$;
• Generate Y $Exp(0,1)$;
(This can be done by the *inverse* method as follows:
• Generate U $uniform(0,1)$;
while $U \leq 0.0000001$ **do** Generate U *uni–form*$(0,1)$;
• $Y := -\log(U)$);
until $U_1 \leq e^{-\frac{Y^2}{2}+Y-0.5}$;
• Take $Z_1 := Y$;
• Generate $U\,uniform\,(0,1)$;
• **if** $U \leq 0.5$ **then** $s := 1$
else $s := -1$; (s *is a random sign*);
• take $Z := sZ_1$. (Z is $N(0,1)$).**Stop**

In the following section we give some results on the implementation of the algorithm SIMSCAN for producing the empirical distribution of the scan statistics when the points (T_i, W_i), $1 \leq i \leq N$ are realizations of a bivariate uniform binomial process on $[0,T] \times [0,W]$ with intensity λ. Comparisons with the results of Alm [1] and with the results of Haiman and Preda [2], in the case of BUPP process, are presented.In order to compare the results with the bivariate uniform binomial process, we use the fact that the binomial distribution $Bin(n,p)$, with n large is approximated by a Poisson distribution $Poisson(\lambda)$, $\lambda = np$. When we refer to the $BUPP(\lambda, I)$ process and to the binomial $BUBP(p, \lambda', I)$ process, both on $I = [0,T] \times [0,W]$, we must distinguish betweem the intensities λ (for Poisson) and λ' (for binomial). In fact, for n large, we must have

$$\lambda WT = \lambda' pWT \tag{5}$$

which gives

$$\lambda' = \frac{\lambda}{p}. \tag{5'}$$

3 Implementation and test results

In this implementation we use one of the programs presented in [8], namely the algorithm called SCAN2, which derives from SIMSCAN. The discrete process used in this implementation is either $BUPP$ or $BUBP$ according to SIMBIN2.

In the following, we underline the main ideas of SCAN2 (see [8]). Fig.1 gives some hints on the construction of SCAN2. In the figure are represented: the map, the scan window and some points to illustrate the scan process;it shows also different positions of the scan window. The first position is on the top-right corner; it moves down until covers the vertical band; then moves to the left,then moves on the new corersponding vertical band, and so on. Some details are explained bellow.

We suppose that the scan surface (i.e the *map*) and the scanning window are rectangles with the sides parallel with the horizontal and vertical axes, having dimensions (T, W) respectively (u, v).

Furthermore we suppose that both the scan surface and the scanning window are defined by two of their corners: the upper right corner and the lower left corner.

We denote these corners by S_{right} and S_{left} for the surface, and W_{right} and W_{left} for the window. Initially $S_{right} = (T, W)$ and $S_{left} = (0, 0)$.

After generating the points $(T_i, W_i), 1 \leq i \leq M$, realizations of an bivariate uniform binomial process on $[0,T] \times [0,W]$ with intensity λ, we begin the scanning process.

First we order the simulated points with respect to coordinates T_i.
Assume that this was already done. Then, let us assume that the first position of the scanning window is characterized by the coordinates:

$$W_{right} = (T, W), \quad W_{left} = (T - u, W - v),$$

i.e. on the top-right corner (see Fig 1a).

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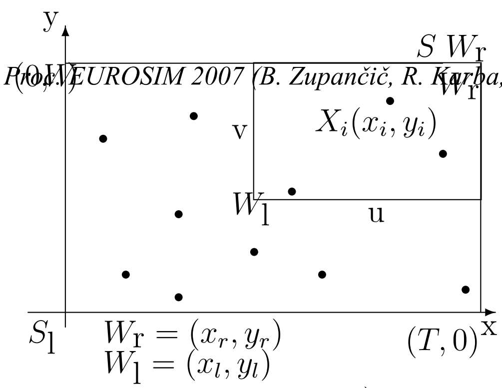

Fig. 1 a)

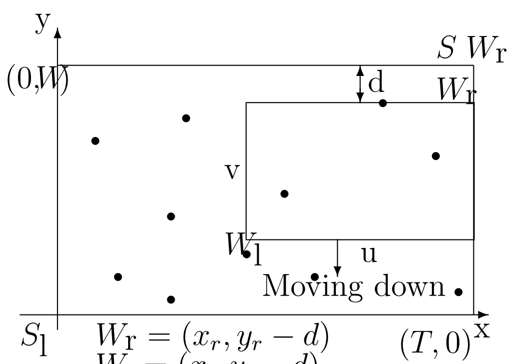

Fig. 1 b)

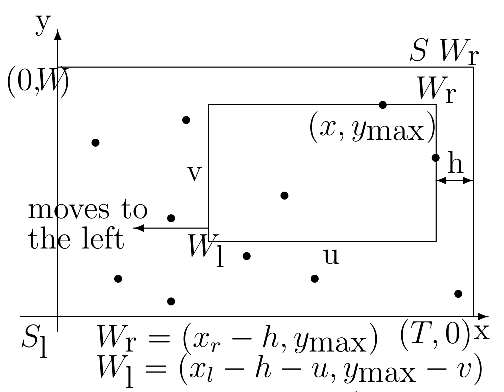

Fig. 1 c)

Fig.1. Hints for scanning algorithm:
($S_l = S_{left}, S_r = S_{right}$..etc for W)
a) initial pozition of scan window;
b) moving down the scan window;
c) moving window to the left.

The scanning window moves as we specified. If we assume that the window is characterized by the coordinates $W_{right} = (x_r, y_r)$, $W_{left} = (x_l, y_l)$, then the following position of the scan window will be: $W_{right} = (x_r, y_r - d)$, $W_{left} = (x_l, y_l - d)$ where $d = \min\{y_r - y_{max}, y_l\}$ and y_{max} is the biggest coordinate y from the band which is smaller than y_r. (See Fig 1 b).

After a band was entirely scanned, the scanning window is moved on the next band in the following way: if the last position on the previous band was characterized by $W_{right} = (x_r, y_r)$, $W_{left} = (x_l, y_l)$, then the present position is characterized by: $W_{right} = (x_r - h, y_{max})$, $W_{left} = (x_r - h - u, y_{max} - v)$ where $h = \min\{x_r - x_{max}, x_l\}$, x_{max} is the biggest coordinate x smaller than x_r, and y_{max} is the maximum value of W_i for the points which have $x_r - h - u \leq T_i \leq x_r - h$. We use this method of scan because the simulated pointshave the coordinates T_i in increasing order (see Fig 1 c).

For each position of the window there is counted the number of points that are in the window and is stored the largest number n_w of points found during the scan process. This maximum n_w is a simulation value of the bivariate scan statistics, (i.e. $n_w = S_w = S$ in the notation of Section 1). By repeating the algorithm SCAN2 for N runs or *iterations* ($N-$large), one determines the empirical distribution of the scan statistics.

The following tables (Tab.1,Tab.2) contain test results. In each table there are also mentioned for comparison, simulated results produced by Alm [1] and approximations produced by a special method due to Haiman and Preda [2]. (Some of the tables are reproduced fron [8]). On the top of each table are mentioned particular values of the input data used, namely:

- λ intensity of the bivariate Poisson process;
- W, T dimensions of the rectangle;
- u, v dimensions of the scanning window;
- N number of simulation runs;
- p and λ' refer to different values of parameters of binomial processes corresponding to the approximate parameter of the Poisson process (determined according to (5),(5')).
- k is the value of scan statistics for which is calculated empirical probability;
- $H\&P$ in the table refers to the results from "Haiman and Preda" [2].
- P refers to BUPP; A refers to Alm; B refers to BUBP; The entries in the following tables represent probabilities $P(S \leq k)$ where $S = S((u, v), T\ , W)$ is the bivariate scan statistics from Definition 1.

The results in the tables show a good agreement between the distributions of scan statistics for all the compared cases (i.e. Poisson, Alm, $H\&P$ and binomial). For values of k of practical interest (see bellow), the values of $P(S \leq k)$ are almost equal for both $BUPP$ and $BUBP$.

Tab.1.Simulated results.Comparisons.

$\lambda = 0.05, W = T = 10, u = v = 1,$
$N = 10000,\ p = 0.1, \lambda' = 0.5$

k	P	H&P	A	$B(p, \lambda')$
2	0.9859	0.9854	0.9905	0.8547
3	0.9998	0.9996	0.9997	0.9798
4	1.0000	0.9999	0.9999	0.9982

$\lambda = 0.1, W = T = 50, u = v = 1,$
$N = 10000, p = 0.01, \lambda' = 10$

k	P	H&P	A	$B(p, \lambda')$
3	0.8762	0.8761	0.9052	0.8719
4	0.9957	0.9957	0.9966	0.9944
5	1.0000	0.9998	0.9999	0.9998

$\lambda = 0.5, W = T = 10, u = v = 1,$
$N = 10000, p = 0.1, \lambda' = 5$

k	P	H&P	A	$B(p, \lambda')$
4	0.7865	0.7938	0.8343	0.7932
5	0.9692	0.9707	0.9759	0.9680
6	0.9968	0.9970	0.9974	0.9971
7	0.9999	0.9997	0.9997	0.9999

$\lambda = 1,\ W = T = 10,\ u = v = 1,$
$N = 10000\ p = 0.1, \lambda' = 10$

k	P	H&P	A	$B(p, \lambda')$
6	0.8396	0.8248	0.8603	0.8335
7	0.9695	0.9468	0.9732	0.9690
8	0.9956	0.9691	0.9959	0.9954

The following tables compare only our results with the results from the implementation of Alm for BUPP. The estimated frequencies (estimated probability distribution) shown in tables refer only to some interesting values of the bivariate scan statistics. These tables are reproduced from [8].

Tab.2. Further comparisons.

$\lambda = 2, W = T = 20, u = v = 1,\ N = 10000$
$p_1 = 0.01, \lambda'_1 = 200, p_2 = 0.1, \lambda'_2 = 20$

k	A	$B(p_1, \lambda'_1)$	$B(p_2, \lambda'_2)$
7	0.0004	0.0002,	0.0001
9	0.5283	0.5100	0.5119
11	0.9640	0.9692	0.9653

Tab2.(continued)

$\lambda = 5, W = T = 20, u = v = 1,\ N = 10000$
$p_1 = 0.01, \lambda'_1 = 500, p_2 = 0.1, \lambda'_2 = 50$

k	A	$B(p_1, \lambda'_1)$	$B(p_2, \lambda'_2)$
13	0.0040	0.0002	0.0005
15	0.2535	0.2645	0.2610
17	0.8442	0.8509	0.8457

During various runs it resulted a convergence of the frequencies to the probabilities calculated by Haiman and Preda [2]. The number of runs $N = 10000$ considered in the tables seems to be large enough to ensure a good estimate of the probability distribution of the scan. Any $N > 10000$ will be recomended.

On the other hand, it was observed that convergence is ensured for large values of the map (i.e. W, T) with respect to the scan window (i.e. u, v); a large ratio of $WT/(uv)$, increases the convergence, for the same N. The tests done here legitimate both assumptions (Poisson or binomial) for defining, via simulation, the critical value of the scan test. Therefore, in the next section (application) we will use the Poisson process. (Runs for BUBP are time consuming!).

BUBP and BUPP processes may be used as equal alternatives in various applications where discrete random (*uniform*) events can occur on some surface of material or geographic area.

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4 An Application to Healthcare

Here we present an application of scan statistics to analyze the cancer disease for children under age 16 in the region North Pas de Callé (north of France). The region consists of two departments, each department contains some *arondisments* and an arondisment consists of *cantons.*

The data consisted in the number of diseased children in each canton (considered the scan window). The total population in the region is about 573500 inhabitants and total number of ill children is $N = 497$. In one canton of the first department was found the largest number of ill children as beeing 9 from a population of $\pi_1 = 1600$ and in other canton of the second department were found 7 ill children from a population of $\pi_2 = 2300$ inhabitants. These two cantons contain the largest figures of ill children.Administrative authorities want to know if these large figures are *natural* or they are determined by some environmental fators of cantons.(The whole region is a mining region!). Therefore, under the *natural hypothesis* (denoted H_0) we asume that number of diseased children in the region is a BUBP (or BUPP) process and we must test the hypotheses H_{01} and H_{02} that the numbers of 9 respectively 7 ill children are considered *normal* or *dangerous* events from the healthcare point of view. Therefore we are in the theoretical situation discussed in the previous sections.

The collection of data for our application follows from the procedure used in [3,4] which defines the dimensions of the hypothetic geographic region (i.e. the map) taking into consideration the seize of population in the region and defines the scan window using the size of population in the cantons with the largest number of ill children. As the geographical map of the region is not a regular one, we consider it as a square $[0, W] \times [0, T]$ with $W = T = \sqrt{P}$ where P is the seize of population of the region (in our case $P = 573500$), hence $W = T = 757.3$. Similarly, the scan windows lengths are $u_1 = v_1 = \sqrt{\pi_1} = \sqrt{1600} = 40$, $u_2 = v_2 = \sqrt{\pi_2} = \sqrt{2300} = 47.95$. The intensity of the Poisson process (over the region) is $\lambda = \frac{N}{P} = 0.0008666$ and the parameters for Poisson processes for the two cantons are $\Lambda_1 = \lambda\pi_1 = 1.384$, $\Lambda_2 = \lambda\pi_2 = 1.9918$. To use the bivariate uniform binomial processes, we need to estimate parameters p_1, p_2. These are simply defined as $p_1 = \frac{7}{N} = 0.014$, $p_2 = \frac{9}{N} = 0.018$. Hence, according to (1.5') we have for BUBP the parameters: $\lambda_1' = \Lambda_1/p_1 = 99, \lambda_2' = \Lambda_2/p_2 = 110.6$. (For BUBP thesefigures are not used).

In order to test the mentioned hypotheses H_{01}, H_{02} we use the simulation procedure presented in the previous sections. We use also the property of the scan statistics which says that

$S = S((u, v), W, T) = S((1, 1), W/u, T/u)$.

Hence, for the first canton $W_1 := W/u_1 = T_1 := T/v_1 = 747/40 = 18.93$, $W_2 := W/u_2 = T := T/47.35 = 757.3/\ 47.35 = 15.77$.

The results of simulation for data under Poisson hypothesis, are resumed in the tables (Tab.3, Tab.4) which contain the values of $S = k$ and corresponding frenquences f for the two cantons:

Tab.3.Results for canton 1.

$W = T = 18.93, u = v = 1, \Lambda = 1.3865,$
$N = 100000 = iterations$, f=frequensy of
$S = k.$

k	6-8	9	10	11-13
f	82137	14302	3006	555

Tab.4. Results for canton 2.

$W = T = 15.77, u = v = 1, \Lambda = 1.99318,$
$N = 100000 = iterations$, f=frequency of
$S = k.$

k	7-8	9	10	11-16
f	23364	44235	23462	7929

From the first table one can see that $P(S \leq 9) \approx 0.96439$. Therefore H_{01} can be accepted with a risk of $\alpha = 0.03561$. (Hence $k_\alpha = 9$ and the critical region of the scan test is $\mathcal{C} = \{k | k > k_\alpha\}$).

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From the second table one can see that $P(S \leq 10) \approx 0.91071$. The hypothesis H_{02} is also accepted with $\alpha = 0.09929$, $k_\alpha = 10$, and the critical region $\mathcal{C} = \{k|k > k_\alpha\}$. Since in the second case (the canton 2) there are 7 ill children, and this is the *second large value* in the region, the frequencies in the second table must be moved one step to the left. Therefore for the second large value (i.e. $k = 7$) the critical region is $\mathcal{C} = \{k|k > k_\alpha\}, k_\alpha = 11, \alpha = 0.01946$ and this gives a better reason to accept the hypothesis H_{02}.

In conclusion, the figures of ill children ($k = 9, k = 6$) are *natural.* There are no problems for authorities, concerning the cancer healtcare.

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