ALGEBRAIC METHODS IN MULTIVARIATE POLYNOMIAL INTERPOLATION

Aihua Li[1], Xiangjuan Min[2]

[1]Montclair State University, Department of Mathematical Science
Montclair, New Jersey, 07043, USA
[2]University of Science and Technology Beijing, Beijing, China

lia@mail.montclair.edu (Aihua Li)

Abstract

In this paper we apply the Gröbner Bases Techniques to solve one type of multivariate polynomial interpolating problem which has applications in modeling finite discrete time series. Consider a set of *n*-dimensional points $T = \{P_1, P_2, \cdots, P_m, P_{m+1}\}$ over the field $\mathbb{R}$ of real numbers**.** The multivariate polynomial interpolation problem with respect to the set $T' = \{P_1, P_2, \cdots, P_m\}$ is stated as: given $\mathbf{b} = (b_1, b_2, \cdots, b_m) \in \mathbb{R}^m$**,** find a polynomial f (in n variables) such that $f(P_i) = b_i$ for all $i = 1, 2, \cdots, m$. We call such an f an interpolator on or on T' targeting at **b**. Further, we view the set *T* as a discrete time series. A polynomial model of *T* is a function $\mathbf{f} = \{f_1, f_2, \cdots, f_n\}$ such that $f(P_i) = P_{i+1}$ for all $i = 1, 2, \cdots, m$**.** Each f_i is an *n*-variable polynomial interpolator on T'. We demonstrate methods using different monomial orders (or term orders) to construct polynomial interpolators of different types. A *Maple* code is developed based on the *Buchberger-Möller Algorithm* to construct separators which play a key role in the interpolation. We show that the set of separators constructed by the algorithm under a fixed order is unique. The behavior of two term orders, the *lex* order and the *graded reverse order*, in the construction of the interpolators are investigated. A relationship between the number of variables appearing in the constructed interpolator and its total degree is given. A method to construct single variable interpolators is discussed. Examples are provided to illustrate the process.

Keywords: Gröbner Bases, separator, polynomial interpolation.

Presenting Author's biography

Aihua Li**.** Dr. Aihua Li is an associate professor at Montclair State University located in New Jersey, USA. She received her Ph. D. from the University of Nebraska-Lincoln in the area of commutative algebra with concentration on structures of commutative Noetherian rings and prime ideals. Her current research interests include solving polynomial and matrix equations, difference equations, ring theory, number theory, algebraic applications in modeling discrete time series and the effective computational methods. Dr. Li's research programs have been funded by various state and national agencies. Currently she has several active research projects with undergraduate students, funded by an NSF Center for Undergraduate Research in Mathematics program. She is an active referee for six journals and a reviewer for the Mathematics Reviews.

1 Introduction

The polynomial interpolating problem is a classic mathematics problem which attracts many researchers. Although much work has been done on single variable polynomial interpolation, multivariate interpolation received much less attention because of its difficulty [1, 2]. Our initial approach is to construct polynomial models in n variables to fit a finite set of n dimensional points over a field, which represents a discrete time series. This is really a multivariate polynomial interpolation problem on the considered time series points. A polynomial model of a discrete time series with n points is an n-variable polynomial which interpolates the points in an iterative way. We apply the Gröbner Bases Techniques [3] and the well-known Buchberger-Möller Algorithm [4] to construct multivariate polynomial interpolators for the time series points which can also serve as a model for the time series. It provides an algebraic approach to solving certain multivariate polynomial interpolation problems. A key element of our construction is to find correlative separators which are different under a different monomial order.

Many real world problems can be formulated as certain time series problems. Motivated by the increasing trend in genetic studies and fast growing computer technologies, the study of discrete time series is gaining more and more attention [5, 6, 7, 8]. Such a time series can be obtained from data achieved from scientific experiments or social activity surveys. We focus on n-dimensional points over the field $\mathbb{R}$ of real numbers. Throughout, we assume the time series in consideration have no identical rows and in most cases, the number of points is less than or equal to n. The setting and noations are as follows:

Consider the field $\mathbb{R}$ and the polynomial ring $R = \mathbb{R}[x_1, x_2, \cdots, x_n]$ $(n > 0)$. Suppose m is a positive integer. Consider a discrete time series $P_1, P_2, \cdots, P_m, P_{m+1}$, where each $P_i = (p_{i1}, p_{i2}, \cdots, p_{in}) \in \mathbb{R}^n$.

Definition 1.1 *A model for the time series* $T = \{P_1, P_2, \cdots, P_m, P_{m+1}\}$ *over* $\mathbb{R}$ *is a function* $\mathbf{f}$ *from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$ *such that* $\mathbf{f}(P_i) = P_{i+1}$ *for each* $i = 1, 2, \ldots, m$. *If all the components of* $\mathbf{f}$ *are polynomials (in* n *variables), we say* $\mathbf{f}$ *is a polynomial model.*

In detail, if $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ is a model for T, then for each j from 1 to n, $f_j(P_i) = p_{(i+1)j}$ for all $i = 1, 2, \ldots, m$. Each component f_j of $\mathbf{f}$ serves as an n-variable polynomial interpolator on the points $P_1, P_2, \cdots, P_m$ with certain special "targets". Finding such a model for the time series is equivalent to solving the following interpolation problem repeatedly for the specified "b"-values stated in the problem below:

Problem 1.2 *The Interpolation Problem*

Given $\mathbf{b} = [b_1, b_2, \cdots, b_m] \in \mathbb{R}^m$, *find a polynomial* $f \in R$ *such that* $f(P_i) = b_i$ *for all* $i = 1, 2, \ldots, m$. *We call such an* f *an interpolator on* $P_1, P_2, \cdots, P_m$ *targeting at* $\mathbf{b}$. *The vector* $\mathbf{b}$ *is also called the target of* f.

Definition 1.3 *Let* $\mathbf{b} = [b_1, b_2, \cdots, b_m]$ *be a vector of real numbers. The set of interpolators on* $P_1, P_2, \cdots, P_m$ *targeting at* $\mathbf{b}$ *is denoted by*

$$(P_1, \cdots, P_m : \mathbf{b}) = \{f \in R \mid f(P_i) = b_i \text{ for all } i\}.$$

Interpolators targeting at the zero vector plays an important role here. It is known that the set $(P_1, P_2, \cdots, P_m : \mathbf{0})$ forms a finitely generated ideal of the ring R. Furthermore,

$$(P_1, P_2, \cdots, P_m : \mathbf{b}) = f_0 + (P_1, P_2, \cdots, P_m : \mathbf{0}),$$

where f_0 is any fixed polynomial in $(P_1, \cdots, P_m : \mathbf{b})$. Thus a particular interpolator and a generating set for the ideal $I = (P_1, P_2, \cdots, P_m : \mathbf{0})$ build all the interpolators. By Gröbner bases thechniques, for each ordering of monomials of R, there exists a unique generating set $g_1, g_2, \ldots, g_r$ of I, called " the reduced Gröbner basis for I" [3]. Every interpolator has the form of

$$f_0 + h_1 g_1 + \cdots + h_r g_r.$$

The basis shown above can provide valuable information about the interpolators. We give a brief background here.

Definition 1.4 *A term order (or monomial order)* $>_\sigma$ *on the set of monomials of* R, $\{\mathbf{x}^\alpha \mid \alpha \in \mathbb{Z}^n_{\geq 0}\}$, *is a total ordering and well ordering such that* $\mathbf{x}^\alpha >_\sigma \mathbf{x}^\beta$ *implies* $\mathbf{x}^{\alpha+\gamma} >_\sigma \mathbf{x}^{\beta+\gamma}$ *for all* $\alpha, \beta, \gamma \in \mathbb{Z}^n_{\geq 0}$.

For example, a term order σ, called Graded Reverse Lex Order, satisfies the following conditions: for any two monomials $\mathbf{x}^\alpha$ and $\mathbf{x}^\beta$, where $\alpha, \beta \in \mathbb{Z}_{\geq 0}$, $\mathbf{x}^\alpha >_\sigma \mathbf{x}^\beta$ if $|\alpha| > |\beta|$ or if $|\alpha| = |\beta|$ but the right most nonzero entry of $\alpha - \beta$ is negative. Under this order, $1 <_\sigma y <_\sigma x <_\sigma y^2 <_\sigma xy <_\sigma x^2 <_\sigma y^3 <_\sigma xy^2 <_\sigma x^2y <_\sigma x^3 <_\sigma y^4 <_\sigma \cdots$ in $\mathbb{R}[x, y]$.

Basically, a term order makes a line-up for all the monomials of R so that one can perform necessary long division of multivariable polynomials similarly to that of single varable polynomials. Several software packages, such as *Singular* [9], *Macaulay* [10, 11], and *CoCoA* [12], are available to perform the computation needed in finding the desired bases. We focus on constructing interplators by finding separators with a pre-selected term order. A Maple code was developed and applied in the computation based on the Buchberger-Möller Algorithm.

The main results, which characterize some properties and structures of the seperators and interpolaors, are given in section 3.

2 The Roles of Seperators and their Supports

Finding a particular interpolator is not a problem. As in the single variable case, we can always find one by computing a product of appropriate linear polynomials. But such an interpolator has a total degree much higher than n, the number of variables. In the single variable case, a traditional method to construct interpolators is through building the separators. Then an interpolator is obtained by taking an appropriate linear combination of the separators. In this section, we show a similar approach. Our focus is on the role of separators in building interpolators.

Definition 2.1 *Let U be a subset of $\mathbb{R}^n$ and P be a point in $\mathbb{R}^n$. A separator of P from U, denoted by s_P, is a polynomial in R such that $s_P(P) = 1$ but $s_P(\mathbf{u}) = 0$ for all $\mathbf{u} \in U \setminus \{P\}$.*

Definition 2.2 *For a given polynomial $f(x)$, the support of f is defined to be the set of all monomials involved in f which have nonzero coefficients. It is denoted by supp(f). For example, supp$(3x_1^2x_2^3 - 5x_3^4 + 2) = \{x_1^2x_2^3, x_3^4, 1\}$.*

Now consider the interpolating problem 1.2. Assume for each i, we have a separator s_{P_i} of P_i from the rest of the points. Then an interpolator f_0 is obtained as a linear combination of the seperators:

$$f_0 = b_1 s_{P_1} + b_2 s_{P_2} + \cdots + b_m s_{P_m} \in (P_1, \ldots, P_m : \mathbf{b}).$$

Thus the interpolator obtained from a set of separators as above is determined by the form of the separators. For instance, if all the separators are linear, then the interpolator produced is linear. If all the spereators are single-variable polynomials involving the same variable, then the interpolator produced is also the same type which must have total degree no more than $m - 1$. The advantage of applying Gröbner bases techniques is that we can pre-select a term order to produce a set of separators in our favor. For example, if we want to get an interpolator with the least total degree, a degree preferred term order, such as the graded reverse lex order, should be chosen.

Another approach is based on the fact that $R/I(P_1, \ldots, P_m) \cong \mathbb{R}^m$ as a vector space over $\mathbb{R}$ [3]. The supports of certain sets of separators for $P_1, \ldots, P_m$ can be used to obtain a vector space basis (over $\mathbb{R}$ for $R/I(P_1, \ldots, P_m)$). Such a basis can also be used to construct an interpolator for Problem 1.2.

3 Applying Buchberger-Möller Algorithm

An effective algorithm, *Buchberger-Möller Algorithm* [4], can be applied to construct interpolators stated in Problem 1.2. We give a brief description of the process. Recall that our propurse is to interpolate the points $P_1, P_2, \ldots, P_m$ at the values $b_1, \ldots, b_m$. Let $T = \{P_1, P_2, \ldots, P_m\}$.

Algorithm 3.1 *Buchberger-Möller Algorithm*

Here is the main precedure of the algorothm.

- Select a term order σ on the monomials of $\mathbb{R}[x_1, x_2, \ldots, x_n]$.
- Input the points in T.
- The output will be:

(1) The unique reduced Gröbner Basis $G = \{g_1, \ldots, g_s\}$ for $I(T)$;

(2) A list of separators $S = \{s_1, \ldots, s_m\}$, where each s_i is a separtor of P_i from T, for $i = 1, \ldots, m$;

(3) A set V of monomials of R which forms a basis for the m-dimensional $\mathbb{R}$-vector space $R/I(T)$.

The computation complexity is Quadratic in the number (n) of variables and cubic in the number (m) of the points [4]. We developed a *Maple* code to perform the computation.

Simultaneously, the algorithm produces the unique Gröbner basis for $I(T)$, a set of separators of P_i from T ($i = 1, 2, \ldots, m$), and a vector space basis V of $R/I(T)$ made of m monomials. All of these products can help constructing interpolators of T and identifying the types of the interpolators. Since these separators are built along with the reduced Gröbner Basis, they have been through the "reduction process". In particular, when a degree-preferred term order is selected, the total degree of the produced interpolator is usually much lower than the ones constructed by other methods. At each step, the algorithm always searches and picks functions with leading terms as "small" (under the selected term order) as possible. Thus one can control the types of resulting interpolating polynomials by selecting appropriate term orders. In addition, the set V is produced form the non-leading terms of the unique Gröbner basis of $I(T)$.

Briefly, the process of the algorithm starts with the constant monomial 1 and updates it to a new function h as a potential function that vanishes all the points in T (i.e., $h(P_i) = 0$ for all $P_i \in T$). At each compuation cycle, if h vanishes all the points in T, then h goes through a reduction procedure and then is added into the set G. If not, the algorithm will search the first point P_i such that $h(P_i) \neq 0$. An appropriate scale multiple of the leading coefficient (under the selected term order) is subtracted from h so that one more point in T vanishes under the modified h. This h is added to the set S and the previous elements in S will be updated based on this new member so that the separating property is reserved. The searching and reduction process makes it possible that all the members produced for G and S have the lowest possible monomial terms under the pre-selected term order. By the design of the algorithm, the set V is simply the set of monomials appearing in the non-leading terms of all members in G (including their factors). We call this the *factorial property*. The support of each separator is also made of elements from V.

We define a relative degree of x_i with respect to S as follows:

Definition 3.2 *Use the above notation, define*

$$\deg_S(x_i) = \max_H \{ r \in \mathbb{Z}_{>0} \mid x_i^r \in H \},$$

where H is the set of all monomial factors of the elements in $\cup_{i=1}^m$Supp(s_j).

Theorem 3.3 *Consider the Problem 1.2. Assume the Buchberger-Möller Algorithm is applied to obtain the sets G, S, and V as above and $f = b_1 s_1 + \cdots + b_m s_m$ be the interpolator obtained. Then*

1. *$V = \cup_{i=1}^m\{$all monomial factors of elements in supp$(s_i)\}$ and it is uniquely determined by the preselected term order;*
2. *The output set S is unique among all the sets of separators with supports included in V;*
3. *$\sum_{i:\ x_i \in V} \deg_S(x_i) \le m - 1$;*
4. *The output interpolator f involves at most $m - 1$ variables and it is the "smallest" possible (in the meaning of the leading monomial is the smallest) interpolator under the given term order;*
5. *The total degree of each s_i and thus that of f is at most $m - 1$.*

Proof.

(1). It is shown in [4] that V is uniquely determined by the term order. Since $R/I(T) \cong \mathbb{R}^m$ it is obvious that if a monomial belongs to V then all the factor monomials are also in V. By the procedure, each separator created has terms chosen from the non-leading monomial terms and their factors of the current members of G. Thus (1) is true.

For (2), assume $S = \{s_1, \ldots, s_m\}$ and $S' = \{s'_1, \ldots, s'_m\}$ are two sets of separators with nonzero terms in V. Then $(s_i - s'_i)(P_i) = 0$ for all i and so $s_i - s'_i \in I(T)$. But $s_i - s'_i$ is a linear combination over $\mathbb{R}$ of monomials in the vector space basis V of $R/I(T)$ as a vector space over $\mathbb{R}$. So all (real number) coefficients in the combination have to be zero. Thus $s_i - s'_i = 0 \Longrightarrow s_i = s'_i$.

(3) If $x_i \in V$ and $\deg_S(x_i) = r > 0$, then $1, x_i, x_i^2, \ldots, x_i^r \in V$, giving at least $r + 1$ elements in V. But V consists of m linearly independent monomials over $\mathbb{R}$ and has the factorial property, so $1 + \sum_{i:\ x_i \in V} \deg_S(x_i) \le m$. Thus (3) is true.

(4) is from the design of the algorithm. In every step, it is searching the "smallest" candidate for the separators that buldt the interpolator.

(5) is an immediate consequence of (3).

An immediate corollary follows.

Corollary 3.4 *Let f be the output interpolator as above. Then*

1. *If f is linear, then f has the form: $f = a_{i_1} x_{i_1} + a_{i_2} x_{i_2} + \cdots + a_{i_{m-1}} x_{i_{m-1}} + a_{i_m}$, where a_{i_j} are nonzero real numbers for each j;*
2. *If f is a single variable polynomial, then $\deg(f) = m - 1$.*

Proof.

For (1), if f is linear then $V \subseteq \{1, x_1, \ldots, x_n\}$ and $\{x_{i_1}, \ldots, x_{i_{m-1}}\} \subseteq V$ because supp(f) is included in V. Also V must contain 1 by the factorial property. Since the size of V is m, then $V = \{1, x_{i_1}, \ldots, x_{i_{m-1}}\}$. Thus f has the indicated form.

For (2), let x_i^r be the leading monomial of f. Then $\{1, x_i, x_i^2 \ldots, x_i^r\} = V$ implies that $\dim(V) = r + 1$. Thus $r = m - 1$.

In [7] and [13], the existence and construction of linear interpolators are discussed. From the above, we see that linear interpolators invole the most number of variables. In contrast, a single variable interpolator has higher degrees. Furthermore, certain single variable interpolators can be constructed easily using the Vandermonde matrix.

Proposition 3.5 *If the jth components of all interpolating points are distinct, then a single variabe interpolator in the variable x_j exists.*

Proof. Consider the set $T = \{P_1, \ldots, P_m\}$, where $P_i = (p_{i1}, p_{i2}, \ldots, p_{in})$, as before. Without loss of generality, assume $p_{11}, p_{21}, \ldots, p_{m1}$ are all distinct. Then the Vandermonde matix

$$M = \begin{bmatrix} 1 & p_{11} & p_{11}^2 & \cdots & p_{11}^{m-1} \\ 1 & p_{21} & p_{21}^2 & \cdots & p_{21}^{m-1} \\ & \cdots & \cdots & \cdots & \\ 1 & p_{m1} & p_{m1}^2 & \cdots & p_{m1}^{m-1} \end{bmatrix}.$$

is invertible. So the matrix equation $M\mathbf{y}^t = \mathbf{b}^t$ has a unique solution, where $\mathbf{b} = [b_1, \ldots, b_m]$ is the target of the interpolation problem. The solution vector, $[a_0, a_1, \ldots, a_{m-1}]^t$, gives the coefficients for the single variable interpolator:

$$f(x_1, \ldots, x_n) = a_{m-1} x_1^{m-1} + \cdots + a_1 x_1 + a_0.$$

It satisfies $f(P_i) = b_i$ for $i = 1, 2, \ldots, m$.

4 Examples and Applications

Example 4.1

Let $T = \{P_1, P_2, P_2, P_4\}$, where

$$\begin{aligned} P_1 &= (0, 0, 0, 0, 0, 0) \\ P_2 &= (1, 10, 3, -5, -2, -1) \\ P_3 &= (10, 10, -6, -14, -2, -10) \\ P_4 &= (-22, -22, 15, 38, 17, 13). \end{aligned}$$

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Let τ be the Lex order and σ be the Degree reverse lex order [3]. For the lex order $<_\tau$, $\mathbf{x}^\alpha <_\tau \mathbf{x}^\beta$ if and only if there exists an i: $\alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i <_\tau \beta_i$. The order τ favors certain variables and it is given by:

$$\begin{aligned} 1 <_\tau x_6 <_\tau x_6^2 <_\tau x_6^3 <_\tau \cdots \\ <_\tau x_5 <_\tau x_5^2 <_\tau \cdots <_\tau x_1 <_\tau x_1^2 <_\tau \cdots \\ <_\tau x_6x_5 <_\tau x_6^2x_5 <_\tau \cdots . \end{aligned}$$

The graded reversed lex order $<_\sigma$ favors polynomials of lower total degrees:

$$\begin{aligned} 1 <_\sigma x_6 <_\sigma x_5 <_\sigma \cdots <_\sigma x_1 \\ <_\sigma x_6^2 <_\sigma x_6x_5 <_\sigma \cdots <_\sigma x_6x_1 \\ <_\sigma x_5^2 <_\sigma x_5x_4 <_\sigma \cdots <_\sigma x_5x_1 <_\sigma \cdots . \end{aligned}$$

Applying the Buchberger-Möller Algorithm we obtain two sets of separators. Under τ, the separators are

$$\begin{aligned} s_1 &= -\tfrac{1}{130}x_6^3 + \tfrac{1}{65}x_6^2 + \tfrac{133}{130}x_6 + 1 \\ s_2 &= \tfrac{1}{126}x_6^3 - \tfrac{1}{42}x_6^2 - \tfrac{65}{63}x_6 \\ s_3 &= -\tfrac{1}{2070}x_6^3 + \tfrac{2}{345}x_6^2 + \tfrac{13}{2070}x_6 \\ s_4 &= \tfrac{1}{4086}x_6^3 + \tfrac{11}{486}x_6^2 + \tfrac{5}{2093}x_6. \end{aligned}$$

Under σ, the separators are

$$\begin{aligned} r_1 &= \tfrac{19}{18}x_4 - \tfrac{29}{18}x_5 - \tfrac{19}{18}x_6 + 1 \\ r_2 &= -\tfrac{8}{9}x_4 + \tfrac{11}{9}x_5 + x_6 \\ r_3 &= -\tfrac{1}{18}x_4 + \tfrac{1}{6}x_5 - \tfrac{1}{18}x_6 \\ r_4 &= -\tfrac{1}{9}x_4 + \tfrac{1}{9}x_5 + \tfrac{1}{9}x_6. \end{aligned}$$

Correspondingly, we can obtain two very different interpolators of T targeting at b_1, b_2, b_3, b_4:

$$f = b_1s_1 + b_2s_2 + b_3s_3 + b_4s_4 \qquad \textit{and}$$

$$g = b_1r_1 + b_2r_2 + b_3r_3 + b_4r_4.$$

It shows that f is a single variable interpolator involving only x_6 but the degree 3 is higher than the total degree 1 of g. While g has the lowest possible total degree, i.e., g is linear, but it involves more variables (x_4, x_5, x_6). The supports of f and g are quite different because of the selection of different term orders. The point is, one should choose an appropriate term order based on the type of interpolator she/he favors. For the above example, the order τ gives interpolators with less number of variables but higher degrees. On the other hand, the order σ produces interpolators with low total degrees but more variables.

Example 4.2 *Let $h(x,y)$ be any real function over $\mathbb{R}^2$. We choose six points in $\mathbb{R}^2$: $P_1 = (3,1)$, $P_2 = (2,2), P_3 = (1,3), P_4 = (1/2,2), P_5 = (2/3,4)$, $P_6 = (3/2,1/2)$. Assume the values of h on the $P_i's$ are $10, 6, 4, 9/4, 40/9, 11/4$ respectively. our goal is to construct an interpolator $f(x,y)$ such that $f(P_i) = h(P_i)$ for all i. The problem is to find an interpolator on the $T = \{P_1, \ldots, P_6\}$ with the target $\mathbf{b} = (10, 6, 4, 9/4, 40/9, 11/4)$.*

By computing the separators using the graded reverse lex order, we obtain the following interpolator:

$$f(x,y) = \frac{59}{9}x^2 + \frac{100}{27}y^2 + \frac{250}{27}xy - \frac{875}{27}x - 24y + \frac{1100}{27}.$$

The next example shows an application of the previously mentioned interpolating methods in modeling a time series from a real data.

Example 4.3 *Consider a data table representing the energy consumption of a region in China from 1980 to 1985 (Tab. 1):*

Tab. 1 Regional Energy Consumption in China

Year	Total	C	P	NG	WE
1980	60257	72.10	20.85	3.06	3.99
1981	24947	72.75	20.00	2.74	4.51
1982	62646	74.02	18.58	2.48	4.92
1983	66040	74.27	18.07	2.40	5.26
1984	70904	75.31	17.45	2.33	4.91
1995	77020	75.92	17.02	2.23	4.83

Here each row represents the data from one year. The first component is the total amount in ten thousand tons of consumption of all four categories of energy. The second, third, fourth, and fifth components are the percentage of the consumption of coal (C), petroleum (P), natural gas (NG), or water/electricity (WE), respectively. We view the set of the first five row vectors, $T = \{P_1, \ldots, P_5\}$, as a discrete time series. For a fixed term order, we apply the Buchberger-Möller Algorithm to interpolate the point set $T' = T \setminus \{P_5\}$ at the five specially selected values so that we can get five interpolators to form a polynomial model $\mathbf{f}$ for the time series T'. That is, $\mathbf{f}(P_i) = P_{i+1}$ for $i = 1, 2, 3, 4$. We use the last data point, P_6, to measure how close P_6 is to the prediction $\mathbf{f}(P_5)$ by the model $\mathbf{f}$.

Using the same two term orders τ and σ as before, we obtain two polynomial models $\mathbf{f} = (f_1, f_2, f_3, f_4, f_5)$ under τ and $\mathbf{g} = (g_1, g_2, g_3, g_4, g_)$ under σ, shown below:

$$\begin{aligned} f_1 &= -\tfrac{16805600}{2091}x_5^2 - \tfrac{141167408}{2091}x_5 + \tfrac{86550846}{425} \\ f_2 &= \tfrac{2276}{697}x_5^2 - \tfrac{525942}{17425}x_5 + \tfrac{6108403}{42500} \\ f_3 &= -\tfrac{1616}{2091}x_5^2 + \tfrac{315947}{52275}x_5 + \tfrac{299259}{42500} \\ f_4 &= -\tfrac{10}{697}x_5^2 - \tfrac{417}{6970}x_5 + \tfrac{12927}{4250} \\ f_5 &= -\tfrac{5182}{2091}x_5^2 - \tfrac{2530013}{104550}x_5 - \tfrac{571733}{10625} \end{aligned}$$

and

$$\begin{aligned} g_1 &= \tfrac{2100700}{139}x_4 + \tfrac{2482800}{139}x_5 - \tfrac{8245552}{139} \\ g_2 &= \tfrac{1707}{278}x_4 + \tfrac{626}{139}x_5 + \tfrac{512693}{13900} \\ g_3 &= -\tfrac{202}{139}x_4 - \tfrac{301}{139}x_5 + \tfrac{449361}{13900} \\ g_4 &= -\tfrac{15}{556}x_4 - \tfrac{59}{278}x_5 + \tfrac{12201}{3475} \\ g_5 &= -\tfrac{2591}{556}x_4 - \tfrac{591}{278}x_5 - \tfrac{189571}{6950}. \end{aligned}$$

One can check that

$$\mathbf{f}(P_i) = \mathbf{g}(P_i) = P_{i+1} \quad \text{for } i = 1, 2, 3, 4.$$

To compare the predictions of these two models with the real data point $P_6 = (77020, 75.92, 17.02, 2.23, 4.83)$, we evaluate each of the models at P_5 and compute the differences between them and P_6:

$$\mathbf{f}(P_5) = (65925.07, 74.25, 18.09, 2.40, 5.26)$$

$$\mathbf{g}(P_5) = (63594.44, 73.30, 18.31, 2.41, 5.98) \quad \text{with}$$

$$\begin{aligned} & d_1 = P_6 - \mathbf{f}(P_5) \\ & \quad = (11094.93, 1.67, -1.07, -0.17, -0.43); \\ & d_2 = P_6 - \mathbf{g}(P_5) \\ & \quad = (11094.93, 1.67, -1.07, -0.17, -0.43). \end{aligned}$$

If we use the regular vector norm (square root of the sum of squares of all components) to measure the differences, we have $\|d_1\| \geq \|d_2\|$. So the order σ gives a better prediction of P_6. We also observe that most errors are from the first component. The error for the other components are relatively much smaller. More work needs to be done to understand the reason for this result.

5 Conclusions

So far we have focused on only two term orders. It shows that the selection of term orders may have significant impact on the output interpolator. There are infinitely many different term orders which we can play with [3]. We will explore other term orders and develop selection criteria for different types of data and desired interpolators. We will develop a more efficient algorithmic code for lager scale computation. We will target real world data and use our interpolating techniques to build needed interpolators or desired polynomial models of the time series involved.

Another direction to go would be to study what set of monomials of size m can form a vector space basis for $R/I(T)$. Such a set has the factorial property and can help construct an interpolator more rapidly.

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